3.3 Phase Portraits for Canonical Systems in Plane:

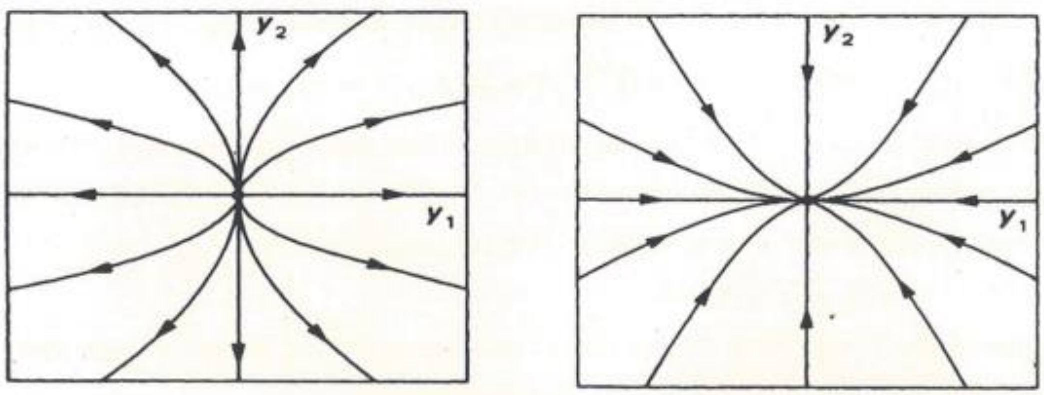
Definition 7: A linear system $\dot{X} = AX$ is said to be simple if the matrix A is non-singular, (i.e. $\det(A) \neq 0$ and A has non-zero eigenvalues).

(a) Real, distinct eigenvalues

$$\dot{Y} = JY \Rightarrow \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \dot{y}_1 = \lambda_1 y_1, \;\; \dot{y}_2 = \lambda_2 y_2,$$

$$y_1 = e^{\lambda_1 t}, \;\; y_2 = e^{\lambda_2 t}, \qquad (2.6)$$

$$\frac{\dot{y}_2}{\dot{y}_1} = \frac{dy_2}{dy_1} = \frac{\lambda_2 y_2}{\lambda_1 y_1}, \frac{dy_2}{y_2} = \frac{\lambda_2}{\lambda_1}\frac{dy_1}{y_1} \Rightarrow \ln y_2 = \frac{\lambda_2}{\lambda_1}\ln y_1 + \ln c \Rightarrow y_2 = c y_1^{\frac{\lambda_2}{\lambda_1}} \; (2.7)$$



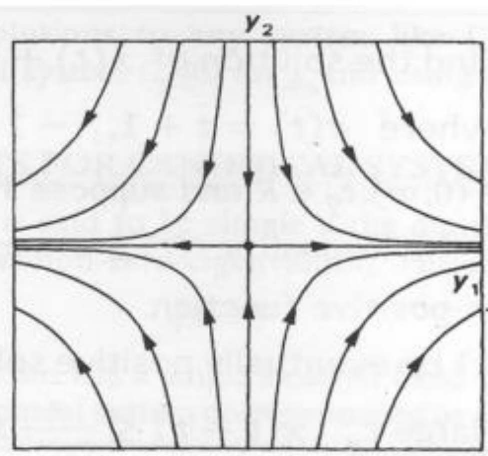


Fig. 2.2. Real eigenvalues of opposite sign (c) ($\lambda_2 < 0 < \lambda_1$) give rise to saddle points.

