Chapter -4-

Linear O.D.E. with constant coefficients

The general form of non-homo. linear O.D.E. with constant coefficients of n order is:-

………(*)$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots \dots\dots\dots\dots a_{n-1} y' + a_n y = f(x)$

Where $a_1, a_2, \dots\dots\dots\dots\dots a_{n-1}, a_n$ are constant

If f(x)=0 then the equation(*) it's become homo.

Ex:-

1- $y'' - 2y' + 3y = 0$

2- $y'' - 4y' + y = e^x$

3- $xy^{(4)} = sinx$

4- $y^{(4)} = \frac{sinx}{x}$, $x \neq 0$

Def.-1- :- The functions $y_1(x), y_2(x), \dots\dots\dots\dots\dots, y_n(x)$ are linearly dependent on I if there is a set of constants $c_1, c_2, \dots\dots\dots\dots\dots, c_n$

Not all zero s.t. $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$ In $a \leq x \leq b \dots\dots..(*)$

The left member of (*) is called a linear combination of the functions $y_1, y_2, \dots\dots. y_n$.

The functions $y_i(x), i = 1,2,3, \dots\dots. n$

Are linearly independent if the only set of constants$c_1, c_2, \dots\dots c_n$ for which (*) holds is the set $c_1 = c_2 = \cdots \dots\dots. = c_n = 0$

Ex.-1- **prove that the functions $\boldsymbol{y_1 = e^x}$, $\boldsymbol{y_2 = e^{2x}}$**

Are linear independent.

Solution:-

$c_1 y_1 + c_2 y_2 = 0$

$c_1 \mathrm{e}^{\mathrm{x}} + \mathrm{c}_2 \mathrm{e}^{2\mathrm{x}} = 0$

(derivative w.r.t x)$c_1 e^x + 2c_2 e^{2x} = 0$

$-c_2 e^{2x} = 0$

$e^{2x} \neq 0 \rightarrow c_2 = 0$

$\therefore c_1 = 0$

$\therefore c_1 = c_2 = 0$

the functions are linear independent for any x$\therefore$

Ex.-2- **prove that $\boldsymbol{f_1(x) = cos^2(x), f_2(x) = sin^2(x),\ f_3 = sec^2(x),}$**

$\boldsymbol{f_4(x) = tan^2(x)}$ **Are linearly dependent on the interval $\boldsymbol{\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)}$**

Solution:

$c_1 cos^2(x) + c_2 sin^2(x) + c_3 sec^2(x) + c_4 tan^2(x) = 0$

$c_1 = c_2 = 1 \rightarrow cos^2(x) + sin^2(x) = 1$

$c_3 = -1, c_4 = 1 \rightarrow -tan^2(x) + sec^2(x) = 1$

$$tan^2(x) - sec^2(x) = -1$$

$cos^2(x) + sin^2(x) - sec^2(x) + tan^2(x) = 0$

The functions are linearly dependent$\therefore$

Theorem-1-

Let $y_1, y_2, \dots \dots, y_n$

Be solutions of the homo. n-th order linear O.D.E. on an interval I then the linear combination $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

Where $c_1, c_2, \dots \dots, c_n$ are arbitrary constants is a solution on I

Corollaries:-

1- A constant multiple $y = c_1 y_1(x)$

of a solution $y_1(x)$ of a homo. linear O.D.E. is also solution

2- A homo. . linear O.D.E. always has the trivial solution $y = 0$

Def.-2- :-The function $y = c_1 y_1 + c_2 y_2 + \cdots \dots + c_n y_n$

Which is a linear combination for solutions is a general solution for homo. linear O.D.E. if $y_1, y_2, \dots \dots, y_n$ is linear independent.

Ex.:- The functions $y_1 = x^2, y_2 = x^2 lnx\ are\ solution\ of\ the\ equation$

0 on $(0, \infty) then x^3 y''' - 2xy' + 4y =$

is a solution of this eq. on $(0, \infty) \mathrm{y} = \mathrm{c_1 x^2 + c_2 x^2 lnx}$

Is called general solution.

Def.-3-:- suppose each of the function $y_1, y_2, \dots \dots y_n$

Has at least (n-1) derivative; the determinate

$$w = w(y_1, y_2, \dots \dots, y_n) = \begin{vmatrix} y_1 & y_2 \dots \dots \dots & \dots y_n \\ y'_1 & y'_2 \dots \dots & \dots \dots\ y'_n \\ & \vdots & \\ y^{(n-1)}{}_1 & y^{(n-1)}{}_2 \dots \dots \dots \dots & y^{(n-1)}{}_n \end{vmatrix}$$

Where the primes denoted derivative is called the **wronskian** of the functions.

<u>Theorem-2-:-</u>

The solutions $y_1, y_2, \dots \dots y_n$ for homo. linear O.D.E. are linear dependent iff $w(y_1, y_2, \dots \dots, y_n) = 0$

And $y_1, y_2, \dots \dots y_n$ are linear independent iff $w(y_1, y_2, \dots \dots, y_n) \neq 0$

<u>Ex.-1-</u>prove that the functions $e^x, 4e^x, 3e^{-2x}$

Are linear dependent for all x.

