الفصل الدراسي الثاني

Theory of Differential Equations

Chapter Two: Systems of differential equations

Theorem 2 (Cayley-Hamilton Theorem) Every $n \times n$ constant matrix satisfies its characteristic equation.

Theorem 2 (Cayley-Hamilton). Let $p(\lambda) = p_0 + p_1\lambda + \ldots + (-1)^n p_n \lambda^n$ be the characteristic polynomial of A. Then,

$$p(A) = p_0 + p_1 A + \ldots + (-1)^n p_n A^n = 0.$$

Example let $A = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ then $p(\lambda) = \lambda^2 + 4\lambda - 1 = 0$ its characteristic equation so $p(A) = A^2 + 4A - I = 0$

Home work

1- Find the solution of

a- $\dot{X} = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} X,$ b- $\dot{X} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} X$

c- $\dot{X} = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix} X,\ X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ d- $\dot{X} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} X,\ X(0) = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}$

1.3 Fundamental matrix solutions $\Phi(t)$; and exponential matrix e^{At}

$$\dot{X} = AX \qquad (1)$$

Definition 2. An $n \times n$ matrix function $\mathbf{\Phi}$ is said to be a fundamental matrix for the vector differential equation (1) provided $\mathbf{\Phi}$ is a solution of the matrix equation (1) on I, often

$$\mathbf{\Phi}(\mathbf{t}) = [X_1\ X_2\ \ldots\ X_n] \rightarrow X(t) = \mathbf{\Phi}(\mathbf{t})\boldsymbol{C} \qquad (2)$$

Definition 3. An n × n matrix function e^{At} is said to be an exponential matrix for the vector differential equation (1) provided

$$X(t) = e^{A(t-t_0)}C \qquad (3)$$

Example 1.a Find a fundamental matrix solution of the system of differential equations $\dot{X} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix} X$

The independent solutions are

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, V_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{\Phi}(\mathbf{t}) = \begin{bmatrix} e^t & -2e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \\ -e^t & 0 & -e^{3t} \end{bmatrix},$$

Example 1.b Find the matrix A from the fundamental matrix $\begin{bmatrix} e^t & -2e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \\ -e^t & 0 & -e^{3t} \end{bmatrix}$,

Sol. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, V_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $(A - \lambda I)V = 0, \Longrightarrow$

$$\begin{bmatrix} a-1 & b & c \\ d & e-1 & f \\ g & h & i-1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0 \to a - c = 1, d - f = 0, g - i = -1$$

$$\begin{bmatrix} a-2 & b & c \\ d & e-2 & f \\ g & h & i-2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0 \Longrightarrow -2a + b = -4, -2d + e = 2, -2g + h = 0,$$

$$\begin{bmatrix} a-3 & b & c \\ d & e-3 & f \\ g & h & i-3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0 \to b - c = 0, e - f = 3, h - i = -3,$$

$$\to b - 2c = -2, b = c = 2, a = 3, -f + 2d = 1, f = d = 1, e = 4, g - h = 2, g = -2, h = -4, i = -1.$$

طريقة أخرى $V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, V_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, M = [V_1 \ \ V_2 \ \ V_3], J = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$MJ = AM \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & -4 & 0 \\ 0 & 2 & 3 \\ -1 & 0 & -3 \end{bmatrix} = \begin{bmatrix} a-c & -2a+b & b-c \\ d-f & -2d+e & e-f \\ g-i & -2g+h & h-i \end{bmatrix}$$

Theorem 3. Let $\mathbf{\Phi}(\mathbf{t})$ be a fundamental matrix solution of the differential equation

$$\dot{X} = AX \qquad (1)$$

Then, $$e^{At} = \Phi(\mathrm{t})\Phi^{-1}(0) \qquad (4)$$

In other words, the product of any fundamental matrix solution of (I) with its inverse at $t = 0$ must yield e^{At}.