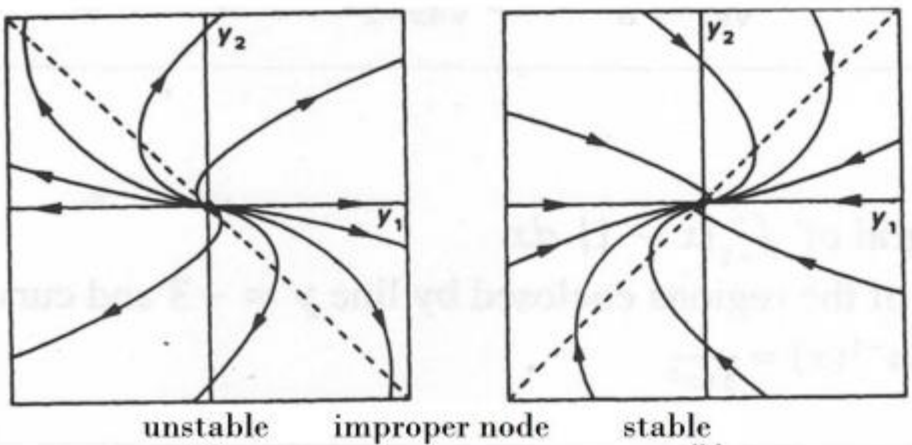
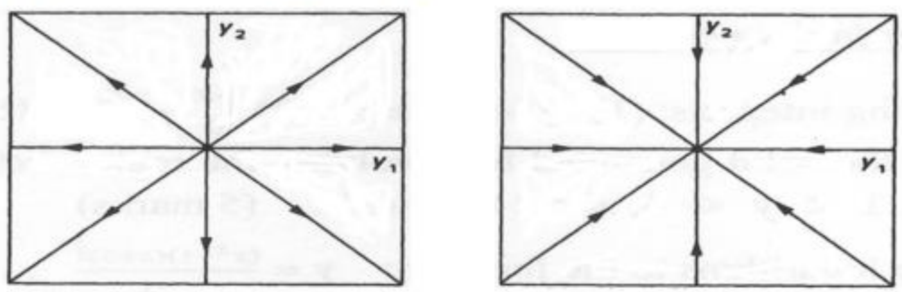


Fig. 2.4. When A is not diagonal, equal eigenvalues indicate that the origin is an improper node: (a) unstable ($\lambda_0 > 0$); (b) stable ($\lambda_0 < 0$).

(b) Equal eigenvalues

If A is diagonal $J = A$, the canonical system has solutions given by Theorem 2-b with $\lambda_1 = \lambda_2 = \lambda_0, \dot{y}_1 = \lambda_0 y_1, \;\; \dot{y}_2 = \lambda_0 y_2, y_1 = c_1 e^{\lambda_0 t}, y_2 = c_2 e^{\lambda_0 t} \rightarrow e^{\lambda_0 t} = \frac{y_1}{c_1},$ $y_2 = \frac{c_2}{c_1} y_1$. Thus (2.7) corresponds to a special node $y_2 = c y_1$, called a star node (stable if $\lambda_0 < 0$; unstable if $\lambda_0 > 0$), in which the non-trivial trajectories are all radial straight lines (as shown in Fig. 2.3).



(a) unstable ($\lambda_0 > 0$); ; (b) stable ($\lambda_0 < 0$).

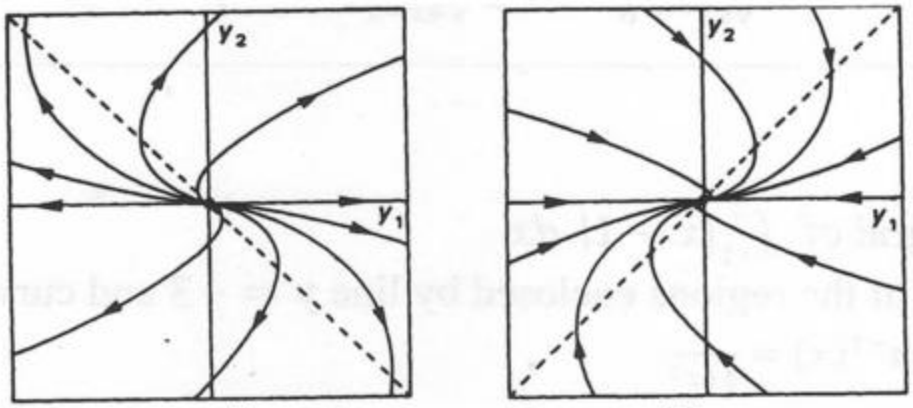
Star node

Fig. 2.3. Equal eigenvalues ($\lambda_1 = \lambda_2 = \lambda_0$) give rise to star nodes (a) unstable; (b) stable; when A is diagonal.

