Ordinary differential equations

Chapter *one*

Basic concepts of differential equations

A differential equation is an equation between specified derivative on an unknown function, its values, and known quantities and functions. Many physical laws are most simply and naturally formulated as differential equations (or DEs, as we will write for short). For this reason, DEs have been studied by the greatest mathematicians and mathematical physicists since the time of Newton.

Ordinary differential equations are DEs whose unknowns are functions of a single variable; they arise most commonly in the study of dynamical systems and electrical networks. They are much easier to treat that partial differential equations, whose unknown functions depend on two or more independent variables

Definition of differential equations

Is an equation content of un known function and derivative or an equation containing the derivatives of one or more dependent variables w.r.t. one or more independent variables.

Types of differential equations

There are two types of differential equations:

1-Ordinary differential equations(O.D.E):

Ordinary DEs are classified according to their order. The order of a DE is defined as the largest positive integer, n, for which an nth derivative occurs in the equation. Thus, an equation of the form

$$\Phi(x, y, y') = 0$$

is said to be of the first order

Or is an equation contains only ordinary derivatives of one or more dependent variables w.r.t. a single independent variable.

Remark:-

Ordinary derivatives will be written by using either Leibniz notation $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \dots\dots\dots\dots.., \frac{d^ny}{dx^n}$

Or the prime notation $y', y'', y''', y^{(4)} \dots\dots\dots.., y^{(n)}$

Example:

$1 - \ 2y' + 3y = 0$

Or $2\frac{dy}{dx} + 3y = 0$

2- $y'' - 4y' + 2y = ex$

Or $\frac{d^2y}{dx^2}{-}4\frac{dy}{dx}{+}2y{=}ex$

3- $y''''{+}2y'''{+}y''{=}sinx$

Or $\frac{d^4y}{dx^4}{+}2\frac{d^3y}{dx^3}{+}\frac{d^2y}{dx^2}{=}sinx$

2-Partial differential equations

Is an equation contains partial derivatives of one or more dependent variables of two or more independent variables.

-:Remark

partial derivatives will be written

$\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \ \frac{\partial^3 y}{\partial x^3}, \dots\dots\dots\dots..$

$\frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^3 z}{\partial y^3}, \dots\dots\dots\dots..$

Example

1- $\frac{\partial^2 u}{\partial x^2}{+}\frac{\partial^2 u}{\partial y^2}{=}0$

2- $\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)-\left(\frac{\partial^2 u}{\partial x\partial y}\right)$

3- $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}$

Order of differential equation

The order of a differential equation is defined to be that of the highest order derivative it contains.

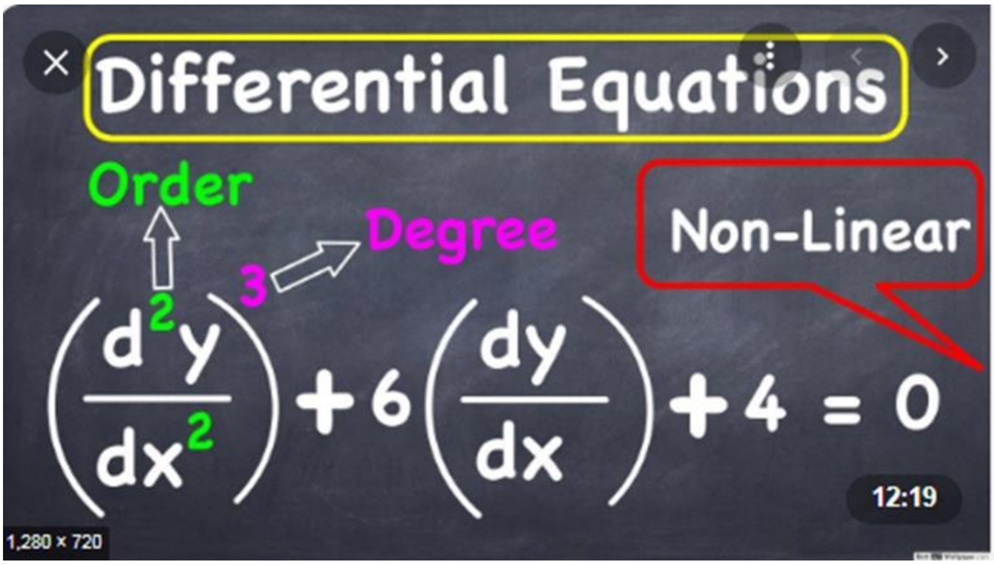
ex:

1- $\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)^3-2y=e^x\ (second\ order)$ -1
2- $y'''-y''-3y=cosx\ (Third\ order)$

Degree of differential equation

The degree of a differential equation is defined as the power to which the highest order derivative is raised. The equation

$(f''')^2+(f'')+f=x$ is an example of a second-degree, third-order differential equation.



