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An Approximate Solution to an Optimal Control Problem of Walking Robot via. Non-Classical Variational Approach

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Abstract: The mathematical model of two-rigid links of one-legged walking robot dynamic control system have been modified and adapted. The state-space model and its equilibrium points are found by using implicit function theorem with Newton-Raphson method. Hence, a local linearized dynamic control systems are obtained. Therefore, an optimal control criterion is designed to achieve some system performance objectives. Since, the resulting system of linear-quadratic optimal control problems, the necessary and sufficient conditions leading to a two point's boundary value problem with non-symmetric linear operator with respect to the usual (classical) bilinear form. Hence, non-classical variational approach is not applicable. So, non-classical variational approach mixing with direct Ritz bases in suitable functional spaces have been developed for solvability of this system. The manipulation to this approach leads to the solution of either linear algebraic equations or unconstrained direct optimization problems. Both direction have been adapted. Illustration to this problem using the physical parameter of have been discussed and solved the approximated solution and their comparisons via. the proposed approach for both directions have been obtained numerically which are showing very high accuracy.

Key words: Robot dynamic system, optimal control problems, non-classical variational approach, direct optimization technique, approximated, solvability

INTRODUCTION

A mathematical model of one-leg of walking robot of two rigid link based on the result (Pannu *et al*., 1996) have been adapted and modified. An experimental system information and configuration were shown by Pannu *et al*. (1996) and Hoifodt (2011). The analysis and design of the linearized system about the critical point using µ-synthesis control for this system was presented by Pannu *et al*. (1996). The stabilizing control for the walking robot use only one leg of the system while the remaining leg follows a command for locomotion where shown by Hoifodt (2011) and Pannu *et al*. (1996). Many research about robotic system and its modeling as well as solvability, stabilization, controllability and optimality can be found by Al-Shuka *et al*. (2014), Campos-Macias *et al*. (2017) and Khusainov *et al*. (2017). A variational formulation to every linear system of equation by modified the classical bilinear forms with a freedom of choice was given by Magri (1974). This direction may be called the invers problem of calculus of variation. In this study, we have mixed this approach with some kinds of basis, for example, Ritz basis of completely continuous functions in a suitable spaces, so that, the solution is transform from non-direct approach to direct one. The non-classical variational approach is developed in a suitable function space regardless of non-symmetry of the governorate linear operator. This approach have been developed for a lot of applications such as integral integro differential equations, partial differential equations, oxygen diffusion in biological tissues, moving boundary value problems with non-uniform initial-boundary conditionanddescriptor system (Jawad, 2007; Makky and Radhi, 1999).

MATERIALS AND METHODS

Mathematical model (robotic problem): The following Mathematical model is developed and adapted the derivation of this can be found in Appendix A. Hence, the dynamic equations of motion in the absence of any fractional forces are:

$$M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) = \tau \qquad (1)$$

Where:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix},\ V = \begin{bmatrix} V_{11}+V_{12} \\ V_{12}+V_{22} \end{bmatrix},\ G = \begin{bmatrix} G_{11} \\ G_{12} \end{bmatrix} \text{ and } \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

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M is assumed to positive definite, (moment of inertia properties), then M^{-1} exist and the description of the variables are given in appendix B.

State-space model, the linearization and the assumption state-space: To completely determines the behavior of the system for any time, a state space representation is defined as follows:

Let:

$$\begin{matrix} p_1 = \theta_1 \\ p_2 = \theta_2 \\ p_3 = \dot{\theta}_1 \\ p_4 = \dot{\theta}_2 \end{matrix} \Rightarrow \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p_3 \\ p_4 \\ M(s)^{-1}(-V(s)-G(s)+\tau) \end{pmatrix} \triangleq \begin{pmatrix} f_1(p_1, p_2, p_3, p_4) \\ f_2(p_1, p_2, p_3, p_4) \\ F(p_1, p_2, p_3, p_4) \end{pmatrix}, F = \begin{pmatrix} f_3(p_1, p_2, p_3, p_4, \tau) \\ f_4(p_1, p_2, p_3, p_4, \tau) \end{pmatrix} \quad (2)$$

Where:

$$\dot{p}^T = (\dot{p}_3, \dot{p}_4),\ s = (p_1, p_2, p = (p_3, p_4)) \in R^4$$

$$\text{and } M^{-1} = \begin{bmatrix} \frac{M_{22}}{\Delta} & \frac{-M_{12}}{\Delta} \\ \frac{-M_{12}}{\Delta} & \frac{M_{11}}{\Delta} \end{bmatrix}$$