<u>Solution:</u>

$$w(\boldsymbol{e^x}, \boldsymbol{4e^x}, \boldsymbol{3e^{-2x}}) = \begin{vmatrix} e^x & 4e^x & 3e^{-2x} \\ e^x & 4e^x & -6e^{-2x} \\ e^x & 4e^x & 12e^{-2x} \end{vmatrix}$$

$$=4 \begin{vmatrix} e^x & e^x & 3e^{-2x} \\ e^x & e^x & -6e^{-2x} \\ e^x & e^x & 12e^{-2x} \end{vmatrix}$$

$$=0$$

The functions $e^x, 4e^x, 3e^{-2x}$ Are linear dependent for all x.∴

<u>Ex.:-</u> prove that the functions x^2, x^3, x^{-2}

Are linear independent solutions for $x^3y''' - 6xy' + 12y = 0$

<u>Solution:-</u>

$$w(\boldsymbol{x^2}, \boldsymbol{x^3}, \boldsymbol{x^{-2}}) = \begin{vmatrix} x^2 & x^3 & x^{-2} \\ 2x & 3x^2 & -2x^{-3} \\ 2 & 6x & 6x^{-4} \end{vmatrix}$$

$$= 20$$

$$\neq 0$$

Theorem-3-:-

If $y = I(x)$ or y_p

Is a particular solution for non-homo. linear O.D.E. and y=c(x) is a complementary function then the general solution for non-homo. linear O.D.E. is y=c(x)+I(x).

Proof:-

………(1)$y^{(n)} + a_1 y^{(n-1)} + \cdots \ldots \ldots + a_n y = f(x)$ $(non - homo.)$

…………(2)$y^{(n)} + a_1 y^{(n-1)} + \cdots \ldots \ldots + a_n y = 0$ $(homo.)$

I(x) is a solution for (1) ; then:

$(\mathrm{I(x)})^{(n)} + a_1(\mathrm{I(x)})^{(n-1)} + \cdots \ldots \ldots \ldots + a_n(\mathrm{I(x)}) = f(x)$

C(x) is a solution for (2) then :-

$(c(\mathrm{x}))^{(n)} + a_1(\mathrm{c(x)})^{(n-1)} + \cdots \ldots \ldots \ldots + a_n(\mathrm{c(x)}) = 0$

$(\mathrm{I(x)} + c(x))^{(n)} + a_1(\mathrm{I(x)} + \mathrm{c(x)})^{(n-1)} + \cdots + a_n(\mathrm{I(x)} + \mathrm{c(x)}) = f(x)$

I(x)+c(x) is a solution for non-homo. O.D.E.

Ex.:- **are the functions sinx ,cosx solution for** $\boldsymbol{y'' + y = 0}$?

and are these functions linear independent?

$\mathrm{y = sinx \rightarrow y' = cosx \rightarrow} \boldsymbol{y''} = -\mathrm{sinx}$

-sinx+sinnx=0

sinx is a solution for this equation∴

$\mathrm{y = cosx \rightarrow y' = -sinx \rightarrow} \boldsymbol{y''} = -\mathrm{cosx}$

-cosx+cosx=0

cosx is a solution for thisequation∴

$=-sin^2x - cos^2x = -1 \neq 0$ $\quad w(sinx, cosx) = \begin{vmatrix} sinx & cosx \\ cosx & -sinx \end{vmatrix}$

Sinx,cosx are linear independent

$y = c_1 sinx + c_2 cosx \ is \ a \ general \ solution \ for \ this \ equation$

Ex.:-the particular solution of eq. $y''' - 6y'' + 11y' - 6y = 3x \text{ is } y_p = \frac{-11}{12} - \frac{1}{2}x$

And general solution of this eq. is $y_c = c_1e^x + c_2e^{2x} + c_3e^{3x}$

Then find the general solution for this eq.

Solution:-

$y = y_p + y_c$

$= c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{11}{12} - \frac{1}{2}x$ is the general solution of given eq.

Dfe.-4-"fundamental set of solutions"

Any set $y_1, y_2 \dots \dots, y_n$ of n linearly independent solutions of the homo. linear n-th order diff. eq. on an interval I is said to be a fundamental set of solutions on the interval I.

Theorem-4-

There exist a fundamental set of solutions for the homo. linear n-th order diff. eq. on I.

Theorem-5-

Let $y_1, y_2 \dots \dots, y_n$ be a fundamental set of solutions of the homo. linear n-th order diff. eq. on interval I .

Then the general solution of the eq. on the interval I is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots \dots + c_ny_n(x)$$

Where $c_i, i = 1,2,3, \dots \dots n$ *are arbitrary constants*

Exercises

1- are the function linear dependent or independent for any interval?

(1) $2x-2x^5, 5x^5-x$ $(2)2x^2, -3x^2, x$ $(3)sinx, x$ $(4)xe^x, e^x, x^2e^x$

(5)$x+5, x+1, x$ $(6)e^x cosx, e^x sinx$ $(7)x^3, x^2, x$ $(8)6-2x, x-3, e^x)$

2-a) prove that e^x, e^{2x}

are solution for $y''-3y'+2y=0$.

Are the functione^x, e^{2x} *are linear dependent or independent*?

b) prove that $y_1=e^{ax}cosbx, y_2=$
$e^{ax}sinbx$ *are solutions for the equation*

$y''-2ay'+(a^2+b^2)y=0$

and prove that y_1, y_2 *are linear independent*.