

Chapter -3-

(1) Solve O.D.E. of the second order

Reduction of order

The general form of non-homo. linear O.D.E. of second order is:

$$y'' + a_1(x)y' + a_2(x)y = f(x) \dots \dots \dots (*)$$

Where a_1, a_2, f are function of x

Let $u(x)$ is a particular solution of homo. linear O.D.E.

$$y'' + a_1(x)y' + a_2(x)y = 0$$

We can reduce of the equation (*) by the following steps:

1- Suppose the solution has the form:-

$$y=uv$$

2- by derivative ,obtain:

$$y=uv$$

$$y' = uv' + vu'$$

$$y'' = uv'' + u'v' + u'v' + u''v$$

$$= uv'' + 2u'v' + u''v$$

Substituted y, y', y'' in (*)

$$uv'' + 2u'v' + u''v + a_1(x)(uv' + vu') + a_2(x)uv = f(x)$$

$$uv'' + [2u' + a_1(x)u]v' + [u'' + a_1(x)u' + a_2(x)u]v = f(x)$$

$$u'' + a_1(x)u' + a_2(x)u = 0$$

$$\therefore uv'' + [2u' + a_1(x)u]v' = f(x) \dots \dots \dots (**)$$

3- Let $p = v'$ then $p' = v''$

Substituted in (**), obtain:

$$up' + (2u' + a_1(x)u)p = f(x)$$

It is non-homo. linear O.D.E. of the first order with dependent variable p independent x ;we can solve it and find the solution of given equation by original variables .

Ex:-1- if $y = e^{-2x}$ is a solution of the homo. equation

$$(1 + x)y'' + (4x + 5)y' + (4x + 6)y = 0$$

then find the solution of the equation

$$(1 + x)y'' + (4x + 5)y' + (4x + 6)y = e^{-2x}$$

Solution:-

$$\text{Let } y = ve^{-2x}$$

$$y' = -2ve^{-2x} + v'e^{-2x}$$

$$y'' = 4ve^{-2x} - 2v'e^{-2x} + v''e^{-2x} - 2v'e^{-2x}$$

Substituted in given equation

$$(1 + x)(4ve^{-2x} - 2v'e^{-2x} + v''e^{-2x} - 2v'e^{-2x}) + (4x + 5)(-2ve^{-2x} + v'e^{-2x}) + (4x + 6)ve^{-2x} = e^{-2x}$$

$$(1 + x)v''e^{-2x} + v'e^{-2x} = e^{-2x}] \div e^{-2x}$$

$$(1 + x)v'' + v' = 1$$

$$p = v' \text{ then } p' = v''$$

$$\therefore (1 + x)p' + p = 1$$

$$(\text{O.D.E. of the first order solving by integrating factor}) \therefore \frac{dp}{dx} + \frac{p}{1+x} dx = \frac{dx}{1+x}$$

$$\text{I.F. is } e^{\int \frac{1}{1+x} dx} = e^{\ln(1+x)} = 1 + x$$

$$d(p(1+x)) = dx$$

by integrating:

$$p(1 + x) = x + c_1$$

$$\therefore p = \frac{x + c_1}{1 + x}$$

$$v' = \frac{x + c_1}{1 + x}$$

$$\therefore dv = \frac{x + c_1}{1 + x} dx$$

By integrating:

$$v = \int \frac{x+1-1+c_1}{1+x} dx + c_2$$

$$\therefore v = \int dx + \int \frac{c_1-1}{1+x} dx + c_2$$

$$v = x + (c_1 - 1) \ln|1 + x| + c_2$$

$$y = e^{-2x} [x + (c_1 - 1) \ln|1 + x| + c_2] \therefore$$

$$y = xe^{-2x} + (c_1 - 1) \ln|1 + x| e^{-2x} + c_2 e^{-2x} \therefore$$

Ex:-2- if $y = x^3$ is a solution of the homo. equation

then find the general solution of this equation and find $y'' + \frac{1}{x}y' - \frac{9}{x^2}y = 0$
the other particular solution of this equation.