Lemma 2. A matrix $\Phi(\mathrm{t})$ is a fundamental matrix solution of (1) if and only if

$$\dot{\Phi}(\mathrm{t}) = A\Phi(\mathrm{t}) \text{ and } \det\Phi(0) \neq 0. \qquad (5)$$

Proof of Lemma 2: Let $X_1(t)X_2(t) \dots X_n(t)$ be linearly independent solution of (1).
Let $\boldsymbol{\Phi}(\mathbf{t}) = [X_1(t)\ X_2(t) \dots X_n(t)]$ then $\boldsymbol{\Phi}(\mathbf{t})$ is Fundamental solution iff
$\dot{\boldsymbol{\Phi}}(\mathbf{t}) = [\dot{X}_1(t)\ \ \dot{X}_2(t)\ \dots\ \dot{X}_n(t)] = [AX_1(t)\quad AX_2(t)\quad \cdots\quad AX_n(t)] =$
$A[X_1(t)\ \ X_2(t)\ \dots\ X_n(t)] = \boldsymbol{A\Phi}(\mathbf{t})$ and

$$\boldsymbol{\Phi}(\mathbf{t}) = [e^{\lambda_1 t}V_1\ e^{\lambda_2 t}V_2 \dots e^{\lambda_n t}V_n] \Longrightarrow \boldsymbol{\Phi}(\mathbf{0}) = [V_1\ V_2 \dots V_n]$$

Since $V_1\ V_2 \dots V_n$ are eigenvectors so they are linearly independent then
$\det \boldsymbol{\Phi}(\mathbf{0}) \neq 0$. □

Example 2.a Show that $\Phi(\mathrm{t}) = \begin{bmatrix} e^t & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix}$ if FM of $\dot{X} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix} X$

Ans. $\Phi'(\mathrm{t}) = \begin{bmatrix} e^t & 3e^{3t} & 5e^{5t} \\ 0 & 6e^{3t} & 10e^{5t} \\ 0 & 0 & 10e^{5t} \end{bmatrix}$, $A\Phi = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} e^t & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix} =$
$\begin{bmatrix} e^t & 3e^{3t} & 5e^{5t} \\ 0 & 6e^{3t} & 10e^{5t} \\ 0 & 0 & 10e^{5t} \end{bmatrix}$, $\det \boldsymbol{\Phi}(\mathbf{0}) = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} = 4 \neq 0.$

Lemma 3. The matrix-valued function

$$e^{At} = I + At + A^2\frac{t^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \qquad (6)$$

is a fundamental matrix solution of (1).

Proof: $\frac{d}{dt}e^{At} = A + A^2t + A^3\frac{t^2}{2} + \cdots = A\left(I + At + A^2\frac{t^2}{2} + \cdots\right) = Ae^{At}$ so e^{At} is a solution of (1),$\det(e^{A0}) = \det(e^0) = \det(I) = 1 \neq 0$

So by Lemma 2 e^{At} is fundamental matrix solution. □

Lemma 4. Let $\Phi(\mathrm{t})$ be a fundamental matrix solution of (1). Then, $\Psi(\mathrm{t}) = \Phi(\mathrm{t})C$ is also a fundamental matrix solution of (1) provided C is constant nonsingular matrix ($\det C \neq 0$).

Proof: Let $\Psi(\mathrm{t}) = \Phi(\mathrm{t})C \rightarrow \Psi'(\mathrm{t}) = \Phi'(\mathrm{t})C, \Psi'(\mathrm{t}) = A\Phi(\mathrm{t})C = A\Psi(\mathrm{t})$,
Then $\Psi(\mathrm{t})$ is a solution of (1)
$\det \Psi(\mathrm{t}) = \det \Phi(\mathrm{t})C = \det \Phi(\mathrm{t}) \det C \rightarrow \det \Psi(0) = \det \Phi(0) \det C \neq 0$
Then $\Psi(\mathrm{t})$ is a fundamental matrix □