Where:

$$\Delta = M_{11}M_{22}-M_{12}^2$$

Assumptions 1; Should be posted on) such that)...0: The equilibrium points of Eq. 2 are then the solutions of the following nonlinear algebraic equations, we have that:

$$\begin{aligned} &\varphi_1(p_1, p_2) \triangleq \cos(p_1+p_2) + \\ &\frac{(m_1L_{c1y}+m_2L_1)\cos(p_1)+(m_1L_{c1x})\cos(p_1)}{m_2L_{c2}} = 0 \\ &\varphi_2(p_1, p_2, T) = (m_2gL_{c2})\cos(p_1+p_2)-T = 0 \\ &p_3 = 0 \text{ and } p_4 = 0 \end{aligned} \quad (3)$$

Implicit function method and Newton-Raphson approach

Implicit function method: In this study, the solution p_2 as a function of p_1 is found by using the implicit function theorem and Newton-Raphson approach. The necessary condition for solvability of the nonlinear algebraic equation as a function of p_1 are found as:

Let:

$$q = (p_2, T; p^*),\ \text{set } \det\left(\frac{\partial\bar{\varphi}}{\partial q}\right) \text{at} (p_2, T; p^*) \neq 0$$

Where:

$$\bar{\varphi}(p_2, T; p^*) = \begin{pmatrix} \varphi_1(p_2, T; p^*) \\ \varphi_2(p_2, T; p^*) \end{pmatrix} = \left(\cos(p_1+p_2) + \frac{(M_1L_{c1y}+M_2L_1)\cos(p_1)+(M_1L_{c1x})\cos(p_1)}{M_2L_{c2}(M_2gL_{c2})\cos(p_1+p_2)-T} \right)$$

Let:

$$J \triangleq \det\left(\frac{\partial\bar{\varphi}}{\partial q}\right) \text{at} (p_2, T; p^*) \neq 0 \Rightarrow$$

$$J \triangleq \begin{vmatrix} \frac{d\varphi_1}{dp_2} & \frac{d\varphi_1}{dT} \\ \frac{d\varphi_2}{dp_2} & \frac{d\varphi_2}{dT} \end{vmatrix}_{(p_2, T; p^*)} \neq 0 \Rightarrow \begin{vmatrix} -\sin(p_1+p_2) & 0 \\ -(M_2gL_{c2})\sin(p_1+p_2) & 1 \end{vmatrix} \neq 0 \Rightarrow -\sin(p_1+p_2) \neq 0 \quad (4)$$

One can set the second assumption (second assumption):

$$p_1+p_2 \in \begin{cases} (-\pi, 0), (-2\pi, -\pi), \ldots \\ (0, \pi), (\pi, 2\pi), \ldots \end{cases} \quad (5)$$

Third assumption is found to be:

$$\left| \frac{m_2L_{c2}}{((m_1L_{c1y}+m_2L_1)+(m_1L_{c1x}))} \right| < 1 \quad (6)$$

The fourth assumption is optional based in the nature of control constraint. Since:

$$T_{eq}^0 = (M_2gL_{c2})\cos(p_1+p_2) \quad (7)$$

p_1^0 is given such that $(p^*_1+p_2)$, 0, B) another choice is also, possible. The restriction on the magnitude of ||T|| may also be given by the following, if one interested in special class of control (Bang-Bong piecewise-constant control). Since:

$$\begin{aligned} &\|T_{eq}^0\| < \|M_2gL_{c2}\|\|\cos(p_1+p_2)\| \triangleq \|T_{eq}^0\| < \|M_2gL_{c2}\| - \\ &M_2gL_{c2} < \|T_{eq}^0\| < M_2gL_{c2} \Rightarrow T_{eq}^0 \in (-M_2gL_{c2}, M_2gL_{c2}) \end{aligned} \quad (8)$$

To define the class of equilibrium points, as:

$$EQ = \left\{ \begin{aligned} &(p_1, p_2, p_3, p_4, T_{eq}) \in \\ &R^5 \left| \begin{aligned} &p_3 = p_4 = 0,\ p_1+p_2 \in (0, \pi) \text{ with 1st,} \\ &\text{2nd and 3d assumption with} \\ &(\text{4th result if needed}) \text{ are satisfy} \end{aligned} \right. \end{aligned} \right\} \quad (9)$$