Solution:-

Let $y = x^3v$

$$y' = 3vx^2 + v'x^3$$

$$y'' = 3v'x^2 + 6xv + 3v'x^2 + v''x^3$$

$$y'' = 6v'x^2 + 6xv + v''x^3$$

Substituted in given equation; obtain:

$$6v'x^2 + 6xv + v''x^3 + \frac{1}{x}(3vx^2 + v'x^3) - \frac{9}{x^2}x^3v = 0$$

$$(6x^2 + x^2)v' + (6x + 3x - 9x)v = 0v''x^3 +$$

$$v''x^3 + 7x^2v' = 0 \dots \dots (*)$$

$$\text{let } p = v' \text{ then } p' = v''$$

Substituted in (*) ;obtain:

$$x^3p' + 7x^2p = 0$$

$$p' + \frac{7}{x}p = 0$$

(O.D.E. of the first order solving by integrating factor) $dp + \frac{7}{x}pdx = 0$

$$\text{I.F. is } e^{\int \frac{7}{x} dx} = e^{7 \ln x} = e^{\ln x^7} = x^7$$

$$d(px^7) = 0$$

$$px^7 = c_1$$

$$p = c_1 x^{-7}$$

$$\therefore v' = c_1 x^{-7}$$

$$dv = c_1 x^{-7} dx$$

By integrating; obtain:

$$v = \frac{-c_1}{6} x^{-6} + c_2$$

$$y = x^3 \left[\frac{-c_1}{6} x^{-6} + c_2 \right]$$

$$y = \frac{-c_1}{6} x^{-3} + c_2 x^3$$

is particular solution of homo. equation $\therefore y_1 = x^3$

is particular solution of homo. equation $y_2 = \frac{-1}{6} x^3$

The linearly combination of y_1, y_2 is the general solution of given equation

Exercise

1- if $y_1 = e^x$ is a solution of the homo. equation $y'' - y = 0$ then use the reduction of order to find a second solution y_2 .

2- if $y=x$ is a solution of the homo. equation $x^3 y'' + xy' - y = 0$ then use the reduction of order to find the general solution.

Remark

We can use the reduction of order to solve O.D.E. of the higher order.

Ex. Solve the following equations:

$$1- xy''' - 2y'' = 0$$

Solution

Let $q = y''$

$$\therefore q' = y'''$$

$$\therefore xq' - 2q = 0$$

$$xdq - 2qdx = 0$$

$$\frac{dq}{q} - \frac{2}{x}dx = 0$$

by integrating

$$\ln q - 2\ln x = \ln a$$

$$\ln q - \ln x^2 = \ln a$$

$$\ln \frac{q}{x^2} = \ln a$$

$$\therefore q = ax^2$$

$$y'' = ax^2$$

By integrating

$$y' = a \frac{x^3}{3} + b$$

By integrating

$$y = a \frac{x^4}{12} + bx + c$$

$$2- y''' - y'' = 1$$

Solution

Let $p = y''$

$$\therefore p' = y'''$$

$$\therefore p' - p = 1$$

$$\frac{dp}{dx} - p = 1$$

$$dp - p dx = dx$$

$$dp = (p + 1)dx$$

$$\frac{dp}{p + 1} = dx$$

by integrating

$$\ln|p + 1| = x + c_1$$

$$\therefore p + 1 = e^{x+c_1}$$

$$\therefore p = Ae^x - 1$$

$$y'' = Ae^x - 1$$

By integrating

$$y' = Ae^x - x + B$$

By integrating

$$y = Ae^x - \frac{x^2}{2} + Bx + c$$

Exercise

Solve the equation $xy'' = y'$

(2) Linear O.D.E. with constant coefficients

Def.:- The functions $y_1(x), y_2(x), \dots, y_n(x)$ are linearly dependent on I if there is a set of the constants $c_1, c_2, \dots, c_n$

Not all zero s.t. $c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$ In $a \leq x \leq b \dots \dots \dots (**)$

And the functions $y_i(x), i = 1, 2, 3, \dots, n$ are linearly independent on I if the constants $c_1, c_2, \dots, c_n$ all zero $c_1 = c_2 = \dots = c_n = 0$

[The left member of (**) is called a linear combination of the functions $y_1, y_2, \dots, y_n$.]

Ex.-1- prove that the functions $y_1 = e^x, y_2 = e^{2x}$

Are linear independent.