Proof of Theorem3**:** Let $\Phi(\mathrm{t})$ be fundamental matrix, by Lemma 3
e^{At} is also a fundamental matrix**,** then by Lemma 4, $e^{At} = \Phi(\mathrm{t})C$ (7)
Let $t = 0$ in (7), $I = \Phi(0)C \rightarrow C = \Phi^{-1}(0) \rightarrow e^{At} = \Phi(\mathrm{t})\Phi^{-1}(0).$ □

$$e^{A(t-t_0)} = \Phi(\mathrm{t})\Phi^{-1}(t_0) \qquad (8)$$

Example 2.b Find e^{At} if $\dot{X} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix} X$ and use it to solve the system

Ans. Our first step is to find 3 linearly independent solutions of the system: $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 5$ and $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ their corresponding eigenvalues, then $\Phi(\mathrm{t}) = \begin{bmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix}$ is FMS from (7), $e^{At} =$

$$\Phi(\mathrm{t})\Phi^{-1}(0) = \begin{bmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix} \frac{1}{4} \begin{bmatrix} 4 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & \frac{-1}{2}e^{t} + \frac{1}{2}e^{3t} & \frac{-1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ 0 & e^{3t} & e^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}$$

Example 3 Find e^{At} and Use it to solve $\dot{X} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} X$,

Ans. The matrix A is lower triangular so $\lambda_1 = 2 = \lambda_2, \lambda_3 = 3$ and $V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, X_1 = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, X_2 = e^{2t} \begin{bmatrix} 1 \\ t \\ -1 \end{bmatrix}, V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, X_3 = e^{3t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\Phi(\mathrm{t}) = \begin{bmatrix} 0 & e^{2t} & 0 \\ e^{2t} & te^{2t} & 0 \\ 0 & -e^{2t} & e^{3t} \end{bmatrix}$ is FMS

$$e^{At} = \Phi(\mathrm{t})\Phi^{-1}(0) = \begin{bmatrix} 0 & e^{2t} & 0 \\ e^{2t} & te^{2t} & 0 \\ 0 & -e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & e^{2t} & 0 \\ e^{2t} & te^{2t} & 0 \\ 0 & -e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ e^{3t} - e^{2t} & 0 & e^{3t} \end{bmatrix}$$

$$X(t) = e^{At}C = \begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ e^{3t} - e^{2t} & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ c_1 te^{2t} + c_2 e^{2t} \\ c_1(e^{3t} - e^{2t}) + c_3 e^{3t} \end{bmatrix}$$

طريقة ثانية لايجاد e^{At}

$$e^{At} = I + At + A^2\frac{t^2}{2} + \cdots = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}2&0&0\\1&2&0\\1&0&3\end{bmatrix}t + \begin{bmatrix}2&0&0\\1&2&0\\1&0&3\end{bmatrix}^2\frac{t^2}{2!} + \cdots$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}2&0&0\\1&2&0\\1&0&3\end{bmatrix}t + \begin{bmatrix}4&0&0\\4&4&0\\5&0&9\end{bmatrix}\frac{t^2}{2!} + \cdots$$

$$= \begin{bmatrix}1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots & 0 & 0\\ t + \frac{4t^2}{2} + \frac{12t^3}{3!} + \cdots & 1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots & 0\\ t + \frac{5t^2}{2} + \frac{19t^3}{3!} + \cdots & 0 & 1 + 3t + \frac{(3t)^2}{2!} + \frac{(3t)^3}{3!} + \cdots\end{bmatrix}$$

$$= \begin{bmatrix}e^{2t}&0&0\\te^{2t}&e^{2t}&0\\e^{3t}-e^{2t}&0&e^{3t}\end{bmatrix}$$

Properties of e^{At}

1- if A is diagonal $\begin{bmatrix}3&0&0\\0&2&0\\0&0&4\end{bmatrix}$ then $e^{At} = \begin{bmatrix}e^{3t}&0&0\\0&e^{2t}&0\\0&0&e^{4t}\end{bmatrix}$