Table 1: Phyical parameter of the one leg of two rigid link system

Parameters units	L_{c1y}(m)	L_{c1x}(m)	L_{c2}(m)	L_1(m)	m_1(kg)	m_2(kg)	I_1(kgm^2)	I_2(kgm^2)	I_m(kg m^2)	n	g
The value	0.298	0.008	0.304	0.508	17.007	8.174	0.559	0.390	0.0020	60	9.81

Physical parameter of one-leg of walking robot of two rigid links control dynamical system have been adopted as given by Pannu *et al*. (1996). Based on the define of EQ in Eq. 9 and the following physical parameter of the one leg of two rigid link system (Table 1) the critical point can be found by using Newton-method.

Newton method for finding the critical points: Based on the result of Yang *et al*. (2005) the following is modified to obtain a generalized formula for solving problem Eq. 1:

$$\bar{\varphi}\left(p_2,T;\ p_1^*\right)= 0 \Rightarrow \begin{matrix}\varphi_1\left(p_2,T;\ p_1^*\right)=0\\ \varphi_2\left(p_2,T;\ p_1^*\right)=0\end{matrix} \text{ with } p_3=p_4=0 \tag{10}$$

On using the Taylor series expansion up to first-order about some estimate point (p^*_1, p^k_2, T^K), EQ, since J...0 one can guarantee that there is only one root to the nonlinear algebraic system for given points $^*_1$, (0, B) and T^0_{eq}, (-M_2gL_{c2}, M_2gL_{c2}):

$$\begin{bmatrix}\varphi_1\left(p_2,T;\ p_1^*\right)\\ \varphi_2\left(p_2,\ T;\ p_1^*\right)\end{bmatrix} \cong \begin{bmatrix}\varphi_1\left(p_2^k,\ T^k;\ p_1^*\right)\\ \varphi_2\left(p_2^k,\ T^k;\ p_1^*\right)\end{bmatrix} + \begin{bmatrix}\dfrac{\partial\varphi_1}{\partial p_2} & \dfrac{\partial\varphi_1}{\partial T}\\ \dfrac{\partial\varphi_2}{\partial p_2} & \dfrac{\partial\varphi_2}{\partial T}\end{bmatrix}_{\left(p_2^k,T^k;p_1^*\right)} \begin{bmatrix}p_2-p_2^k\\ T-T^K\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \tag{11}$$

using some manipulations one can obtain:

$$\begin{bmatrix}p_2^{k+1}\\ T^{K+1}\end{bmatrix} = \begin{bmatrix}p_2^k\\ T^K\end{bmatrix} - \begin{bmatrix}\dfrac{-1}{\sin\left(p_2^*+p_2^*\right)} & 0\\ -(m_2gL_{c2}) & 1\end{bmatrix}\begin{bmatrix}\varphi_1\left(p_2^k,\ T^k;\ p_1^*\right)\\ \varphi_2\left(p_2^k,\ T^k;\ p_1^*\right)\end{bmatrix} \tag{12}$$

Hence, the critical point of the system is (p^k_2, T^k, p_3, p_4; p^*_1) k^* where a suitable number of iteration designed is based on some accuracy criterion. A modified Newton-Raphson method is then adapted to solve the nonlinear-algebraic Eq. 12 for a given the initial points which are selected such that:

$$0<p_1+p_2<\pi,\ \Delta \neq 0$$

$$\text{and}\left|\frac{m_2L_{c2}}{\left(\left(m_1L_{c1y}+m_2L_1\right)+\left(m_1L_{c1x}\right)\right)}\right|<$$

$$1\left\|T_{eq}^k\right\|<\left\|M_2gL_{c2}\right\|$$

Linearization: Once the class of equilibrium points in EQ is obtained, it is then necessary to approximate the nonlinear dynamic control system by linearization scheme about some point belonging to EQ. Given the non-linear state-space system control (Eq. 2) and the equilibrium point p^* = [p^*_1 = 2$^*_1$, p^*_2= 2$^*_2$, p^*_3 = 0, p^*_4 = 0] from E Q and u^* = T^*_{eq}:

$$x \triangleq \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}p_1-p_1^*\\p_2-p_2^*\\p_3-p_3^*\\p_4-p_4^*\end{bmatrix} \text{ and } u \triangleq T_{eq}-T_{eq}^*$$