Solution:-

$$c_1 y_1 + c_2 y_2 = 0$$

$$c_1 e^x + c_2 e^{2x} = 0$$

$$(\text{derivative w.r.t } x) c_1 e^x + 2c_2 e^{2x} = 0$$

$$-c_2 e^{2x} = 0$$

$$e^{2x} \neq 0 \rightarrow c_2 = 0$$

$$\therefore c_1 = 0$$

$$\therefore c_1 = c_2 = 0$$

The functions are linear independent for any x :

Theorem-1-

Let $y_1, y_2, \dots, y_n$

Be solutions of the homo. n-th order linear O.D.E. on an interval I Then the linear combination

Where $c_1, c_2, \dots, c_n$ Are arbitrary constants

Is a solution on I.

Corollaries:-

1- A constant multiple $y = c_1 y_1(x)$

Of a solution $y_1(x)$ Of a homo. linear O.D.E.

Is also solution.

2- A homo. linear O.D.E. always has the trivial solution $y = 0$

Def.:- The function $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

Which is a linear combination for solutions is a general solution for homo. linear O.D.E. if $y_1, y_2, \dots, y_n$ is linear independent.

Ex.:- The functions $y_1 = x^2, y_2 = x^2 \ln x$

Are solution of

$$0x^3 y''' - 2xy' + 4y =$$

On $(0, \infty)$ then

$$y = c_1 x^2 + c_2 x^2 \ln x$$

is a solution of this eq. on $(0, \infty)$

Is called the general solution.

Def.:- suppose each of the function $y_1, y_2, \dots, y_n$

Has at least $(n-1)$ derivative; the determinate

$$w = w(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y'_1 & y'_2 & \dots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y^{(n-1)}_1 & y^{(n-1)}_2 & \dots & y^{(n-1)}_n \end{vmatrix}$$

Where the primes denoted derivative is called the **wronskian** of the functions.

Theorem-2:-

The solutions $y_1, y_2, \dots, y_n$ for homo. linear O.D.E. are linear dependent
iff $w(y_1, y_2, \dots, y_n) = 0$

And $y_1, y_2, \dots, y_n$ are linear independent iff $w(y_1, y_2, \dots, y_n) \neq 0$

Ex.-1-prove that the functions $e^x, 4e^x, 3e^{-2x}$

Are linear dependent for all x.

Solution:

$$\begin{aligned}
 w(e^x, 4e^x, 3e^{-2x}) &= \begin{vmatrix} e^x & 4e^x & 3e^{-2x} \\ e^x & 4e^x & -6e^{-2x} \\ e^x & 4e^x & 12e^{-2x} \end{vmatrix} \\
 &= 4 \begin{vmatrix} e^x & e^x & 3e^{-2x} \\ e^x & e^x & -6e^{-2x} \\ e^x & e^x & 12e^{-2x} \end{vmatrix} \\
 &= 0
 \end{aligned}$$

The functions $e^x, 4e^x, 3e^{-2x}$ are linear dependent for all x.∴

Ex.-2- prove that the functions x^2, x^3, x^{-2}

Are linear independent solutions for $x^3y''' - 6xy' + 12y = 0$

Solution:-

$$\begin{aligned}
 w(x^2, x^3, x^{-2}) &= \begin{vmatrix} x^2 & x^3 & x^{-2} \\ 2x & 3x^2 & -2x^{-3} \\ 2 & 6x & 6x^{-4} \end{vmatrix} \\
 &= 20 \\
 &\neq 0
 \end{aligned}$$

Ex.-3- are the functions $\sin x, \cos x$ solution for $y'' + y = 0$?

and are these functions linear independent?

$$y = \sin x \rightarrow y' = \cos x \rightarrow y'' = -\sin x$$

$$-\sin x + \sin x = 0$$

$\sin x$ is a solution for this equation.:

$$y = \cos x \rightarrow y' = -\sin x \rightarrow y'' = -\cos x$$

$$-\cos x + \cos x = 0$$

$\cos x$ is a solution for this equation.:

$$\begin{aligned} w(\sin x, \cos x) &= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \\ &= -\sin^2 x - \cos^2 x = -1 \neq 0 \end{aligned}$$

$\sin x, \cos x$ are linear independent

$$y = c_1 \sin x + c_2 \cos x$$

Is a general solution for this eq.

"Solution of the homo. Linear diff.eq. with constant coefficients"

The general form of homo. linear diff. eq. with constant coefficients is:-

$$\frac{dy}{dx} + a_n y = 0 \quad (\text{n order}) \quad \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} y = 0$$

Where $a_1, a_2, \dots, a_{n-1}, a_n$ are constants.

