Chapter one (introduction)

(1) Linear and Nonlinear Integral Equations Abdul-Majid Wazwaz 2011

2- Integral Equations and their Applications M. Rahman 2007

Standard Integral equation

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt \qquad (1.1)$$

λ is constant

$k(x,t)$ is called the kernel of integral (1.1) $\int_{-1}^{x+1} e^{-2x}e^{t}dt$

Abel in 1825 italian mathematician, first produced an integral equation.

Integro-differential equation

$$u^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt. \qquad (1.2)$$

$$u''(x) = u'(x) + u(x) + 3 + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt\,.$$

1.2 Leibnitz Rule for Differentiation of Integrals

One of the methods that will be used to solve integral equations is the conversion of the integral equation to an equivalent differential equation. The conversion is achieved by using the well-known Leibnitz rule [4,6,7] for differentiation of integrals. Let $f(x,t)$ be continuous and $\frac{\partial f}{\partial t}$ be continuous , and let

$$F(x) = \int_{g(x)}^{h(x)} f(x,t)dt, \qquad (1.3)$$

then differentiation of the integral in (1.3) exists and is given by

$$F'(x) = \frac{dF}{dx} = f(x,h(x))\frac{dh(x)}{dx} - f(x,g(x))\frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}dt. \, (1.4)$$

If $g(x) = a$ and $h(x) = b$ where a and b are constants, then the Leibnitz rule (1.4) reduces to

$$F'(x)=\int_a^b \frac{\partial f(x,t)}{\partial x}dt. (1.5)$$

Ex Find $\frac{d}{dx}\int_{\sin x}^{\cos x}\sqrt{1+t^3}dt$

$$=0+(-\sin x)\sqrt{1+\cos^3 x}-(\cos x)\sqrt{1+\sin^3 x}$$

Ex Find $\frac{d}{dx}\int_x^{x^2}(x-t)\cos t\,dt$

$$=\int_x^{x^2}\cos t\,dt+2x(x-x^2)\cos x^2-0=\sin x^2-\sin x+2x^2(1-x)\cos x^2$$

Ex Find $\frac{d}{dx}\int_0^x xtu(t)\,dt, \frac{d^2}{dx^2}\int_0^x xt\,u(t)dt$

$$F'(x)=x^2u(x)+\int_0^x tu(t)dt\,,\;\; F''(x)=x^2u'(x)+3xu(x)$$

H.W. Find $F'(x), F''(x), F'''(x)$ and $F''''(x)$ for the following integral

$$F(x)=\int_0^x (x-t)^3\,u(t)dt.$$

1.4 Reducing multiple integrals to single integrals

Show that

$$\int_0^x\int_0^s f(t)dtds=\int_0^x(x-t)f(t)dt \qquad (*)$$

Ans.: method 1: $\frac{d}{dx}\int_0^x(x-t)f(t)dt=\int_0^x f(t)dt+0-0$

integrating both sides from 0 to x we get

$$\int_0^x(x-t)f(t)dt=\int_0^x\left(\int_0^s f(t)dt\right)ds$$

method 2: $\int_0^x udv=uv|_0^x-\int_0^x vdu$

Let $u=\int_0^s f(t)dt\,,\; dv=ds\rightarrow du=f(s)ds, v=s$

$$\int_0^x\int_0^s f(t)dtds=\int_0^s f(t)dt\,s|_0^x-\int_0^x sf(s)ds$$

$$= x\int_0^x f(t)dt - \int_0^x sf(s)ds$$

$$= \int_0^x xf(t)dt - \int_0^x tf(t)dt = \int_0^x (x-t)f(t)\,dt$$

general form

$$\int_0^x \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_{n-1}} u(x_n)dx_n dx_{n-1} \cdots dx_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}u(t)dt. \quad (1.6)$$

Or

$$\overbrace{\int_0^x \int_0^x \int_0^x \cdots \int_0^x (x-t)u(t)dtdt \cdots dt}^{n \text{ integrals}} = \frac{1}{n!}\int_0 (x-t)^n u(t)dt \quad (1.7)$$

Example 1.4 Convert the following multiple integral to a single integral:

$$\int_0^x \int_0^{x_1} \int_0^{x_2} u(x_3)dx_2 dx_1 dx\ . \qquad (1.8)$$

Using the formula (1.6) we obtain

$$\int_0^x \int_0^{x_1} \int_0^{x_2} u(x_3)dx_2 dx_1 dx = \frac{1}{2!}\int_0^x (x\ -\ t)^2 u(t)dt.$$