2- if A is upper (or lower)triangular $A = \begin{bmatrix}2&a\\0&3\end{bmatrix} \to e^{At} = \begin{bmatrix}e^{2t}&ae^{3t}-ae^{2t}\\0&e^{3t}\end{bmatrix}$

$\begin{bmatrix}3&0&0\\0&2&0\\0&0&4\end{bmatrix}$ then $e^{At} = \begin{bmatrix}e^{3t}&0&0\\0&e^{2t}&0\\0&0&e^{4t}\end{bmatrix}$, $\begin{bmatrix}3&1&2\\0&2&-1\\0&0&4\end{bmatrix}$, $e^{At} = \begin{bmatrix}e^{3t}&0&0\\0&e^{2t}&0\\0&0&e^{4t}\end{bmatrix}$

H.W. 1: without solving the system find e^{At} from the following

i. $A = \begin{bmatrix}a&b\\0&c\end{bmatrix}, A = \begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}, A = \begin{bmatrix}a&0\\-b&a\end{bmatrix}, a, b.c \neq 0$

1.4 The nonhomogeneous equation; variation of parameters

Let the matrix $\Phi(t) = [X_1(t)\ X_2(t)\ \cdots\ X_n(t)]$ be FMS of the homogenous system

$$\dot{X}(t) = AX(t) \qquad (1)$$

Then the system

$$\dot{X}(t) = AX(t) + H(t) \qquad (9)$$

Is the nonhomogenous system,

Theorem 4 Let $\Phi(t)$ be FM and e^{At} be exponential matrix then the general solution satisfying $X(t_0) = X_0$ of (9) is

$$X(t) = e^{A(t-t_0)}X_0 + e^{At}\int_{t_0}^{t} e^{-As}H(s)\,ds$$

Proof: We have to seek a solution in the form

$$X(t) = \Phi(\mathrm{t})U(t). \qquad (10)$$
$$U(t) = \Phi^{-1}(\mathrm{t})X(t) \qquad (11)$$

Differentiating (3) we get $\dot{X}(t) = \dot{\Phi}(\mathrm{t})U(t) + \Phi(\mathrm{t})\dot{U}(t)$,

$$AX(t) + H(t) = \dot{\Phi}(\mathrm{t})U(t) + \Phi(\mathrm{t})\dot{U}(t) = A\Phi(\mathrm{t})U(t) + \Phi(\mathrm{t})\dot{U}(t)$$
$$= AX(t) + \Phi(\mathrm{t})\dot{U}(t),$$
$$H(t) = \Phi(\mathrm{t})\dot{U}(t) \to \dot{U}(t) = \Phi^{-1}(\mathrm{t})H(t)$$

Integrating this expression between t_0 and t gives

$$U(t) - U(t_0) = \int_{t_0}^{t} \Phi^{-1}(\mathrm{s})H(s)\,ds$$

$$U(t) = \Phi^{-1}(t_0)X(t_0) + \int_{t_0}^{t} \Phi^{-1}(\mathrm{s})H(s)\,ds$$

$$\Phi(\mathrm{t})U(t) = \Phi(\mathrm{t})\Phi^{-1}(t_0)X_0 + \Phi(\mathrm{t})\int_{t_0}^{t} \Phi^{-1}(\mathrm{s})H(s)\,ds$$

$$X(t) = \Phi(\mathrm{t})\Phi^{-1}(t_0)X_0 + \Phi(\mathrm{t})\int_{t_0}^{t} \Phi^{-1}(\mathrm{s})H(s)\,ds \qquad (12)$$

$$X(t) = e^{A(t-t_0)}X_0 + e^{A(t-t_0)}\Phi(t_0)\int_{t_0}^{t} e^{-A(s-t_0)}\Phi^{-1}(t_0)H(s)\,ds$$