The linearized state-space model of nonlinear control system (Eq. 2) becomes:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \dfrac{h_{22}g_{11}-h_{12}g_{12}}{\Delta} & \dfrac{h_{22}g_{12}-h_{12}g_{12}}{\Delta} & 0 & 0\\ \dfrac{h_{11}g_{12}-h_{12}g_{11}}{\Delta} & \dfrac{h_{11}g_{12}-h_{12}g_{12}}{\Delta} & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_3\end{bmatrix} + \begin{bmatrix}0\\0\\ \dfrac{-h_{12}}{\Delta}\\ \dfrac{-h_{11}}{\Delta}\end{bmatrix}u+\text{height order terms} \tag{13}$$

Which can approximate by:

$$\dot{x}=Ax+Bu,\ y=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \text{ with } x\left(t_0\right)= \tag{14}$$

$$x_0 \in R^4,\ (\text{A and B are given in Eq. 13})$$

Where:

Table 2: Critical points in radian

Parameters	p^*_1 (given)	p^*_2	T^*_{eq}	p^*_3	p^*_4
Root1	1.501	0.3355	-6.4012	0	0
Root2	1.4839	0.4198	-7.9660	0	0
Root3	1.466	0.5096	-9.6014	0	0
Root4	1.4485	0.5996	-11.1973	0	0
!	!	!	!	!	!

(The critical points is found for given p^*_1 when p_3, p_4) where the result is obtained for some n 0 N of iteration up to some accuracy 0 = 10^{-6}

$$h_{11} = I_1+I_2+m_1\left(\left(L_{c1x}\right)^2+\left(L_{c1y}\right)^2\right)+$$
$$m_2\left(\left(L_1\right)^2+\left(L_1\right)^2+2\left(L_1\right)\left(L_{c2}\right)\cos\left(p_2-p_2^*\right)\right)$$
$$h_{22} = I_2+n^2I_m+m_2\left(L_{c2}\right)^2$$
$$h_{21} = I_2+m_2\left(\left(L_{c2}\right)^2+L_1\left(L_{c2}\right)\cos\left(p_2-p_2^*\right)\right)$$
$$h_{12} = I_2+m_2\left(\left(L_{c2}\right)^2+L_1\left(L_{c2}\right)\cos\left(p_2-p_2^*\right)\right)$$
$$-M_2g\left[-L_1\sin\left(p_1-p_1^*\right)+L_{c2}\cos\left(\left(p_1-p_1^*\right)+\left(p_2-p_2^*\right)\right)\right]$$
$$g_{12} = -M_2g\left[L_{c2}\sin\left(\left(p_1-p_1^*\right)+\left(p_2-p_2^*\right)\right)\right]$$

and $\Delta = h_{11}h_{22}-h_{12}^2$, $\Delta \neq 0$ form (4.2.2)

All are evaluated at p^*, T^*_{eq} from Table 2. Since, we are interesting to transfer the system from an arbitrary initial state to the origin while minimizing some performance measure, the controllability may be interpreted as necessary and (sufficient) condition for the existence of the solution. The system (Eq. 14) is locally controllability about the critical point if and only if:

$$\begin{bmatrix} \left.\frac{\partial f}{\partial u}\right|_{p^*,\,T^*_{eq}}, \left.\frac{\partial f}{\partial x}\right|_{p^*,\,T^*_{eq}} \left.\frac{\partial f}{\partial u}\right|_{p^*,\,T^*_{eq}}, \left.\left(\frac{\partial f}{\partial x}\right)^2\right|_{p^*,\,T^*_{eq}} \\ \left.\frac{\partial f}{\partial u}\right|_{p^*,\,T^*_{eq}}, \left.\left(\frac{\partial f}{\partial x}\right)^3\right|_{p^*,\,T^*_{eq}} \left.\frac{\partial f}{\partial u}\right|_{p^*,\,T^*_{eq}} \end{bmatrix} = 4$$

where, $f(f_1,\ f_2,\ f_3,\ f_4)^T$ hence, the following is need for optimality points view.