Example 1.5 Convert the following multiple integral to a single integral:

$$\int_0^x \int_0^x \int_0^x (x-t)u(t)dtdtdt. \qquad (1.9)$$

Using the formula (1.6) we obtain

$$\int_0^x \int_0^x \int_0^x (x-t)u(t)dtdtdt = \frac{1}{3!}\int_0^x (x\ -\ t)^3 u(t)dt$$

1.4: Classification of integral equations

There are four major types of integral equations

- Fredholm integral equations
- Volterra integral equations
- Integro-differential equations

- Singular integral equations

1. Fredholm integral equations

$$\phi(x)u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \qquad (1.10)$$

where a, b are constant, if $\phi(x) = 1$ then (1.10) becomes

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \qquad (1.11)$$

which known as Fredholm integral equation of second kind. If $\phi(x) = 0$ then (1,11) becomes

$$f(x) + \lambda \int_a^b k(x,t)u(t)dt = 0$$

which is called fredholim integral equation of First kind

2. Volterra integral equations

$$\phi(x)u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt, \qquad (1.12)$$

where $g(x), h(x)$ are functions, if $\phi(x) = 1$ then (1.12) becomes

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt, \qquad (1.13)$$

which known as Volterra integral equation of second kind. If $\phi(x) = 0$ then (1,13) becomes

$$f(x) + \lambda \int_0^x k(x,t)u(t)dt = 0$$

which is called Volterra integral equation of First kind

Examples of the Volterra integral equations of the first kind are

$$xe^{-x} = \int_0^x e^{t-x}u(t)dt, \qquad (1.14)$$

and

$$5x^2 + x^3 = \int_0^x (5 + 3x - 3t)u(t)dt. \qquad (1.15)$$

However, examples of the Volterra integral equations of the second kind are

$$u(x) = 1 - \int_0^x u(t)\, dt\,, \qquad (1.16)$$

and

$$u(x) = x + \int_0^x (x-t)u(t)\, dt\,, \qquad (1.17)$$

1.1.3 Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt, \quad (1.18)$$

and

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F\big(x,t,\xi,\tau,u(\xi,\tau)\big)d\xi d\tau\,, (x,t) \in \Omega \times [0,T\,], (1.19)$$

1.1.4 Singular Integral Equations

Volterra integral equations of the first kind

$$f(x) = \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt,$$

or of the second kind

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt,$$

are called singular if one of the limits of integration $g(x), h(x)$ or both are infinite. Moreover, the previous two equations are called singular if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration. In this text we will focus our concern on equations of the form:

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)\, dt, \;\; 0 < \alpha < 1, \qquad (1.20)$$

or of the second kind:

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u(t)\, dt, \;\; 0 < \alpha < 1. \;\; (1.21)$$

1.5 Converting IVP to Volterra Integral Equation

In this section, we will study the technique that will convert an initial value problem (IVP) to an equivalent Volterra integral equation and Volterra integro-differential equation as well. We will apply this process to a second order initial value problem given by

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x) \qquad (1.22)$$

Subject to the initial conditions: $y(0) = \alpha, y'(0) = \beta,$ (1.23)

Where α and $\beta$are constants. The functions $p(x)$and $q(x)$ are analytic functions, and $g(x)$ is continuous through the interval of discussion. To achieve our goal we first set $y''(x) = u(x),$ (1.24)

where $u(x)$ a continuous function. Integrating both sides of (1.24) from 0 to x yields

$$y'(x) - y'(0) = \int_0^x u(t)dt, \qquad (1.25)$$

or equivalently

$$y'(x) = \beta + \int_0^x u(t)dt, \qquad (1.26)$$

Integrating both sides of (1.26) from 0 to x yields

$$y(x) - y(0) = \beta x + \int_0^x \int_0^x u(t)dtdt, \qquad (1.27)$$

or equivalently by (1.6)

$$y(x) = \alpha + \beta x + \int_0^x (x-t)u(t)dt, \qquad (1.28)$$

Substituting (1.24), (1.26), and (1.28) into the initial value problem (1.22) yields the Volterra integral equation:

$$u(x) + p(x)\left(\beta + \int_0^x u(t)dt\right) + q(x)\left(\alpha + \beta x + \int_0^x (x-t)u(t)dt\right) = g(x). \qquad (1.29)$$

The last equation can be written in the standard Volterra integral equation form:

$$u(x) = f(x) - \int_0^x K(x,t)u(t)dt, \qquad (1.30)$$

where

$$K(x,t) = p(x) + q(x)(x-t), \tag{1.31}$$

and

$$f(x) = g(x) - [\beta p(x) + \alpha q(x) + \beta x q(x)]\,. \tag{1.32}$$

Differentiating Volterra equation (1.29) with respect to x, using Leibnitz rule, we obtain an equivalent Volterra integro-differential equation in the form:

$$u'(x) + K(x,x)u(x) = f'(x) - \int_0^x \frac{\partial K(x,t)}{\partial x} u(t)dt, \;\; u(0) = f(0). \tag{1.33}$$

