

## **Chapter -2-**

### **"Methods to solve O.D.E. of the first order"**

#### **Introduction**

In this chapter we will studied a solution for O.D.E. of the first order.

The general form of O.D.E. of the first order and degree is:

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

$$\text{or } M(x, y)dx + N(x, y)dy = 0$$

$$\text{or } \frac{dy}{dx} = \frac{-M(x, y)}{N(x, y)} = f(x, y)$$

**Ex.**

$$1 - (y - x) + x^2 \frac{dy}{dx} = 0$$

$$2 - (y - x) + xy \frac{dy}{dx} = 0$$

$$3 - (x^2y + 2x)dx + (3x - \cos x)dy$$

$$4 - \frac{dy}{dx} = \frac{y}{x} + 2\sin x$$

#### **Remark**

1- There is no rule to solve all O.D.E.

2- We will be solving the following equations:

1- Equations with separated and separable variables

2- Exact equations and integrating factors

3- Linear equations

4- Bernoulli's Equation

## 1- Equations with separated and separable variables

### Definition

A first order O.D.E. of the form:

$$\frac{dy}{dx} = f_1(x)f_2(y)$$

Is said to be separable or to have separable variables

### Ex.

1-  $\frac{dy}{dx} = y^2 x e^{3x+4y}$  Separable variables

2-  $\frac{dy}{dx} = y + \sin x$  not Separable variables

### Ex. Solve the following equations:

1-  $xydy + (2x^2 - 1)(y + 2)dx = 0$

### Solution

$$\frac{y}{y+2} dy + \frac{2x^2 - 1}{x} dx = 0$$

$$\frac{y+2-2}{y+2} dy + \left[2x - \frac{1}{x}\right] dx = 0$$

$$\left[1 - \frac{2}{y+2}\right] dy + \left[2x - \frac{1}{x}\right] dx = 0$$

$$y - 2 \ln(y+2) + x^2 - \ln x + c = 0$$

2-  $xdx + ydy = 0$

### Solution

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2}$$

$$x^2 + y^2 = c \text{ (circles with center at the origin)}$$

$$3- \frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = -3$$

### Solution

$$ydy = -x dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{c_1}{2}$$

$$x^2 + y^2 = c$$

$$16 + 9 = c \rightarrow \therefore c = 25$$

$$x^2 + y^2 = 25$$

$$y = \pm \sqrt{25 - x^2}$$

$$y = \varphi_1(x) = \sqrt{25 - x^2}, \quad y = \varphi_2(x) = -\sqrt{25 - x^2}$$

### Exercises

**Solve the following:**

$$1- (1 + x)dy - ydx = 0$$

$$2- (e^{2y} - y)\cos x \frac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0$$

### **2- Exact O.D.E.**

**Exact differential:-** The exact differential for  $f(x,y)$  in  $x,y$  has the form:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

**Exact O.D.E.** The differential equation  $M(x,y)dx + N(x,y)dy = 0$  is called exact if  $M(x,y)dx + N(x,y)dy$  is exact differential for a function  $f(x,y)$

i.e. O.D.E. is called exact if there exist a function  $f(x,y)$  s.t.

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N$$

$$\therefore df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = Mdx + Ndy$$

Then we will get the solution of exact O.D.E. by integrating  $df=0$

$f(x,y)=c$  is a solution of exact O.D.E.  $\therefore$

**Def.:-** A differential expression  $Mdx+Ndy$  is an exact differential in a region  $R$  of the  $xy$ -plane if it corresponds to the differential of some function  $f(x,y)$  defined in  $R$ ; A first differential equation  $Mdx+Ndy=0$  is said to be an exact equation if the expression on the left –hand side is an exact differential.

**Theorem:-** Let  $M(x,y)$  and  $N(x,y)$  be continuous and have continuous first partial derivative in a rectangular  $R$  defined by  $a < x < b, c < y < d$ .

Then a necessary and sufficient condition that  $Mdx+Ndy$  be an exact differential is  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$  [O. D. E.  $Mdx + Ndy = 0$  is exact iff  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$  ]

**Ex.:-** Solve the following equations

(1)  $ydx+xdy=0$

**Solution**

$$\frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The O.D.E. is exact  $\therefore$

$$\exists f(x,y) \text{ s.t. } df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = Mdx + Ndy \therefore$$

$$\frac{\partial f}{\partial x} = M = y, \frac{\partial f}{\partial y} = N = x \therefore$$

Integrating  $\frac{\partial f}{\partial x}$  w.r. t.  $x$

$$\int_x \frac{\partial f}{\partial x} dx = \int_x y dx + g(y) \quad (g(y) \text{ is arbitrary function})$$

$$f(x,y)=xy+g(y)$$

$$\frac{\partial f}{\partial y} = x + g'(y)$$

$$\frac{\partial f}{\partial y} = N = x$$

$$x + g'(y) = x \therefore$$

$$\therefore g'(y) = 0$$

*integrating  $g'(y)$  w.r.t.  $y$*

$$\therefore g(y) = 0$$

$$f(x,y)=xy \therefore$$

The solution of O.D.E. is  $f(x,y)=c$

$xy=c$  is the general solution  $\therefore$

$$(2) (6x^2 + 4xy + y^2)dx + (2x^2 + 2xy - 3y^2)dy = 0$$

### Solution

$$\frac{\partial M}{\partial y} = 4x + 2y, \quad \frac{\partial N}{\partial x} = 4x + 2y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The O.D.E. is exact  $\therefore$

$$\exists f(x,y) \text{ s.t. } df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = Mdx + Ndy \therefore$$

$$\frac{\partial f}{\partial x} = M = 6x^2 + 4xy + y^2, \quad \frac{\partial f}{\partial y} = N = 2x^2 + 2xy - 3y^2 \therefore$$

Integrating  $\frac{\partial f}{\partial x}$  w.r.t.  $x$

$$\begin{aligned} f(x,y) &= \int \frac{\partial f}{\partial x} dx = \int (6x^2 + 4xy + y^2) dx + g(y) \\ &= 2x^3 + 2x^2y + xy^2 + g(y) \quad (g(y) \text{ is arbitrary function}) \end{aligned}$$

$$\frac{\partial f}{\partial y} = 2x^2 + 2xy + g'(y) = N = 2x^2 + 2xy - 3y^2$$

$$\therefore g'(y) = -3y^2$$

integrating  $g'(y)$  w.r.t.  $y$

$$\therefore g(y) = -y^3$$

the general solution of given equation is:

$$f(x,y)=c$$

$$\therefore 2x^3 + 2x^2y + xy^2 - y^3 = c$$

### Exercises

**(1) are the following equations exact?**

1)  $(y^2 - x)dx + (x^2 - y)dy = 0$

2)  $x \cos y dx + y \cos x dy = 0$

**(2) solve the following equations:-**

1)  $2xy dx + (x^2 - 1)dy = 0$

2)  $(3x^2 + 2y \sin 2x)dx + (2 \sin^2 x + 3y^2)dy = 0$

3)  $\left(3x^2 + \frac{2y}{x}\right)dx + \left(2 \ln 3x + \frac{3}{y}\right)dy = 0, x > 0, y \neq 0$