An Existence and Uniqueness Theorem

For first-order differential equations the answers to the existence and uniqueness questions we have just posed are fairly easy. We have an Existence and Uniqueness Theorem—simple conditions that guarantee one and only one solution of an IVP.

Theorem "existence of a unique solution"

1-$y'=(x,y)$ **for**

Let R be a rectangular region in the xy-plane defined by $a \le x \le b,\ c \le y \le d$ That contains the point (x_0, y_0)in its interior. Then there exists some interval If $f(x,y) and \frac{\partial f}{\partial y} are$ continuous on R then there exists some interval

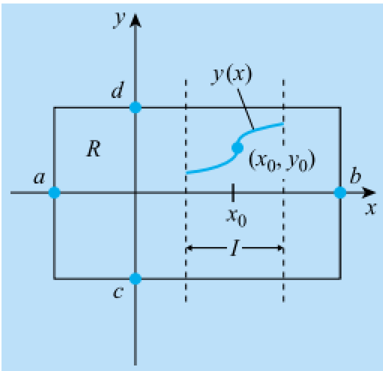
$I_0: (x_0 - h, x_0 + h)\ , h > 0\ contained\ in\ [a,b]$

And a unique function y(x) defined on I_0

That is a solution of the I.V.P. $y' = (x,y), y(x_0) = y_{\ 0}$

Existence and Uniqueness Theorem

Let R be a rectangular region in the x-y plane described by the two inequalities $a \le x \le b$ and $c \le y \le d$. Suppose that the point (x_0, y_0) is inside R. Then if $f(x, y)$ and the partial derivative $\frac{\partial f}{\partial y}(x, y)$ are continuous functions on R, there is an interval I centered at $x = x_0$ and a unique function $y(x)$ defined on I such that y is a solution of the IVP $y' = f(x, y)$, $y(x_0) = y_0$.



gives an idea of what such a region R and interval I in the Existence and Uniqueness Theorem may look like.

Remark

1- The geometric meaning of the theorem consist in the fact that there exists one and only one such function $y = \varphi(x)$ the graph of which passes through the point (x_0,y_0)

2- The condition (x_0)= y_0 $\quad or \quad$ $y|_{x=x_0}$ =x_0=y_0

It's called the initial condition.

2-.for linear O.D.E

Let $a(x), a_{n-1}(x), \dots \dots ., a_1(x), a_0(x)$

And f(x) be continuous on an interval I and let $a_0(x) \neq 0$ for every x in this interval I, if $x = x_0$ is any point in this interval the solution y(x) of the I.V.P. exists on the I and is unique.

Ex.1

The I.V.P. 3y'''+5y''−y'+7y=0 , y(1)=0 ,y'(1)=0 ,y''(1)=0

Has a unique solution y=0 on any interval containing x=1.

Ex.2

The function $y = 3e^{2x} + e^{-2x} - 3x$ is a solution of the I.V.P.
$y'' - 4y = 12x \ , y(0) = 4 \, , y'(0) = 1$

On interval I containing x=0 has unique solution on I.

Singular point

It is a point in plane not satisfies one or more necessary conditions in theorem of existence and uniqueness.

Ex.1

$$y' = y$$

is a real and continuous function where y≥0 $f(x, y) = \sqrt{y}$

is a real and continuous function where y>0 ,$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$

∴every point (x_0, y_0) $s.t. y_0 > 0$ ∃ $aunique$ solution

∴every point(x_0, y_0) $s.t. y_0 \leq 0 are\ singular\ point$

Ex.2

$\boldsymbol{y'}$=$\boldsymbol{2x}$

is a real and continuous function on (−∞,∞) f(x,y)=2x

is a real and continuous function on (−∞,∞) $\frac{\partial f}{\partial y}$=0

∴every point (x_0, y_0) on (−∞,∞) ∃ a unique solution

∴There are no singular points.