The technique presented above to convert initial value problems to equivalent Volterra integral equations can be generalized by considering the general initial value problem:

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = g(x), \tag{1.34}$$

with the initial conditions

$$y(0) = c_0, y'(0) = c_1, y''(0) = c_2, \ldots, y^{(n-1)}(0) = c_{n-1}. \tag{1.35}$$

Let $u(x)$ be a continuous function on the interval of discussion, and we consider the transformation: $y^{(n)}(x) = u(x)$. (1.36). Integrating both sides with respect to x gives

$$y^{(n-1)}(x) = c_{n-1} + \int_0^x u(t)dt. \tag{1.37}$$

Integrating again both sides with respect to x yields

$$\begin{aligned} y^{(n-2)}(x) &= c_{n-2} + c_{n-1}x + \int_0^x \int_0^x u(t)dtdt \\ &= c_{n-2} + c_{n-1}x + \int_0^x (x-t)u(t)dt\,, \end{aligned} \tag{1.38}$$

obtained by reducing the double integral to a single integral. Proceeding as before we find

$$\begin{aligned} y^{(n-3)}(x) &= c_{n-3} + c_{n-2}x + \frac{1}{2}\, c_{n-1}x^2 + \int_0^x \int_0^x \int_0^x u(t)dtdtdt \\ &= c_{n-3} + c_{n-2}x + \frac{1}{2}\, c_{n-1}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt\,. \end{aligned} \tag{1.39}$$

Continuing the integration process leads to

$$y(x) = \sum_{k=0}^{n-1} \frac{c_k}{k!} x^k + \frac{1}{(n\ -\ 1)!} \int_0^x (x-t)^{n-1}\, u(t)dt. \quad (1.40)$$

Substituting (1.36)–(1.40) into (1.34) gives

$$u(x) = f(x) - \int_0^x K(x,t)u(t)dt, \qquad (1.41)$$

where $K(x,t) = \sum_{k=1}^{n} \frac{an}{(k\ -\ 1)!}(x-t)^{k-1}$, (1.42)

$$\text{and} \quad f(x) = g(x) - \sum_{j=0}^{n} a_j \,(\sum_{k=1}^{n-1} \frac{c_{n-k}}{(j-k)!} x^{j-k}. \qquad (1.43)$$

Notice that the Volterra integro-differential equation can be obtained by differentiating (1.34) as many times as we like, and by obtaining the initial conditions of each resulting equation. The following examples will highlight the process to convert initial value problem to an equivalent Volterra integral equation.

Example 1.1 Convert the following initial value problem to an equivalent Volterra integral equation: $y'(x) - 2xy(x) = e^{x^2},\ y(0) = 1.$ (1.44)

We first set $y'(x) = u(x)$. (1.45) Integrating both sides of (1.44), using the initial condition $y(0) = 1$ gives

$$y(x) - y(0) = \int_0^x u(t)dt, \quad (1.46)$$

$$y(x) = 1 + \int_0^x u(t)dt, \quad (1.47)$$

Substituting (1.45) and (1.47) into (1.44)gives the equivalent Volterra integral equation:

$$u(x) = 2x + e^{x^2} + 2x\int_0^x u(t)dt. \quad (1.48)$$

Example 1.2 Convert IVP to an equivalent Volterra integral equation:

$$y''(x) - y(x) = \sin x\,,\ \ y(0) = 0,\ \ y'(0) = 0. \quad (1.49)$$

Let $y''(x) = u(x).$ (1.50)

Integrating both sides of (1.50), using the initial condition $y'(0) = 0$ gives $y'(x) = \int_0^x u(t)dt$ (1.51). Integrating (1.51) again, using the initial condition $y(0) = 0$, yields

$$y(x) = \int_0^x \int_0^x u(t)dtdt = \int_0^x (x-t)u(t)dt \qquad (1.52)$$

obtained upon using the rule to convert double integral to a single integral. Inserting $(1.50) - (1.52)$ into (1.49) leads to the following Volterra integral equation:

$$u(x) = \sin x + \int_0^x (x-t)u(t)dt. \qquad (1.53)$$

Example 1.3 Convert IVP to an equivalent Volterra integral equation:

$$y''' - y'' - y' + y = 0, y(0) = 1, y'(0) = 2, y''(0) = 3. (1.53)$$

We first set $y'''(x) = u(x)$, (1.54) where by integrating both sides of (1.54) and using the initial condition $y''(0) = 3$ we obtain $y'' = 3 + \int_0^x u(t)dt$. (1.55)
Integrating again and using the initial condition $y'(0) = 2$ we find

$$y'(x) = 2 + 3x + \int_0^x \int_0^x u(t)dtdt = 2 + 3x + \int_0^x (x-t)u(t)dt. \qquad (1.56)$$