$$X(t) = e^{A(t-t_0)}X_0 + e^{At}\int_{t_0}^{t} e^{-As}H(s)\,ds \qquad (13)$$

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Multiply (2) by $e^{-At} \to e^{-At}\dot{\mathrm{X}}(\mathrm{t}) = e^{-At}AX(t) + e^{-At}H(t)$

$$e^{-At}\dot{\mathrm{X}}(\mathrm{t}) - e^{-At}AX(t) = e^{-At}H(t) \to e^{-At}\dot{\mathrm{X}}(\mathrm{t}) - Ae^{-At}X(t) = e^{-At}H(t)$$
$$\Longrightarrow e^{-At}\mathrm{X}'(\mathrm{t}) + (e^{-At})'X(t) = e^{-At}H(t) \Longrightarrow (e^{-At}X(\mathrm{t}))' = e^{-At}H(t)$$

Integrating this expression between t_0 and t gives

$$e^{-At}X(\mathrm{t}) - e^{-At_0}X(t_0) = \int_{t_0}^{t} e^{-As}H(s)ds$$

$$e^{-At}X(\mathrm{t}) = e^{-At_0}X(t_0) + \int_{t_0}^{t} e^{-As}H(s)ds$$

$$X(t) = e^{A(t-t_0)}X_0 + e^{At}\int_{t_0}^{t} e^{-As}H(s)\,ds\,.$$

Example 1. Solve the initial-value problem

$$\dot{X} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ e^{t}\cos 2t \end{bmatrix}, \qquad X(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

في البداية نحل النظام المتجانس $\dot{X} = AX$ وذلك باستخراج القيم الذاتية

$$\det\begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda + 5) = 0 \to \lambda_1 = 1, \lambda_{2,3} = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$$1.\ \lambda_1 = 1 \to \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \to 2a - 2c = 0, 3a + 2b = 0, c = a, b = -\frac{3}{2}a$$

$$V_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, X_1 = e^t\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2e^t \\ -3e^t \\ 2e^t \end{bmatrix}$$

$$2.\ \lambda = 1 + 2i \to \begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \to -2ia = 0 \to a = 0,$$

$$2a - 2ib - 2c = 0,\ \ ib + c = 0 \to V = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix}$$

$$X = e^{(1+2i)t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix}\right) = e^t(\cos 2t + i\sin 2t)\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= e^t[\cos 2t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin 2t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i(-\cos 2t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \sin 2t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})]$$

$$X_2 = e^t[\cos 2t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin 2t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}], X_3 = e^t[-\cos 2t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \sin 2t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}]$$

$$X_2 = e^t\begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}, X_3 = e^t\begin{bmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{bmatrix}$$

$$\Phi(\mathrm{t}) = \begin{bmatrix} 2e^t & 0 & 0 \\ -3e^t & e^t\cos 2t & e^t\sin 2t \\ 2e^t & e^t\sin 2t & -e^t\cos 2t \end{bmatrix}, \Phi(0) = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\Phi^{-1}(0) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$e^{At} = \Phi(\mathrm{t})\Phi^{-1}(0) = \begin{bmatrix} 2e^t & 0 & 0 \\ -3e^t & e^t\cos 2t & e^t\sin 2t \\ 2e^t & e^t\sin 2t & -e^t\cos 2t \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3}{2}e^t + \frac{3}{2}e^t\cos 2t + e^t \sin 2t & e^t\cos 2t & -e^t \sin 2t \\ e^t + \frac{3}{2}e^t \cos 2t - e^t \sin 2t & e^t \sin 2t & e^t \cos 2t \end{bmatrix}$$