Assumption 5:

$$\begin{bmatrix} 0 & \frac{-h_{12}}{\Delta} & 0 & \frac{\left(h_{12}\left(h_{12}g_{12}-h_{22}g_{11}\right)\right)-\left(h_{11}\left(h_{12}*g_{12}-h_{22}g_{12}\right)\right)}{\Delta^2} \\ 0 & \frac{h_{11}}{\Delta} & 0 & \frac{\left(h_{11}\left(h_{11}g_{12}-h_{12}g_{12}\right)\right)-\left(h_{12}\left(h_{11}g_{12}-h_{12}g_{11}\right)\right)}{\Delta^2} \\ \frac{-h_{12}}{\Delta}, & 0 & ,\frac{\left(h_{12}\left(h_{12}g_{12}-h_{22}g_{11}\right)\right)-\left(h_{11}\left(h_{12}*g_{12}-h_{22}g_{12}\right)\right)}{\Delta^2}, & 0 \\ \frac{h_{11}}{\Delta} & 0 & \frac{\left(h_{11}\left(h_{11}g_{12}-h_{12}g_{12}\right)\right)-\left(h_{12}\left(h_{11}g_{12}-h_{12}g_{11}\right)\right)}{\Delta^2} & 0 \end{bmatrix} \text{has rank 4, for } p^*,\ T^*_{eq} \in EQ$$

Optimal control of linear quadratic: The first aim is to minimize velocity and position of the linearized state space systemandits applied torque with energy consumption. Hence, the optimal control problem is formulated as a quadratic optimization with the performance measure J(u) of the form:

$$J(u) = \frac{1}{2}x^T\left(t_f\right)S_f x\left(x_f\right)+\frac{1}{2}\int_{to}^{tf} L(x,\ u)dt \qquad (15)$$

and the Lagrangian:

$$L\left(x,\ u\right) = x^T\left(t\right)Qx\left(t\right)+u^T\left(t\right)Ru\left(t\right) =$$
$$\left(x^T,\ u^T\right)\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\left(x,\ u\right)$$

With the following requirements; The approximate motion is given by system (Eq. 14). The optimal control aim is to transfer the arbitrary initial state to the zero as state quickly as possible. The control variable u is weighted with a given positive definite matrix $R = R^T>0$ (u^T (t) R (t) u (t) (which guarantees smoothness of operation and x is weighted with a given positive semi definite matrix $Q = Q^T>0$ as well as $S_f = S^t_f>0$.

From the requirements 1-3 and the objective function (Eq. 15) there exist optimal control solution (x, u); x is response corresponding to smooth controller (Lee and Markus, 1967). Hence, the necessary and sufficient condition of optimality are derived by using Euler-Lagrangian equations as follows:

$$J\left(u\right) = \frac{1}{2}x^T\left(t_f\right)S_f x\left(t_f\right)+$$
$$\frac{1}{2}\int_{to}^{tf}\left[L\left(x,\ u\right)+\lambda^T\left(t\right)\left(A\left(t\right)x\left(t\right)+B\left(t\right)u\left(t\right)\right)\right]dt$$

Define a scalar function H (the Hamiltonian) from (Eq. 15 and 14) as follows:

$$H = \frac{1}{2}\left[x^T(t)Q(t)x(t)+u^T(t)R(t)u(t)\right]+ \lambda^T(t)\left[A(t)x(t)+B(t)u(t)\right] \quad (16)$$

Where:

$$\dot{\lambda}^T = \frac{\partial H}{\partial x}, \dot{\lambda}^T - \frac{\partial H}{\partial x} \triangleq \dot{\lambda} = -\frac{\partial\left[x^T(t)Q(t)x(t)+u^T(t)R(t)u(t)\right]^T}{\partial x} - \frac{\partial\left[A(t)x(t)+B(t)u(t)\right]^T}{\partial x} \text{ and} \quad (17)$$

$$\lambda(t_f) = S_f x(t_f), t_f \text{ is fixed}$$

Since, u is uisassumed to be unbounded smooth controller (without using the forth assumptions), the necessary condition for optimality becomes:

$$0 = \frac{\partial H}{\partial u} \Rightarrow 0 = Ru+B^T\lambda \rightarrow u = -R^{-1}B^T\lambda \quad (18)$$

hence,

$$\dot{x} = Ax(t)+Bu(t) \quad (19)$$

$$\dot{\lambda} = -Qx(t)-A^T?(t) \quad (20)$$

$$x(t_0) = x_0 \in R^4 \text{ and } \lambda(t_f) = S_f x(t_f), 0<t_0<t_f$$

t_f given. (t_f may be arbitrary point of interval). Hence, the two-point boundary-value problem is obtained as:

$$\begin{bmatrix}\dot{x}\\ \dot{\lambda}\end{bmatrix} = \begin{bmatrix}A & -BR^{-1}B3v^T\\ -Q & -A\end{bmatrix}\begin{bmatrix}x\\ \lambda\end{bmatrix}$$

