

"Integrating factors"

Def.:- When an O.D.E. is not exact then multiplying by a factor; obtain an exact O.D.E. this factor is called an integrating factor of this equation.

i.e. if $M(x,y)dx + N(x,y)dy = 0$ inexact O.D.E.

if there exist a factor $u(x,y)$ s.t. $uM(x,y)dx + uN(x,y)dy = 0$ exact O.D.E.

$$\therefore \frac{\partial(uM)}{\partial y} = \frac{\partial(uN)}{\partial x}$$

$U(x,y)$ is called integrating factor.

The combination	Integrating factor	Exact differential
$x dy - y dx$	$\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2 + y^2},$ $\frac{1}{x^2 - y^2}, \dots, \frac{1}{ax^2 + bxy + cy^2}$	$\frac{y}{x}, \frac{-x}{y}, \ln \frac{y}{x}, \dots$
$P y dx + Q x dy$	$x^{p-1} y^{q-1}$	$x^p y^q$
$y dx + x dy$	1	$x dy - xy$
$dy + p(x)y dx$	$e^{\int p(x) dx}$	$y e^{\int p(x) dx}$

Ex.:- Solve the following equations:

(1) $x dy - y dx = x^2 y^3 dx$

Solution

multiply the equation by the integrating factor (I. F.) $\frac{1}{x^2}$ or $\frac{1}{y^2}$ or $\frac{1}{xy}$

$$\frac{x dy - y dx}{x^2} = y^3 dx$$

$$d\left(\frac{y}{x}\right) = y^3 dx$$

$$\text{Let } z = \frac{y}{x} \rightarrow y = xz$$

$$dz = x^3 z^3 dx$$

$$\frac{dz}{z^3} = x^3 dx$$

By integrating

$$\frac{-1}{2z^2} = \frac{x^4}{4} + c$$

$$\frac{-1}{2\left(\frac{y}{x}\right)^2} = \frac{x^4}{4} + c \text{ the general solution}$$

$$(2) xdy - \left(3y + \frac{x^4}{y}\right) dx = 0$$

Solution

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ is inexact}$$

$$xdy - 3ydx - \frac{x^4}{y} dx = 0$$

$$xdy - 3ydx = \frac{x^4}{y} dx$$

$$p=-3, q=1$$

$$\text{The integrating factor (I.F.) is } x^{p-1}y^{q-1} = x^{-4} = \frac{1}{x^4}$$

$$\therefore d(x^{-3}y) = -3x^{-4}ydx + x^{-3}dy$$

$$= x^{-4}(-3ydx + xdy)$$

$$\therefore \frac{xdy - 3ydx}{x^4} = \frac{dx}{y}$$

$$d\left(\frac{y}{x^3}\right) = \frac{1}{y} dx$$

$$z = \frac{y}{x^3} \rightarrow y = x^3 z$$

$$dz = \frac{1}{x^3 z} dx$$

$$z dz = x^{-3} dx$$

By integrating

$$\frac{z^2}{2} = \frac{x^{-2}}{-2} + c$$

$$\frac{1}{2} \left(\frac{y}{x^3}\right)^2 = \frac{-1}{2x^2} + c \text{ the general solution}$$

$$(3) \frac{dy}{dx} + \frac{y}{x} = 3x + 2$$

Solution

$$dy + \frac{y}{x} dx = (3x + 2) dx$$

The integrating factor (I.F.) is $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$$\therefore x \left(dy + \frac{y}{x} dx \right) = (3x^2 + 2x) dx$$

$$d(xy) = (3x^2 + 2x) dx$$

By integrating

$$\therefore xy = x^3 + x^2 + c \text{ the general solution}$$

Exercises

Solve the following equations by using integrating factor:-

(1) $4ydx + xdy = xy^2dx$

(2) $2ydx + 3xdy = 3x^{-1}dx$

(3) $xdy - 2ydx = x^3y^4dx$

(4) $dy + \frac{3}{x}ydx = x^{-3}e^x dx$

(5) $\frac{dy}{dx} + 2xy = xe^{-x^2}$

3-Linear O.D.E.