Integrating again and using $y(0) = 1$ we obtain

$$y(x) = 1 + 2x + \frac{3}{2}x^2 + \int_0^x \int_0^x \int_0^x u(t)dtdtdt$$
$$= 1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt \quad (1.57)$$

Substituting $(1.54) - (1.57)$ leads to the Volterra integral equation:

$$u(x) = 4 + x + \frac{3}{2}x^2 + \int_0^x \left[1 + (x-t) - \frac{1}{2}(x-t)^2\right] u(t)dt \quad (1.58)$$

Remark We can also show that if $y^{(\text{iv})}(x) = u(x)$, then

$$
\begin{aligned}
y'''(x) &= y'''(0) + \int_0^x u(t)dt, \\
y''(x) &= y''(0) + xy'''(0) + \int_0^x (x-t)u(t)dt \\
y'(x) &= y'(0) + xy''(0) + \frac{1}{2}x^2y'''(0) + \frac{1}{2}\int_0^x (x-t)^2u(t)dt \\
y(x) &= y(0) + xy'(0) + \frac{1}{2}x^2y''(0) + \frac{1}{6}x^3y'''(0) + \frac{1}{6}\int_0^x (x-t)^3u(t)dt
\end{aligned}
\quad , \qquad (1.59)
$$

1.6 Converting Volterra Integral Equation to IVP

The resulting initial value problems can be solved easily by using ODEs methods that were summarized in Chapter 1. The conversion process will be illustrated by discussing the following examples.

Example 1.4 Find the initial value problem equivalent to the Volterra integral equation:

$$u(x) = e^x + \int_0^x u(t)dt. \qquad (1.60)$$

Differentiating both sides of (1.60) and using Leibnitz rule we find

$$u'(x) = e^x + u(x). \qquad (1.61)$$

Since there is no integral so there is no need for differentiating again. To determine the initial condition, we substitute $x = 0$ into both sides of (1.60) to find $u(0) = 1$. This in turn gives the initial value problem:

$$u'(x) - u(x) = e^x, \;\; u(0) = 1. \qquad (1.62)$$

Notice that the resulting ODE is a linear inhomogeneous equation of first order.

Example 1.5 Find the initial value problem equivalent to the Volterra integral equation:

$$u(x) = x^2 + \int_0^x (x-t)u(t)dt. \qquad (1.63)$$

Differentiating both sides of (1.63) and using Leibnitz rule we find

$$u'(x) = 2x + \int_0^x u(t)dt. \qquad (1.64)$$

To get rid of the integral sign we should differentiate (1.64) and by using Leibnitz rule we obtain the second order ODE:

$$u''(x) = 2 + u(x). \qquad (1.65)$$

To determine the initial conditions, we substitute x = 0 into both sides of (1.63) and (1.64) to find $u(0) = 0$ and $u'(0) = 0$. This in turn gives the initial value problem:

$$u''(x) - u(x) = 2, \ \ u(0) = 0, \ \ u'(0) = 0.$$

Example 1.6 Find the initial value problem equivalent to the Volterra integral equation:

$$u(x) = \sin x - \frac{1}{2}\int_0^x (x-t)^2 u(t)dt. \qquad (1.66)$$

Differentiating both sides of (1.66) three times to get rid of the integral sign to find

$$u'(x) = \cos x - \int_0^x (x-t)u(t)dt, \ \ u''(x) = -\sin x - \int_0^x u(t)dt,$$

$$u'''(x) = -\cos x - u(x).$$

Substituting $x = 0$ gives the initial conditions: $u(0) = 0, u'(0) = 1, u''(0) = 0$. In view of the last results, the initial value problem equivalent to the Volterra integral equation is a third order inhomogeneous ODE given by

$$u'''(x) + u(x) = -\cos x\,, u(0) = 0, u'(0) = 1, u''(0) = 0. \qquad (1.67)$$

Exercises 2.5

Convert each of the following IVPs to an equivalent Volterra integral equation:

1. $y' - 4y = 0, y(0) = 1$
2. $y' + 4xy = e^{-2x^2}\,, \ \ y(0) = 0$
3. $y'' + 4y = 0, y(0) = 0, y(0) = 1$
4. $y''' - y' = 0, y(0) = 2, y'(0) = y''(0) = 1$
5. $y^{(\text{iv})} + y'' + y = x, y(0) = y'(0) = 1, y''(0) = y'''(0) = 0$