Ex.3

Is there exist a unique solution for $y = \frac{y-2}{x-1}\ in\ (1,2)$

Solution

$f(x, y) = \frac{y-2}{x-1}$ is continuous at every point (x_0, y_0) in plane s.t.$x_0 \neq 1$

$\frac{\partial f}{\partial y} = \frac{1}{x-1}$is continuous at every point (x_0, y_0) in plane s.t. $x_0 \neq 1$

∴every point (x_0, y_0) s.t.xo≠1 ∃ a unique solution

∴every point (x_0, y_0) s.t. $x_0 = 1\ are\ singular\ point$

∴There is no a unique solution on (1,2)

Ex.4 $(y')^2 = \frac{-1}{2}$, $y(0) = 1$

$$f(x, y) = \sqrt{\frac{-1}{2}} \text{ is not real}$$

There is no solution for every point in plane.∴

Ex.5 **Is there exist a unique solution for $y' = y$ and in (1,-2)?(H.W.)**

Ex.6 **Is there exist a unique solution for $y' = \sqrt{y - x}$, $y(0) = -1$? (H. W.)**

Remark

1- f(x,y)=y and $\frac{df}{dy} = 1$ are continuous in xy−plane

2- The condition in theorem existence and uniqueness are sufficient but not necessary.

Exercises

1- Prove that:

1- $y = x^2 + cx$ *is a solution of* $xy' = x^2 + y$

2- $y = A\sin 2x + B\cos 2x$ *is a solution of* $y'' + 4y = 0$

3- $y = Ae^{-x} + Be^{-2x}$ *is a solution of* $y'' + 3y' + 2y = 0$

2-Prove that

$y = 2e^x \ , y = 3x \ \ , y = Ae^x + Bx \ \ (A, B \ are \ constant)$

Are solutions of $(1-x)y'' + xy' - y = 0$

3- Find A when $y = Ax^3 \ is \ a \ solution \ of \ x^2y'' + 6xy' + 5y = x^3$

4- Find the O.D.E. from

1- $y = Ae^x - x$

2- $y = Ax^2 + Bx + C$

3- $y = Ax^2 + A^2$

$4 - y = Asinx + Bcosx$

Where A,B,C are constants

5- Is the solution of the I.V.P. $y' = 5y + 3e^{5x}, y(0) = 8 \ unique$?

6- Is there exist unique solution in (1,-1) for $y' = \frac{x}{y}$?

<u>An Existence and Uniqueness Theorem generalized for n-th order</u>

Existence and uniqueness theorem for nth order linear ODEs.

The general nth order linear ODE if there is given by

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t) = b(t), \tag{1}$$

where $a_j : I \to \mathbf{R}$, $b : I \to \mathbf{R}$ are functions defined on some open interval $I \subset \mathbf{R}$. The ODE becomes an initial value problem if we further impose that

$$y(t_0) = y_0, \ldots, \frac{d^{n-1}y}{dt^{n-1}}(t_) = y_{n-1} \tag{2}$$

for some $t_0 \in I$ and $y_0, \ldots, y_{n-1} \in \mathbf{R}$. The fundamental theorem concerning solutions of such problems is as follows.

Theorem 0.1. *Suppose that the coefficient functions $a_j : I \to \mathbf{R}$ and the right side $b(t)$ in 1 are all continuous on I. Then there is a unique n-times differentiable function $y : I \to \mathbf{R}$ satisfying both 1 and 2.*

Theorem

Set of differential equations

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, y_3, \dots, y_n) \qquad ; i = 1,2,3, \dots, n$$

There is only one set of continuous solutions

$$y_1(x), y_2(x), \dots, y_n(x)$$

So that solutions take values

$$y^\circ{}_1, y^\circ{}_2, y^\circ{}_3, \dots, y^\circ{}_n$$

When $x = x_\circ$ Provided that the functions

$$f_1, f_2, f_3, \dots, f_n, \frac{\partial f_i}{\partial y_1}, \frac{\partial f_i}{\partial y_2}, \dots, \frac{\partial f_i}{\partial y_n} \qquad ; i = 1,2,3, \dots, n$$

Is continues and have existence of a unique solution in the reign R

$$|x - x_0| \le a, \left|y_1 - y_1^0\right| \le b_1, \dots, |y_n - y_n^0| \le b_n$$

$$where\ a, b_1, b_2, \dots., b_n\ is\ positive.$$