The general form of non-homo. linear O.D.E. of the first order is:-

$$\frac{dy}{dx} + p(x)y = \varphi(x) \dots \dots \dots (*)$$

Where p, φ are functions of x

We can solve this equation (*) by using the integrating factor, as follows:-

$$dy + p(x)ydx = \varphi(x)dx$$

The integrating factor is $e^{\int p(x)dx}$

$$\therefore e^{\int p(x)dx} (dy + p(x)ydx) = e^{\int p(x)dx} \varphi(x)dx$$

$$\therefore d(ye^{\int p(x)dx}) = e^{\int p(x)dx} \varphi(x)dx$$

by integrating the general solution of (*) is:-

$$\therefore ye^{\int p(x)dx} = \int e^{\int p(x)dx} \varphi(x)dx + c$$

$$\text{Or } y = e^{-\int p(x)dx} \int e^{\int p(x)dx} \varphi(x)dx + ce^{-\int p(x)dx}$$

Ex.:- Solve the following equations:

$$(1) \frac{dy}{dx} + \frac{y}{x} = x^2 - 3$$

Solution

$$dy + \frac{1}{x}ydx = (x^2 - 3)dx$$

$$\text{I.F. } e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$x(dy + \frac{1}{x}ydx) = (x^3 - 3x)dx$$

$$d(yx) = (x^3 - 3x)dx$$

By integrating the general solution is

$$yx = \frac{x^4}{4} - \frac{3}{2}x^2 + c \rightarrow \therefore y = \frac{x^3}{4} - \frac{3}{2}x + \frac{c}{x}$$

$$(2) x \frac{dy}{dx} + 2y = x^3$$

Solution

$$xdy + 2ydx = x^3dx$$

$$dy + \frac{2}{x}ydx = x^2dx$$

$$\text{I.F. } e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$d(yx^2) = x^4dx$$

by integrating the general solution is:-

$$yx^2 = \frac{x^5}{5} + c$$

$$y = \frac{x^3}{5} + \frac{c}{x^2}$$

Exercises

Solve the following equations:-

$$(1) \frac{dy}{dx} + 3xy = -2$$

$$(2) y' + x^2 y^2 = x$$

$$(3) xy' - \frac{y}{x^2} = \sin x$$

$$(4) y' - 2xy = e^{x^2}$$

4- Bernollie's equation

This equation has the form:-

$$\frac{dy}{dx} + p(x)y = y^n \varphi(x) \dots \dots (*)$$

If $n=0 \rightarrow non - homo. Linear O.D.E.$

If $n=1 \rightarrow homo. Linear O.D.E.$

The equation (*) is non-Linear transfer it to Linear and we can solve this equation by the following steps:-

$$(1) \left[\frac{dy}{dx} + p(x)y = y^n \varphi(x) \right] \div y^n$$

$$y^{-n} \frac{dy}{dx} + p(x)y^{1-n} = \varphi(x)$$

$$(2) \text{ let } z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n)y^{-n} \frac{dy}{dx}$$

$$\therefore y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dz}{dx}$$

(3) Substituted (2) in (1); obtain:-

$$\frac{1}{1-n} \frac{dz}{dx} + p(x)z = \varphi(x)$$

$$(4) \frac{dz}{dx} + (1-n)p(x)z = (1-n)\varphi(x) \text{ non-homo. Linear O. D. E.}$$

We can solve this equation by integrating factor

$$(5) \text{ I.F. } e^{\int (1-n)p(x)dx}$$

$$(6) d(z e^{\int (1-n)p(x)dx}) = (1-n)\varphi(x) e^{\int (1-n)p(x)dx}$$

(7) by integrating; obtain:-

$$z e^{\int (1-n)p(x)dx} = \int (1-n)\varphi(x) e^{\int (1-n)p(x)dx} dx + c$$

$$(8) \text{ Substituted } z=y^{1-n} \text{ in (7)}$$

Ex. Solve the following equation:-

$$xy - \frac{dy}{dx} = y^3 e^{-x^2}$$

Solution

Bernoulli's equation

$$\left[\frac{dy}{dx} - xy = -y^3 e^{-x^2} \right] \div y^3$$

$$(1) y^{-3} \frac{dy}{dx} - xy^{-2} = -e^{-x^2}$$

$$(2) z=y^{-2}$$

$$\frac{dz}{dx} = -2y^{-3} \frac{dy}{dx}$$

$$y^{-3} \frac{dy}{dx} = \frac{-1}{2} \frac{dz}{dx}$$

$$(3) \frac{-1}{2} \frac{dz}{dx} - xz = -e^{-x^2}$$

$$\frac{dz}{dx} + 2xz = 2e^{-x^2}$$

$$(4) dz + 2xzdx = 2e^{-x^2} dx$$

$$\text{I.F. is } e^{\int 2x dx} = e^{x^2}$$

$$(5) e^{x^2} (dx + 2xzdx) = 2dx$$

$$(6) d(ze^{x^2}) = 2dx$$

By integrating

$$ze^{x^2} = 2x + c$$

The general solution is:-

$$y^{-2}e^{x^2} = 2x + c$$

Exercises

$$(1) y' + xy = \frac{x}{y^3}, y \neq 0$$

$$(2) xdy - ydx = y^2 dx$$

$$(3) xy' + y = y^2 \ln x$$

$$(4) 2y' - \frac{y}{x} = 5x^3 y^3, x \neq 0$$