Ex (1).

1- $(y'''')^2 \ + y'' = 0 \ (furth\ order\ and\ second\ degree)$

2- $\left(\frac{dy}{dx}\right)^3 \ + 2ytanx = sinx \ (first\ order\ and\ third\ degree)$

Ex.(2)

Consider the following equations:

1- $cos\left(\frac{dy}{dx}\right) = \frac{dy}{dx} + x$

2- $lny' = x3 + 2$

3- $siny'' - y = 0$

Does not have a degree

Remark :-

O.D.E. has the general form:

$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \dots \dots \dots \frac{d^ny}{dx^n})$=0

$or\ F(x, y, y', y'', \dots \dots \dots y^n) = 0$

$n - th$ order O.D.E. where F is a real –valued function of (n+2) variables x,y,y',y'',………$y(n)$

$\frac{dy}{dx} = f(x, y)$

$\frac{d^2y}{dx^2}$=(x,y,y').

.

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. $\frac{d^ny}{dx^n}$= (x, ,y',y'',……..,$y(n-1)$)

.Linear O.D.E

A linear equation or polynomial, with one or more terms, consisting of the derivatives of the dependent variable with respect to one or more independent variables is known as a linear differential equation.

A general first-order differential equation is given by the expression:

$$\boldsymbol{dy/dx + Py = Q}$$

where y is a function and dy/dx is a derivative.

The general form of linear O.D.E. is:-

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots \ldots \ldots . + a_1(x)y' + a_0 y = f(x) \ldots \ldots \ldots . (*)$$

Where $a_0, a_1, \ldots \ldots, a_n, f$ are functions to x, $x \in [a, b]$

- if an≠0

The equation (*) is called of the n-th order.

- if $a_0, a_1, \ldots \ldots, a_n$ are constants then the equation (*) it is called of the constant coefficients.

- if f(x)=0 then the equation(*) it is called homogenous

- if $f(x)$≠0 then the equation(*) it is called non homogenous

Definition 5.3. Linearity of a DE. A ***linear*** differential equation can be written in the form

$$F_n(x)\frac{d^n y}{dx^n} + F_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + F_2(x)\frac{d^2y}{dx^2} + F_1(x)\frac{dy}{dx} + F_0(x)y = G(x)$$

where $F_i(x)$ and $G(x)$ are functions of x. Otherwise, we say that the differential equation is ***non-linear***.

Non-Linear Differential Equation:-

When an equation is not linear in unknown function and its derivatives, then it is said to be a nonlinear differential equation. It gives diverse solutions which can be seen for chaos.

Ex.:-

1- $4xy' + 2e^{x}y = sinx$ (linear ,non homo.,non with constant coefficients)

2- y'''+$cosy$=0 (non-linear)

3- (1−y)$''$+4y=lnx (non-linear)

4- y''+y'=0 (linear, homo., with constant coefficients)

Exercises

Classify the following equations:

1- $y'' + 4y = x^3$

2- $(x + y)'' + 4y = 2$

3- $(2 + 3(y')^3)^{1/3} = 2y''$

5- $\sqrt{y'} = x + y$

6- $y'' + y'tanx = lnx$

7- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2z$

8- $cosy'' + y'' = x + ey'$

9- $y2 + 3xyy'' + (yy')3 = x$

10- $\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0$

Solution of differential equation

A solution of a differential equation is an expression for the dependent variable in terms of the independent one(s) which satisfies the relation. The general solution includes all possible solutions and typically includes arbitrary constants (in the case of an ODE) or arbitrary functions (in the case of a PDE.)

Solution of differential equation

any function φ defined on an interval I and possessing at least n derivatives that are continuous on I ,which when substitute into an n-th order O.D.E. reduces the equation to an identity, is said to be a **solution** of the equation on the interval I .

is a solution of $F(x,y,y',y'',\ldots\ldots\ldots,y(n))=0$ if $\varphi(x)$

Then $F(x,\varphi(x),\varphi'(x),\varphi''(x),\ldots\ldots..\varphi(n)(x))=0$ for all x in I.

Example 1:

prove that $y = e^3x$ is a solution of $y'-3y=0$

Solution:-

$y'=3e3x$ Substituted in given equation obtain: $3e3x-3e3x=0$

Example 2:

Prove that $y = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$?