Convert each of the following Volterra integral equation to an equivalent IVP:

6. $u(x) = x + 2\int_0^x u(t)dt$

7. $u(x) = 1 + x^2 + \int_0^x (x-t)u(t)dt$

8. $u(x) = 1 - \cos x + 2\int_0^x (x-t)^2 u(t)dt$

9. $u(x) = 2 + \sinh x + \int_0^x (x-t)^2 u(t)dt$

10. $u(x) = 1 + e^x + \int_0^x (1+x-t)^3 u(t)dt$

1.6 Converting BVP to Fredholm Integral Equation

Type I

We first consider the following boundary value problem:

$$y''(x) + g(x)y(x) = h(x), \ \ 0 < x < b, \qquad (1.68)$$

with the boundary conditions: $y(0) = \alpha, \ y(b) = \beta. (1.69)$ We start as in the previous section and set $y''(x) = u(x)$. (1.70) Integrating both sides of (1.70) from 0 to x we obtain

$$\int_0^x y''(t)dt = \int_0^x u(t)dt,$$

that gives

$$y'(x) = y'(0) + \int_0^x u(t)dt, \qquad (1.71)$$

where the initial condition $y'(0)$ is not given in a boundary value problem. The condition $y'(0)$ will be determined later by using the boundary condition at $x = b$. Integrating both sides of (1.71) from 0 to x gives

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x u(t)dtdt,$$

or equivalently by reducing double integral to a single integral.

$$y(x) = \alpha + xy'(0) + \int_0^x (x-t)u(t)dt, \qquad (1.72)$$

To determine $y'(0)$, we substitute $x = b$ into both sides of (1.72) and using the boundary condition at $y(b) = \beta$ we find

$$y(b) = \alpha + by'(0) + \int_0^b (b-t)u(t)dt,$$

that gives

$$by'(0) = \beta - \alpha - \int_0^b (b-t)u(t)dt,$$

This in turn gives

$$y'(0) = \frac{1}{b}(\beta - \alpha) - \frac{1}{b}\int_0^b (b-t)u(t)dt, \qquad (1.73)$$

Substituting (1.73) into (1.72) gives

$$y(x) = \alpha + (\beta - \alpha)\frac{x}{b} - \int_0^b \frac{x}{b}(b-t)u(t)dt + \int_0^x (x-t)u(t)dt. \,(1.74)$$

Substituting (2.70) and (1.74) into (1.68) yields

$$u(x) + [\alpha + (\beta - \alpha)\frac{x}{b}]g(x) - \int_0^b \frac{x}{b}g(x)(b-t)u(t)dt$$

$$+ \int_0^x g(x)(x - t)u(t)dt = h(x). \,(1.75)$$

From calculus we can use the formula:

$$\int_0^b (\cdot) = \int_0^x (\cdot) + \int_x^b (\cdot),$$

to carry Eq. (1.75) to

$$u(x) = h(x) - [\alpha + (\beta - \alpha)\frac{x}{b}]g(x) + \int_0^x \frac{g(x)}{b}(b-x)tu(t)dt$$

$$+ \int_x^b \frac{xg(x)}{b}(b-t)u(t)dt,$$

that leads to the Fredholm integral equation:

$$u(x) = f(x) + \int_0^b K(x,t)u(t)dt, \qquad (1.76)$$

Where

$$f(x) = h(x) - \left(\alpha + (\beta - \alpha)\frac{x}{b}\right) g(x), \qquad (1.77)$$

and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} \dfrac{g(x)}{b}(b-x)t \text{ for } & 0 \le t \le x \\ \dfrac{xg(x)}{b}(b-t), \text{for } & x \le t \le b \end{cases} \qquad (1.78)$$

Example 1.7 Convert the following BVP to an equivalent Fredholm integral equation: $y''(x) + 9y(x) = \cos x, \;\; y(0) = y(1) = 0.$

$\beta = 0, \alpha = 0, \; g(x) = 9$ and $h(x) = \cos x$. This in turn gives $f(x) = \cos x$. Substituting this into (1.76) gives the Fredholm integral equation:

$$u(x) = \cos x + \int_0^1 K(x,t)u(t)dt,$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 9t(1-x), & \text{for } \; 0 \le t \le x \\ 9x(1-t), & \text{for } \; x \le t \le 1 \end{cases}.$$

Example 1.8 Convert the following BVP to an equivalent Fredholm integral equation: $y''(x) + xy(x) = 0, \;\; y(0) = 0, \;\; y(3) = 2.$