(c) Equal eigenvalues, A is non-diagonal, $\lambda_1 = \lambda_2 = \lambda_0$ hence

$$J = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}, y_1 = (c_1 + c_2 t) e^{\lambda_0 t}, \;\; y_2 = c_2 e^{\lambda_0 t}, \quad (2.8)$$

$$\frac{dy_1}{dy_2} = \frac{\lambda_0 y_1 + y_2}{\lambda_0 y_2} = \frac{1}{y_2} y_1 + \frac{1}{\lambda_0} \Rightarrow \; y_1 = c_1 y_2 + \frac{y_2}{\lambda_0} \ln y_2, \quad (2.9)$$



unstable improper node stable

(a) (b)

Fig. 2.4. When A is not diagonal, equal eigenvalues indicate that the origin is an improper node: (a) unstable ($\lambda_0 > 0$); (b) stable ($\lambda_0 < 0$).

(d) Complex eigenvalues

$$J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \lambda_{1,2} = \alpha \pm i\beta, \ \ \dot{y}_1 = \alpha y_1 - \beta y_2, \dot{y}_2 = \beta y_1 + \alpha y_2$$

Using p

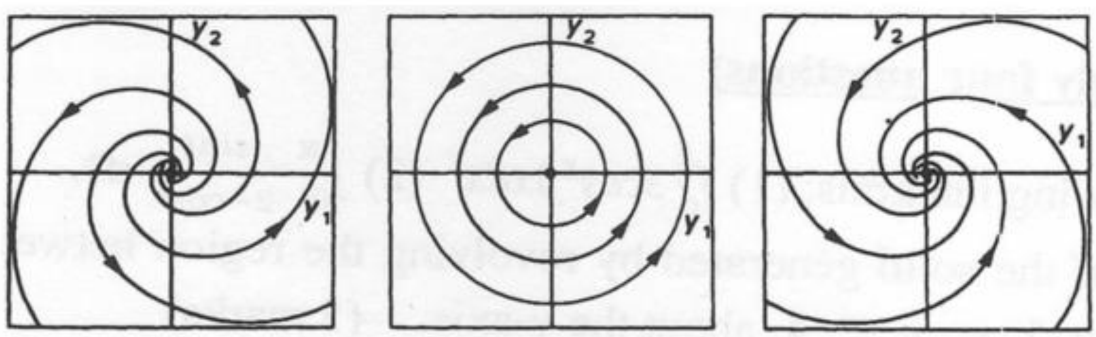
olar coordinate's $r^2 = y_1^2 + y_2^2, \ \ \tan\theta = \frac{y_2}{y_1}$

$$\rightarrow \dot{r} = \alpha r, \qquad \dot{\theta} = \beta \qquad (2.10)$$

$$r(t) = r_0 e^{\alpha t}, \qquad \theta(t) = \beta t + \theta_0 \qquad (2.11)$$

$$\text{if } \alpha < 0 \rightarrow \text{spiral(focus)stable, if } \alpha > 0 \rightarrow \text{spiral unstable,}$$

$$\text{if } \alpha = 0 \rightarrow \text{centre (stable)}$$



(a) spiral unst. centers (b) spiral stable (c)

Fig. 2.5. Complex eigenvalues give rise to (a) unstable foci ($\alpha > 0$), (b) centres ($\alpha = 0$) and (c) stable foci ($\alpha < 0$).

Example 3.15 Sketch the phase portrait of the system

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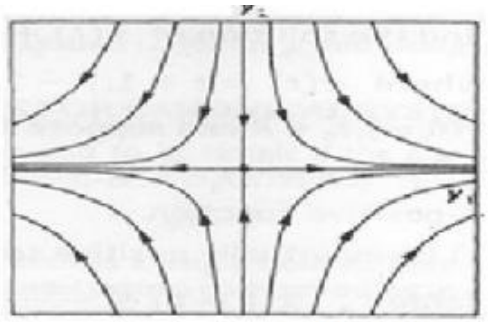
$$y_1' = 2y_1, \ \ y_2' = -2y_2; \text{ and } \ y_1' = -2y_2, \ \ y_2' = 2y_1 \qquad (2.12)$$

and the corresponding phase portraits in the x_1-x_2 plane where

$$M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, M_5 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, M_6 = \begin{bmatrix} 1 & 1 \\ 4 & -4 \end{bmatrix} \qquad (2.13)$$

$$J = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \lambda_1 = 2, \lambda_2 = -2$$

Jordan canonical form رسم صورة الطورالى



$\lambda_1 = 2, \lambda_2 = -2$: Saddle point

رسم صورة الطور الخاصة بالمصفوفات $M_1, M_2, M_3, M_4, M_5, M_6$