Where:

$$A_{4\times4}, B_{4\times1}, Q_{4\times4}, R_{1\times1}, x = [x_1, x_2, x_3, x_4]$$

and

$$\lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4]$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\\ \dot{\lambda}_1(t)\\ \dot{\lambda}_2(t)\\ \dot{\lambda}_3(t)\\ \dot{\lambda}_4(t)\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ \frac{h_{22}g_{11}-h_{12}g_{12}}{\Delta} & \frac{h_{22}g_{12}-h_{12}g_{12}}{\Delta} & 0 & 0 & 0 & 0 & \frac{-(h_{12})^2}{\Delta} & \frac{(h_{12}h_{11})}{\Delta}\\ \frac{h_{11}g_{12}-h_{12}g_{11}}{\Delta} & \frac{h_{11}g_{12}-h_{12}g_{12}}{\Delta} & 0 & 0 & 0 & 0 & \frac{(h_{11}h_{12})}{\Delta} & \frac{(-h_{11})^2}{\Delta}\\ -q_{11} & 0 & 0 & 0 & 0 & 0 & -\frac{h_{22}g_{11}-h_{12}g_{12}}{\Delta} & -\frac{h_{22}g_{12}-h_{12}g_{12}}{\Delta}\\ 0 & -q_{22} & 0 & 0 & 0 & 0 & \frac{h_{11}g_{12}-h_{12}g_{11}}{\Delta} & \frac{h_{11}g_{12}-h_{12}g_{12}}{\Delta}\\ 0 & 0 & -q_{33} & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -q_{44} & 0 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\\ \lambda_1(t)\\ \lambda_2(t)\\ \lambda_3(t)\\ \lambda_4(t)\end{bmatrix} \quad (21)$$

$$x(t_0)\text{given}, \lambda(t_f) \triangleq S_f x(t_f), q_{ii}, i = 1, 2, 3, 4 \text{ are the main dignal elemat of Q} \quad (22)$$

The aim is than to solve this problem by using non-classical variational approach to obtain an approximate solution of the original optimal control problem.

RESULTS AND DISCUSSION

Two boundary value problem solution by non-classical variational approach: The difficulty of finding compact form solution to general two-boundary value problem with a non-symmetric linear (differential) operator d/dt with respect to the classical inner product bilinear form $\langle v_1, v_2\rangle = \int v_1 v_2 \, dt$ have led to formulate a non-classical variational approach to this problem, so that, the solution is equivalent to the critical point of some variational function under some necessary condition (Dyer and McReynolds, 1970). Consider the two-boundary value problem (Eq. 21), define the linear operator L as follow:

$$L(w) \triangleq L(x_1, x_2, x_3, x_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

$$\triangleq \left[\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}, \frac{d\lambda_1}{dt}, \frac{d\lambda_2}{dt}, \frac{d\lambda_3}{dt}, \frac{d\lambda_4}{dt}\right)^T - Aw\right] \quad (23)$$

Where:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{h_{22}g_{11}\text{-}h_{12}g_{12}}{\Delta} & \frac{h_{22}g_{12}\text{-}h_{12}g_{12}}{\Delta} & 0 & 0 & 0 & 0 & \frac{-(h_{12})^2}{\Delta} & \frac{(h_{12}h_{11})}{\Delta} \\ \frac{h_{11}g_{12}\text{-}h_{12}g_{11}}{\Delta} & \frac{h_{11}g_{12}\text{-}h_{12}g_{12}}{\Delta} & 0 & 0 & 0 & 0 & \frac{(h_{11}h_{12})}{\Delta} & \frac{(-h_{11})^2}{\Delta} \\ -q_{11} & 0 & 0 & 0 & 0 & 0 & -\frac{h_{22}g_{11}\text{-}h_{12}g_{12}}{\Delta} & -\frac{h_{22}g_{12}\text{-}h_{12}g_{12}}{\Delta} \\ 0 & -q_{22} & 0 & 0 & 0 & 0 & \frac{h_{11}g_{12}\text{-}h_{12}g_{11}}{\Delta} & \frac{h_{11}g_{12}\text{-}h_{12}g_{12}}{\Delta} \\ 0 & 0 & -q_{33} & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_{44} & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\text{Domain}(L) = \left\{ \begin{matrix} w = (x, \lambda) \in R^4 \times R^4 \mid x(t_0) = x_0 \in R^4,\ \lambda(t_f) = x_{tf}\, S_{tf} \in R^4,\ (x(.), \lambda(.)) \in C^1[t_0, t_f] \times C^1[t_0, t_f] \\ \text{when } \Delta \neq 0 \text{ and } p^*,\ T^*_{eq} \in EQ \end{matrix} \right\} \subset H \quad (24)$$