$\alpha = 0, \beta = 2, b = 3, g(x) = x$ and $h(x) = 0$. This in turn gives $f(x) = h(x) - \left(\alpha + (\beta - \alpha)\frac{x}{b}\right) g(x) = -\frac{2x^2}{3}$. Substituting this into (1.76) gives the Fredholm integral equation:

$$u(x) = -2x^2 + \int_0^3 K(x,t)u(t)dt,$$

Where the kernel $K(x,t)$ in (1.78) is given by

$$K(x,t) = \begin{cases} \dfrac{tx}{3}(3-x), & \text{for } \; 0 \le t \le x \\ \dfrac{x^2}{b}(3-t), & \text{for } \; x \le t \le 1 \end{cases}.$$

Type II

We next consider the following boundary value problem:

$$y''(x) + g(x)y(x) = h(x), \ \ 0 < x < b, \qquad (1.68)$$

with the boundary conditions: $y(0) = \alpha_1, y'(b) = \beta_1. (1.79).$ We again set

$$y''(x) = u(x). \qquad (1.70)$$

Integrating both sides of (1.70) from 0 to x we obtain

$$\int_0^x y''(t)dt = \int_0^x u(t)dt$$

that gives

$$y'(x) = y'(0) + \int_0^x u(t)dt, \qquad (1.80)$$

Where the initial condition $y'(0)$ is not given. The condition $y'(0)$ will be derived later by using the boundary condition at $y'(b) = \beta_1$. Integrating both sides of (1.80) from 0 to x gives

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x u(t)dtdt,$$

or equivalently using the condition $y(0) = \alpha_1$ and by reducing double integral to a single integral.

$$y(x) = \alpha_1 + xy'(0) + \int_0^x (x - t)u(t)dt, \ \ (1.81)$$

To determine $y'(0)$, we first differentiate (1.81) with respect
to x to get

$$y'(x) = y'(0) + \int_0^x u(t)dt, \qquad (1.82)$$

where by substituting $x = b$ into both sides of (1.82) and using the boundary condition at $y'(b) = \beta_1$ we find

$$y'(b) = y'(0) + \int_0^b u(t)dt,$$

that gives

$$y'(0) = \beta_1 - \int_0^b u(t)dt.$$

$$y'(0)x = \beta_1 x - \int_0^b xu(t)dt. \quad (1.83)$$

Using (1.83) into (1.81) gives

$$y(x) = \alpha_1 + \beta_1 x - \int_0^b xu(t)dt + \int_0^x (x - t)u(t)dt. \quad (1.84)$$

Substituting (1.70) and (1.84) into (1.68) yields

$$u(x) + \alpha_1 g(x) + \beta_1 xg(x) - \int_0^b xg(x)u(t)dt + \int_0^x g(x)(x - t)u(t)dt$$
$$= h(x). (1.85)$$

or

$$u(x) = h(x) - (\alpha_1 + \beta_1 x)g(x) + xg(x)[\int_0^x u(t)dt + \int_x^b u(t)dt$$
$$- g(x)\int_0^x (x - t)u(t)dt.$$

The last equation can be written as

$$u(x) = h(x) - (\alpha_1 + \beta_1 x)g(x) + \int_0^x g(x)tu(t)dt + \int_x^b xg(x)u(t)\,dt. \quad (1.86)$$

that leads to the Fredholm integral equation:

$$u(x) = f(x) + \int_0^b K(x,t)u(t)dt, \quad (1.87)$$

Where

$$f(x) = h(x) - (\alpha_1 + \beta_1 x)g(x). \quad (1.88)$$

and the kernel K(x, t) is given by

$$K(x,t) = \begin{cases} tg(x) \text{ for } & 0 \le t \le x \\ xg(x), \text{for } & x \le t \le b \end{cases} \quad (1.89)$$

Example 2.9 Convert the BVP to an equivalent Fredholm integral equation:

$$y''(x) + y(x) = 0, y(0) = y'(b) = 0.$$

We can easily observe that $\alpha_1 = \beta_1 = 0, g(x) = 1$ and $h(x) = 0$. This in turn gives $f(x) = 0$. Substituting this into equation gives the homogeneous Fredholm integral equation:

$$u(x) = \int_0^b K(x,t)u(t)dt,$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} t & \text{for} \quad 0 \le t \le x \\ x, & \text{for} \quad x \le t \le b \end{cases}$$

Example 2.10 Convert BVP to an equivalent Fredholm integral equation:

$$y''(x) + 2y(x) = 4, \quad y(0) = 0, y(2) = 1.$$

We can easily observe that $\alpha_1 = 0, \beta_1 = 1, g(x) = 2$ and $h(x) = 4$. This in turn gives $f(x) = 4 - 2x$. Then the inhomogeneous Fredholm integral equation:

$$u(x) = 4 - 2x + \int_0^2 K(x,t)u(t)dt,$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 2t & \text{for} \quad 0 \le t \le x \\ 2x, & \text{for} \quad x \le t \le 2 \end{cases}$$

Converting Fredholm Integral Equation to BVP.