We can write this eq. by using polynomial's operator:-

$$F(D)y = (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n)y = 0$$

$$\text{Let } F(m) = m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_{n-1} m + a_n$$

$$\text{Let } F(m) = 0$$

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_{n-1} m + a_n = 0 \therefore$$

Characteristic eq. for the homo. linear diff. eq.

$$(m - m_1)(m - m_2)(m - m_3) \dots (m - m_n) = 0 \therefore$$

are roots for Characteristic eq. $m_1, m_2, \dots, m_n$

Remark-1-

The general form the homo. linear diff. eq. with constant coefficients of the second order is:-

$$\frac{dy}{dx} + a_2y = 0 \dots \dots \dots (*) \frac{d^2y}{dx^2} + a_1$$

$$D^2y + a_1Dy + a_2y = 0$$

$$(D^2 + a_1D + a_2)y = 0$$

$$F(D)y = 0 \quad \text{where } F(D) = D^2 + a_1D + a_2 \therefore$$

$$F(m) = m^2 + a_1m + a_2$$

$$\text{Let } F(m)=0$$

$$\therefore m^2 + a_1m + a_2 = 0$$

$$(m - m_1)(m - m_2) = 0$$

Where m_1, m_2 are roots

We can solve the homo. linear O.D.E. with constant coefficients by the following:-

Case-1-

If the roots are real and $m_1 \neq m_2$; then the general solution for(*) is:-

$$y(x) = c_1e^{m_1x} + c_2e^{m_2x}$$

Case-2-

If the roots are real and $m_1 = m_2 = m$; then the general solution for(*) is:-

$$y(x) = (c_1x + c_2)e^{mx}$$

Case-3-

If the roots are complex $m_1 = a + ib$, $m_2 = a - ib$; then the general solution for(*) is:-

$$y(x) = e^{ax}(A\cos bx + B\sin bx)$$

Ex.:- solve the following eq.'s:-

$$1-y'' + y' - 2y = 0$$

Solution:-

$$D^2y + Dy - 2y = 0$$

$$(D^2 + D - 2)y = 0$$

$$F(D)y = (D^2 + D - 2)y = 0$$

$$F(m) = m^2 + m - 2$$

$$\text{Let } F(m)=0$$

$$\therefore m^2 + m - 2 = 0$$

$$(m - 1)(m + 2) = 0$$

$$\text{are roots } m_1 = 1, m_2 = -2$$

$$m_1 \neq m_2$$

$$\therefore y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\therefore y(x) = c_1 e^x + c_2 e^{-2x}$$

$$2-y'' - 4y' + 4y = 0$$

Solution:-

$$D^2y - 4Dy + 4y = 0$$

$$(D^2 - 4D + 4)y = 0$$

$$F(D)y = (D^2 - 4D + 4)y = 0$$

$$F(m) = m^2 - 4m + 4$$

$$\text{Let } F(m)=0$$

$$\therefore m^2 - 4m + 4 = 0$$

$$(m - 2)(m - 2) = 0$$

are roots $m_1 = 2, m_2 = 2$

$$m_1 = m_2$$

$$\therefore y(x) = e^{mx}(c_1x + c_2)$$

$$\therefore y(x) = e^{2x}(c_1x + c_2)$$

$$\mathbf{3-y'' + 2y' + 2y = 0}$$

Solution:-

$$D^2y + 2Dy + 2y = 0$$

$$(D^2 + 2D + 2)y = 0$$

$$F(D) = D^2 + 2D + 2 \quad F(D)y = (D^2 + 2D + 2)y = 0 \rightarrow$$

$$F(m) = m^2 + 2m + 2$$

$$\text{Let } F(m)=0$$

$$\therefore m^2 + 2m + 2 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = \frac{2(-1 \pm i)}{2} = -1 \pm i$$

are roots $(a=-1, b=1)m_1 = -1 + i, m_2 = -1 - i$

$$\therefore y(x) = e^{ax}(A\cos bx + B\sin bx)$$

$$\therefore y(x) = e^{-x}(A\cos x + B\sin x)$$

Exercises:- solve the following eq.'s:-

$$\mathbf{1-y'' - 4y' + 3y = 0}$$

$$\mathbf{2-y'' + y' = 0}$$

$$\mathbf{3-y'' - 2y' + y = 0}$$

$$\mathbf{4-y'' + y' + y = 0}$$