Then by (13) we get

$$X(t) = \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3}{2}e^t + \frac{3}{2}e^t\cos 2t + e^t \sin 2t & e^t\cos 2t & -e^t \sin 2t \\ e^t + \frac{3}{2}e^t \cos 2t - e^t \sin 2t & e^t \sin 2t & e^t \cos 2t \end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}$$
$$+ \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3}{2}e^t + \frac{3}{2}e^t\cos 2t + e^t \sin 2t & e^t\cos 2t & -e^t \sin 2t \\ e^t + \frac{3}{2}e^t \cos 2t - e^t \sin 2t & e^t \sin 2t & e^t \cos 2t \end{bmatrix}\int_0^t \begin{bmatrix} e^s & 0 & 0 \\ -\frac{3}{2}e^s + \frac{3}{2}e^t\cos 2s + e^s \sin 2s & e^s\cos 2s & -e^s \sin 2s \\ e^s + \frac{3}{2}e^s \cos 2s - e^s \sin 2s & e^s \sin 2s & e^s \cos 2s \end{bmatrix}\begin{bmatrix}0\\0\\e^s\cos 2s\end{bmatrix}ds$$

$$X(t) = \begin{bmatrix} e^t \\ e^t\cos 2t - e^t \sin 2t \\ e^t \cos 2t + e^t \sin 2t \end{bmatrix}$$
$$+ \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3}{2}e^t + \frac{3}{2}e^t\cos 2t + e^t \sin 2t & e^t\cos 2t & -e^t \sin 2t \\ e^t + \frac{3}{2}e^t \cos 2t - e^t \sin 2t & e^t \sin 2t & e^t \cos 2t \end{bmatrix}\int_0^t \begin{bmatrix} 0 \\ -e^{2s}\cos 2s \sin 2s \\ e^{2s}\cos^2 2s \end{bmatrix}ds$$

$$X(t) =$$

Example 2 Solve the initial-value problem $\dot{X} = \begin{bmatrix}3 & -4\\0 & 3\end{bmatrix}X + \begin{bmatrix}1\\0\end{bmatrix}e^t,\ X(0) = \begin{pmatrix}0\\1\end{pmatrix}$

$$\det(A - \lambda I) = 0 \to \det\begin{bmatrix}3-\lambda & -4\\0 & 3-\lambda\end{bmatrix} = 0$$

$$(3-\lambda)^2 = 0 \to \lambda_1 = \lambda_2 = 3,$$

$$\lambda_1 = 3 \to (A - 3I)V_1 = 0 \to \begin{bmatrix}0 & -4\\0 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = 0 \to b = 0, V_1 = \begin{bmatrix}1\\0\end{bmatrix}$$

$$X_1 = e^{3t}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\lambda_1 = 3 \to (A - 3I)V_2 = V_1 \to \begin{bmatrix}0 & -4\\0 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}, -4b = 1, b = \frac{-1}{4}, V_2 = \begin{bmatrix}0\\\frac{-1}{4}\end{bmatrix}$$

$$X_2 = e^{3t}\left[\begin{bmatrix}0\\\frac{-1}{4}\end{bmatrix} + t\begin{bmatrix}0 & -4\\0 & 0\end{bmatrix}\begin{bmatrix}0\\\frac{-1}{4}\end{bmatrix}\right] = e^{3t}\begin{bmatrix}t\\\frac{-1}{4}\end{bmatrix}$$

$$\Phi(t) = e^{3t}\begin{bmatrix}1 & t\\0 & \frac{-1}{4}\end{bmatrix} \to \Phi^{-1}(0) = -4\begin{bmatrix}\frac{-1}{4} & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -4\end{bmatrix}$$

$$e^{At} = \Phi(t)\Phi^{-1}(0) = e^{3t}\begin{bmatrix}1 & t\\0 & \frac{-1}{4}\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -4\end{bmatrix} = e^{3t}\begin{bmatrix}1 & -4t\\0 & 1\end{bmatrix}$$