2.6.1 Converting Fredholm Integral Equation to BVP

Type I

We first consider the Fredholm integral equation given by

$$u(x) = f(x) + \int_0^b K(x,t)u(t)dt, \qquad (1.76)$$

$$u(x) = f(x) + \int_0^x \frac{tg(x)}{b}(b-x)u(t)dt + \int_x^b \frac{xg(x)}{b}(b-t)u(t)dt. \quad (1.76)$$

Where

$$f(x) = h(x) - \left(\alpha + (\beta - \alpha)\frac{x}{b}\right)g(x), \qquad (1.77)$$

and the kernel $K(x,t)$ is given by

$$K(x,t)=\begin{cases}\dfrac{g(x)}{b}(b-x)t, & \text{for } \ 0\le t\le x\\ \dfrac{xg(x)}{b}(b-t), & \text{for } \ x\le t\le b\end{cases} \qquad (1.78)$$

For simplicity reasons, we may consider $g(x)=\lambda$ where λ is constant. Equation (1.76) can be written as

$$u(x)=f(x)+\frac{\lambda}{b}(b-x)\int_0^x tu(t)\,dt+\frac{\lambda x}{b}\int_x^b (b-t)u(t)\,dt. \quad (1.79)$$

Each term of the last two terms at the right side of (1.79) is a product of two functions of x. Differentiating both sides of (1.79), using the product rule of differentiation and using Leibnitz rule we obtain

$$u'(x)=f'(x)+\frac{\lambda}{b}(b-x)xu(x)-\frac{\lambda}{b}\int_0^x tu(t)\,dt-\frac{\lambda}{b}(b-x)xu(x)$$

$$+\frac{\lambda}{b}\int_x^b (b-t)u(t)dt,$$

$$u'(x)=f'(x)-\frac{\lambda}{b}\int_0^x tu(t)dt+\frac{\lambda}{b}\int_x^b (b-t)u(t)dt. \quad (1.80)$$

To get rid of integral signs, we differentiate both sides of (1.80) again with respect to x to find that

$$u''(x)=f''(x)-\frac{\lambda}{b}xu(x)-\frac{\lambda}{b}(b-x)u(x),$$

that gives the ordinary differential equations:

$$u''(x)+\lambda u(x)=f''(x),$$

The related boundary conditions can be obtained by substituting $x=0$ and $x=1$ in (1.79) to find that $u(0)=f(0), u(b)=f(b)$. Gives the boundary value problem

$$y''(x)+\lambda y(x)=f''(x),\ \ u(0)=f(0), u(b)=f(b) \ (1.81)$$

Example 2.11 Convert the Fredholm integral equation

$$u(x)=e^x+\int_0^3 k(x,t)u(t)dt\,,\ \ k(x,t)=\begin{cases}3t(3-x) \text{ for } 0\le t\le x\\ 3x(3-t) \text{ for } x\le t\le 3\end{cases}$$

$$u(x) = e^x + \int_0^x 3t(3-x)u(t)dt + \int_x^3 3x(3-t)u(t)dt$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$u(x) = e^x + 3(3-x)\int_0^x t\,u(t)dt + 3x\int_x^3 (3-t)u(t)dt, \qquad (*)$$

Differentiating twice with respect to x gives

$$u'(x) = e^x - 3\int_0^x t\,u(t)dt + 3x(3-x)u(x) + 3\int_x^3 (3-t)u(t)dt - 3x(3 - x)u(x)$$

$$u'(x) = e^x - 3\int_0^x t\,u(t)dt + 3\int_x^3 (3-t)u(t)dt. \quad (**)$$

and $u''(x) = e^x - 9u(x)$. From $(*)$ $u(0) = f(0) = 1,\ u(3) = f(3) = e^3$, the BVP is

$$y''(x) + 9y(x) = e^x, \quad y(0) = 1, \ y(3) = e^3.$$

Example 2.12 Convert the Fredholm integral equation

$$u(x) = x^3 + \int_0^2 k(x,t)u(t)dt, \quad k(x,t) = \begin{cases} 4(2-x)t \text{ for } 0 \le t \le x \\ 4x(2-t) \text{ for } x \le t \le 2 \end{cases}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$u(x) = x^3 + 4(2-x)\int_0^x t\,u(t)dt + 4x\int_x^2 (2-t)u(t)dt, \qquad (*)$$