And the range of linear operator define as:

$$\begin{matrix} \text{Range (L)} \subset V = C^0(t_0, t_f) \subset H \\ (\text{H is a suitable Hilbert space}) \end{matrix} \quad (25)$$

Assumption 6: H is a suitable Hilbert space (may be define as C^1 [0, T] with max inner product law). Set F [w] = 1/2<Lw, w>-<f, w> defined on the domain Domain (L). The bilinear form $<w_1, w_2>$ is assumed to be non-degenerate on H, i.e.; If for every w_1, H, $\langle w_1, \overline{w}_2 \rangle = 0$ $w_2 = 0$, If for every w_2 , H, $\langle \overline{w}_1, w_2 \rangle = 0$ then $w_1 = 0$. If the linear operator L: Domain (L) dH6Range (L) d H is symmetric with respect to the chosen bilinear form $<w_1, w_2>$ i.e., $<Lw_1, w_2> = <w_1, Lw_2>$, hence, define F [w] $\triangleq$ 1/2 <Lw, w>-<f, w> otherwise in H,one can choose the symmetric product (w_1, w_2) as bilinear form on H, therefor $<w_1, w_2> \triangleq (w_1, Lw_2)Y(Lw_1, Lw_2) = (Lw_2, Lw_1) \triangleq <w_2, Lw_1>$, (symmtrics).

Remarks: All critical points of F [w]be a solution of Eq. 21 when the bilinear form $<w_1, w_2>$ be symmetric on the Range (L) of the given linear operator L (Magri, 1974; Reiss and Haug, 1978). The linear operate L is positive definite, ensures that the solution of Eq. 23 is the minimum part of F [w] (Reiss and Haug, 1978). Since, the operator L is define by Eq. 15 and due to the present of d/dt, L is not symmetric, linear operator with the usual bilinear from $<w_1, w_2>$ hence, $<w_1, w_2>$ is redefine as:

$$\langle w_1, w_2 \rangle \triangleq (w_1, Lw_2) \quad (26)$$

For given symmetric inner product bilinear form. One can suppose that the range of given the linear operator L (Range (L) ,H) be dense in the linear space V, i.e., Range (L) = Vd H, for approximation point of view. Due the present differential operator, an integral bilinear form is the best suggesting as $<w_1, w_2> = \int_{to}^{tf} w_1(t)\, w_2(t)\, dt$ which clear that L is not symmetric because of d/dt operator appearing in the L operator. Therefor:

$$F[w] = \frac{1}{2} <Lw, w> \triangleq (Lw, Lw) = \frac{1}{2}\int_{to}^{tf} Lw(t)(Lw(t))^T dt$$

$$F[w] = \frac{1}{2}\int_{t_0}^{t_f} \left[\left[\begin{pmatrix} \frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}, \\ \frac{d\lambda_1}{dt}, \frac{d\lambda_2}{dt}, \frac{d\lambda_3}{dt}, \frac{d\lambda_4}{dt} \end{pmatrix}^T \text{-Aw}\right]\right] \left[\left[\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}, \frac{d\lambda_1}{dt}, \frac{d\lambda_2}{dt}, \frac{d\lambda_3}{dt}, \frac{d\lambda_4}{dt}\right)^T \text{-Aw}\right]\right]^T \quad (27)$$

Theorem (5.1): Consider the nonlinear robotic system (Eq. 1) the following are assumed:

A_1: The state space represented by nonlinear transformation Eq. 2.

A_2: The class of critical point EQ is given by Eq. 9.

A_3: The linearization is found by linearized the nonlinear state space system (Eq. 2) about the critical point $(p^*_1, p^*_2, p^*_3, T^*)$, EQ when the EQ is the class of equilibrium define in Eq. 9. The approximate equation of motion then is found by Eq. 14.