Then by (13) we get

$$X(\mathrm{t}) = e^{3t}\begin{bmatrix}1 & -4t\\ 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} + e^{3t}\begin{bmatrix}1 & -4t\\ 0 & 1\end{bmatrix}\int_0^t e^{3s}\begin{bmatrix}1 & -4s\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}ds$$

$$= e^{3t}\begin{bmatrix}-4t\\1\end{bmatrix} + e^{3t}\begin{bmatrix}1 & -4t\\ 0 & 1\end{bmatrix}\int_0^t \begin{bmatrix}e^{3s}\\0\end{bmatrix}ds$$

$$= e^{3t}\begin{bmatrix}-4t + \frac{1}{3}[e^{3t} - 1]\\ 1\end{bmatrix}$$

Homework

1. Solve the initial-value problem $\dot{X} = \begin{bmatrix}2 & -5\\ 1 & -2\end{bmatrix}X + \begin{bmatrix}\sin t\\ \tan t\end{bmatrix}$, $X(0) = \begin{pmatrix}0\\0\end{pmatrix}$
2. Solve the initial-value problem $\dot{X} = \begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}X + \begin{bmatrix}1\\1\end{bmatrix}e^t$, $X(0) = \begin{pmatrix}1\\1\end{pmatrix}$

3.13 Solving systems by Laplace transforms

$$\dot{\mathrm{X}}(\mathrm{t}) = AX(t) + H(t),\ \ X(0) = X_0 \qquad (14)$$

$$\mathbf{X}(s) = \begin{bmatrix}X_1(s)\\ \vdots\\ X_n(s)\end{bmatrix} = \mathcal{L}\{\mathbf{x}(t)\} = \begin{bmatrix}\int_0^\infty e^{-st}x_1(t)dt\\ \vdots\\ \int_0^\infty e^{-st}x_n(t)dt\end{bmatrix} \qquad (15)$$

$$\mathbf{F}(s) = \begin{pmatrix}F_1(s)\\ \vdots\\ F_n(s)\end{pmatrix} = \mathcal{L}\{\mathbf{f}(t)\} = \begin{pmatrix}\int_0^\infty e^{-st}f_1(t)dt\\ \vdots\\ \int_0^\infty e^{-st}f_n(t)dt\end{pmatrix} \qquad (16)$$

Taking Laplace transforms of both sides of (1) gives

$$\mathcal{L}\{\dot{\mathrm{X}}(t)\} = \mathcal{L}\{AX(t) + H\} = A\mathcal{L}\{X(t)\} + \mathcal{L}\{H\} \rightarrow$$

$$\begin{bmatrix}\mathcal{L}\{\dot{x}_1(t)\}\\ \vdots\\ \mathcal{L}\{\dot{x}_n(t)\}\end{bmatrix} = A\begin{bmatrix}\mathcal{L}\{x_1(t)\}\\ \vdots\\ \mathcal{L}\{x_n(t)\}\end{bmatrix} + \begin{bmatrix}\mathcal{L}\{h_1(t)\}\\ \vdots\\ \mathcal{L}\{h_n(t)\}\end{bmatrix}$$

$$\begin{bmatrix}s\mathcal{L}\{x_1(t)\} - x_1(0)\\ \vdots\\ \mathcal{L}\{x_n(t)\} - x_n(0)\end{bmatrix} = A\begin{bmatrix}\mathcal{L}\{x_1(t)\}\\ \vdots\\ \mathcal{L}\{x_n(t)\}\end{bmatrix} + \begin{bmatrix}\mathcal{L}\{h_1(t)\}\\ \vdots\\ \mathcal{L}\{h_n(t)\}\end{bmatrix} \qquad (17)$$

Example 1. Solve the initial-value problem

$$\dot{\boldsymbol{X}} = \begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix}\boldsymbol{X} + \begin{pmatrix}1\\1\end{pmatrix}e^t,\ \boldsymbol{X}(0) = \begin{pmatrix}2\\1\end{pmatrix}.$$