Proceeding as before we find $\frac{g(x)}{b} = \frac{g(x)}{2} = 4 \rightarrow g(x) = 8$

$$u''(x) = 6x - 8u(x).$$

From (*) $u(0) = f(0) = 0,\ u(2) = f(2) = 2^3 = 8$, the BVP is

$$y''(x) + 8y(x) = 6x, \quad y(0) = 0, \qquad y(2) = 8.$$

Type II

We next consider the Fredholm integral equation given by

$$u(x) = f(x) + \int_0^b K(x,t)u(t)dt, \qquad (1.76)$$

Where

$$f(x) = h(x) - (\alpha_1 + \beta_1 x)g(x). \qquad (1.88)$$

and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} tg(x) \text{ for } & 0 \leq t \leq x \\ xg(x), \text{for } & x \leq t \leq b \end{cases} \qquad (1.93)$$

For simplicity reasons, we may consider $g(x) = \lambda$ where λ is constant. Equation (1.76) can be written as

$$u(x) = f(x) + \lambda \int_0^x tu(t)\, dt + \lambda x \int_x^b u(t)\, dt.$$

Each term of the last two terms at the right side is a product of two functions of x. Differentiating both sides, using the product rule of differentiation and using Leibnitz rule we obtain

$$u'(x) = f'(x) + \lambda \int_x^b u(t)dt$$

To get rid of integral signs, we differentiate both sides again with respect to x to find that

$$u''(x) = f''(x) - \lambda u(x),$$

that gives the ordinary differential equations:

$$u''(x) + \lambda u(x) = f''(x),$$

The related boundary conditions can be obtained by substituting $x = 0$ and $x = 1$ in (1.90) to find that $u(0) = f(0), u(b) = f(b)$. gives the boundary value problem

$$y''(x) + \lambda y(x) = f''(x), u(0) = f(0), u(b) = f(b) \ (1.92)$$

Example 2.13 Convert the Fredholm integral equation:

$$u(x) = e^x + \int_0^2 k(x,t)u(t)dt\,, \quad k(x,t) = \begin{cases} 2t \text{ for } 0 \leq t \leq x \\ 2x \text{ for } x \leq t \leq 2 \end{cases}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as $tg(x) = 2t$

$$u(x) = e^x + 2\int_0^x t\,u(t)dt + 2x\int_x^2 u(t)dt, \qquad (*)$$

Differentiating twice with respect to x gives

$$u'(x) = e^x + 2\int_x^2 u(t)dt$$

and $u''(x) = e^x - 2u(x)$. From (*) $u(0) = f(0) = 1,\ u(2) = f(2) = e^2$, the BVP is

$$y''(x) + 2y(x) = e^x, \quad y(0) = 1, \qquad y(2) = e^2.$$

Example 2.14 Convert the Fredholm integral equation

$$u(x) = x^2 + \int_0^3 k(x,t)u(t)dt\,, \quad k(x,t) = \begin{cases} 2t \text{ for } 0 \le t \le x \\ 2x \text{ for } x \le t \le 3 \end{cases}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as $tg(x) = tg(x) = 2t$

$$u(x) = x^3 + 2\int_0^x t\,u(t)dt + 2x\int_x^3 u(t)dt, \qquad (*)$$

Proceeding as before we find

$$u''(x) = 2 - 2u(x).$$

From (*) $u(0) = f(0) = 0,\ u(3) = f(3) = 3^2 = 9$, the BVP is

$y''(x) + 2y(x) = 2, \quad y(0) = 0, \qquad y(2) = 9.$

Exercises 2.6

Convert the following BVPs into an equivalent Fredholm integral equation:

1. $y'' + xy = 0,\ y(0) = y(1) = 0$
2. $y'' + 2y = x,\ 0 < x < 1, y(0) = 1, y(1) = 0$
3. $y'' + xy = 0,\ y(0) = 0, y'(1) = 0$
4. $y'' + 4y = x,\ 0 < x < 1, y(0) = 1, y'(1) = 0.$

Convert the following Fredholm integral equation to an equivalent BVP:

5. $u(x) = e^{2x} + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 3t(1-x) \text{ for } 0 \le t \le x \\ 3x(1-t) \text{ for } x \le t \le 1 \end{cases}$

6. $u(x) = 3x^2 + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} t(1-x) \text{ for } 0 \leq t \leq x \\ x(1-t) \text{ for } x \leq t \leq 1 \end{cases}$

7. $u(x) = \sinh x + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 4t(1-x) \text{ for } 0 \leq t \leq x \\ 4x(1-t) \text{ for } x \leq t \leq 1 \end{cases},$

8. $u(x) = x^4 + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 6t \text{ for } 0 \leq t \leq x \\ 6x \text{ for } x \leq t \leq 1 \end{cases}.$