Solution:-

$$y' = 2x \,,\, y'' = 2$$

Substituted in given equation obtain: $2x2-6x2+4x2=0$

$\therefore y=x2$ *is a solution of given equation*

Example 3:

Prove that $y = xlnx - x$ ***is a solution of*** $xy' = x + y \,;\, x > 0$

Solution

$y' = 1 + lnx - 1 = lnx$

$\therefore xlnx = x + xlnx - x$

$\therefore xlnx = xlnx \ \rightarrow \therefore$

$y = xlnx - x \ is\ a\ solution\ of\ given\ equation$

Example 4

is $x = ce^{-\mathrm{kt}}$ **a solution of** $\frac{\boldsymbol{dx}}{\boldsymbol{dt}} = -\boldsymbol{kx}$?

-:Solution

$x' = -cke^{-\mathrm{kt}} \quad \therefore -cke^{-\mathrm{kt}} = -kce^{-\mathrm{kt}}$

$\therefore x = ce^{-\mathrm{kt}} \ is\ a\ Solution\ of\ \frac{\boldsymbol{dx}}{\boldsymbol{dt}} = -kx$

Exercises

1- prove that :

a- $y = \sqrt{1 - x2}\ is\ a\ solution\ of\ yy' + x = 0\ on(-1,1).$

b- $\boldsymbol{y}=\frac{1}{16}x^4$ $\boldsymbol{is\ a\ solution\ of}$ $y' = xy^{1/2}$

2- Is $y = e^{5x}$ $\boldsymbol{a\ solution\ of}$ $y'' - y' + y = 0$?

3- Is $y = xe^{x}$ $\boldsymbol{a\ solution\ of\ y''-2y'+y=0}$?

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Types of the solution for O.D.E

1-The general solution

Is a solution of O.D.E. has arbitrary constants.

.Ex

1- $y=sinx+c$ *is a general solution of* $y'=cosx$

2- $y = x^4 + xc_1 + c_2$ *is a general solution of* $y'' = 12x^2$

3- $y = ce^x$ *is a general solution of* $y' = y$

2-The particular solution

Is a solution of O.D.E. we get it from the general solution.

Ex.1

$y=sinx$ $(c=0)$

$y=sinx+12$ $(c=12)$

$y=sinx-1$ $(c=-1)$

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Are particular solutions of $y'=cosx$

Ex.2

$y = x^4$ $(c_1 = c_2 = 0)$

$y = x^4 + x - 1$ $(c_1 = 1, c_2 = -1)$

$y = x^4 + \sqrt{2x} - 3$ $(c_1 = \sqrt{2}, c_2 = -3$

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Are particular of $y'' = 12x^2$

Ex.3

$y = e^x \qquad (c = 1)$

$y = -e^x \qquad (c = -1)$

$y = \sqrt{2}e^x \qquad (c = \sqrt{2})$

.

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Are particular solutions 0f y'=y

<u>3-The singular solution</u>

Any solution of O.D.E. it is not get it from the general solution.

Ex.1

y=0 is a singular solution of $y' = y$

Ex.2

y=0 is a singular solution of $y' = x\sqrt{y}\ on$ (−∞,∞)

$y = \left(\frac{1}{4}x^2 + c\right)^2$ is a general solution of $y' = x\sqrt{y}$

If c=0 then $y = \frac{1}{16}x^4$ is particular solution of $y' = x\sqrt{y}$

<u>4-The complete solution</u>

The general solution of O.D.E. is called complete if all solution for O.D.E. is a particular solution from the general solution.

Ex.1

The general solution $y = sinx + c\ of\ y' = cosx$ is complete

Ex.2

The general solution $y = cx - c^2$ is not complete since $y = \frac{x^2}{4}$ is a singular solution of O.D.E.

$$(y')^2 - xy' + y = 0$$

5-**Implicit solution of an O.D.E .**

A relation $G(x, y) = 0$ is said to be an implicit solution of an O.D.E. $F(x, y, y', y'', \dots \dots, y(n)) = 0$

On an interval I, provided that there exists at least one function φ that satisfies the relation as well as the O.D.E. on I.

Note:

Implicit solution means a solution in which dependent variable is not separated and explicit means dependent variable is separated.

EX .

Let us consider a differential equation

$$x + yy' = 0$$

The relation $x^2 + y^2 = 25$

Is an implicit solution of the $x + yy' = 0\, , for\ all\ x \in\ (-5{,}5)$

$\therefore y = \varphi_1(x) = \sqrt{25 - x2}\, , y = \varphi_2(x)$