Solution. Taking Laplace transforms of both sides of the differential equation gives

$$\begin{bmatrix}s\mathcal{L}\{x_1(t)\} - 2\\ s\mathcal{L}\{x_2(t)\} - 1\end{bmatrix} = \begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix}\begin{pmatrix}\mathcal{L}\{x_1(t)\}\\ \mathcal{L}\{x_2(t)\}\end{pmatrix} + \frac{1}{s-1}\begin{pmatrix}1\\1\end{pmatrix}$$

or

$$(s-1)\mathcal{L}\{x_1(t)\}-4\mathcal{L}\{x_2(t)\}=2+\frac{1}{s-1} \qquad (s-1)X_1(s)-4X_2(s)=2+\frac{1}{s-1}$$
$$-\mathcal{L}\{x_1(t)\}+(s-1)\mathcal{L}\{x_2(t)\}=1+\frac{1}{s-1}. \quad -X_1(s)+(s-1)X_2(s)=1+\frac{1}{s-1}.$$

$$((s-1)^2-4)\mathcal{L}\{x_1(t)\}=2(s-1)+5+\frac{4}{s-1}$$
$$((s-1)^2-4)\mathcal{L}\{x_1(t)\}=\frac{2s-2}{(s-3)(s+1)(s-1)}+\frac{5s-1}{}$$

The solution of these equations is

$$\mathcal{L}\{x_1(t)\}=\frac{2}{s-3}+\frac{1}{s^2-1},\ \mathcal{L}\{x_2(t)\}=\frac{1}{s-3}+\frac{s}{(s-1)(s+1)(s-3)}$$

Now,

$$\frac{2}{s-3}=2\mathcal{L}\{e^{3t}\}, \qquad \frac{1}{s^2-1}=\mathcal{L}\{\sinh t\}=\mathcal{L}\left\{\frac{e^t-e^{-t}}{2}\right\}$$
$$\mathcal{L}\{x_1(t)\}=2\mathcal{L}\{e^{3t}\}+\mathcal{L}\left\{\frac{e^t-e^{-t}}{2}\right\}=\mathcal{L}\left\{2e^{3t}+\frac{e^t-e^{-t}}{2}\right\}$$
$$x_1(t)=2e^{3t}+\frac{e^t-e^{-t}}{2}$$
$$\frac{s}{(s-1)(s+1)(s-3)}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s-3}$$
$$s=A(s^2+2s-3)+B(s^2-4s+3)+C(s^2-1)$$
$$A+B+C=0, 2A-4B=1, -3A+3B-C=0$$
$$A=-\frac{1}{4}, B=-\frac{1}{8}, C=\frac{3}{8},$$
$$\mathcal{L}\{x_2(t)\}=\mathcal{L}\{e^{3t}\}-\frac{1}{4}\mathcal{L}\{e^t\}-\frac{1}{8}\mathcal{L}\{e^{-t}\}+\frac{3}{8}\mathcal{L}\{e^{3t}\}$$
$$x_2(t)=\frac{11}{8}e^{3t}-\frac{1}{4}e^t-\frac{1}{8}e^{-t}$$

Homework

1. $\dot{\mathbf{x}}=\begin{pmatrix}1 & -1\\ 5 & -3\end{pmatrix}\mathbf{x},\ \mathbf{x}(0)=\begin{pmatrix}1\\2\end{pmatrix}$
2. $\dot{x}=\begin{pmatrix}3 & -2\\ 2 & -2\end{pmatrix}\mathbf{x}+\begin{pmatrix}t\\3e^t\end{pmatrix},\ \mathbf{x}(0)=\begin{pmatrix}2\\1\end{pmatrix}$
3. $\dot{\mathbf{x}}=\begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix}\mathbf{x}+\begin{pmatrix}2\\-1\end{pmatrix}e^t,\ \mathbf{x}(0)=\begin{pmatrix}0\\0\end{pmatrix}$
4. $\dot{\mathrm{x}}=\begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}x+\begin{pmatrix}\sin t\\ \tan t\end{pmatrix},\ x(0)=\begin{pmatrix}-1\\0\end{pmatrix}$