A_4: The optimal preform index (cost function) is define by Eq. 15 which defines the optimal control problem together with the initial and boundary condition, the necessary and sufficient conditions for optimality leads to the two point boundary value problem Eq. 21 and 22.

A_5: The nonlinear control system is locally controllable.

Then the approximate solution to the original optimal control problem (Eq. 1) with the assumptions (A1-A5) is the critical point of the following functional Eq. 21 and vise-versa .Where the bilinear form $\langle w_1, w_2\rangle \triangleq \int_{to}^{tf} Lw_1(t)\,(Lw_2(t))^T\,dt$ is symmetric and non-degenerate bilinear form.

Proof: The proof is easy to be derived by using the step of [A1-A5] and the direction proof of (Magri, 1974; Zaboon and Abd, 2015).

Application approach (robotic problem): Based on the result of theorem (5.1) with a suitable Hilbert space (may be separable Hilbert space, for optimization point of view if the selected bilinear form is positive definite one) if the following are assumed.

$$x_i(t) = a_0^i(t)+\sum_{j=1}^{Ni} a_j^i G_j^i(t);\ i = 1, 2, 3, 4 \qquad (28)$$

$$G_j^i(t) \text{ is linearly independent function with } G_j^i(t_0) = 0 \qquad (29)$$

$$a_0^i(t) \text{ is found such that of } x^i(t_0) = x_0^i \Leftrightarrow a_0^i(t_0) = x_0^i \qquad (30)$$

$$\lambda_k(t) = b_0^k(t_f)+\sum_{s=1}^{Mk} b_s^k H_s^k(t);\ k = 1, 2, 3, 4 \qquad (31)$$

$$H_s^k(t) \text{ is linearly independent function with } H_s^k(t_f) = 0 \qquad (32)$$

$$b_0^k(t_f) \text{ is found such that of } \lambda^i(t_f) = x_{tf}^i S_{tf}^i \Leftrightarrow b_0^k(t_f) = x_{tf}^i S_{tf}^i \qquad (33)$$

The functional F [x, 8] becomes function of the variable $\bar{a}, \bar{b}$ where $\bar{a} = (a^1_j, a^2_j, a^3_j, a^4_j)$ where j = 1, 2, ..., N_i and $\bar{b} = (b^1_s, b^2_s, b^3_s, b^4_s)^T$ where s = 1, 2, ..., M_k:

$$F[\bar{a}, \bar{b}] = \frac{1}{2}\langle Lw, w\rangle - f, w \triangleq \frac{1}{2}\langle L(x,\lambda), (x,\lambda)\rangle \triangleq \frac{1}{2}\Big(L(\bar{a},\bar{b}), L(\bar{a},\bar{b})\Big)\text{form} =$$

$$\frac{1}{2}\int_{to}^{tf}\left[\begin{aligned}
&\Big(\sum_{j=1}^{Ni} a_j^1 G_{jt}^1(t)\Big) - \Big(a_0^3(t)+\sum_{j=1}^{Ni} a_j^3 G_j^3(t)\Big)\Big)^2 + \Big(\sum_{j=1}^{Ni} a_j^2 G_{jt}^2(t)\Big) - \Big(a_0^4(t)+\sum_{j=1}^{Ni} a_j^4 G_j^4(t)\Big)\Big)^2 + \\
&\Big(\sum_{j=1}^{Ni} a_j^3 G_{jt}^3(t)\Big) - \Big(\frac{h_{22}g_{11}-h_{12}g_{12}}{\Delta}\Big) a_0^1(t)+\sum_{j=1}^{Ni} a_j^1 G_j^1(t) - \Big(\frac{h_{22}g_{12}-h_{12}g_{12}}{\Delta}\Big)\Big(a_0^2(t)\sum_{j=1}^{Ni} a_j^2 G_j^2(t)\Big) - \\
&\Big(\frac{-(h_{12}\times h_{12})}{\Delta}\Big)\Big(b_0^3(t)+\sum_{s=1}^{Mk} b_s^3 H_s^3(t)\Big) - \Big(\frac{(h_{12}\times h_{11})}{\Delta}\Big)\Big(b_0^4(t)+\sum_{s=1}^{Mk} b_s^4 H_s^4(t)\Big)\Big)^2 + \Big(\sum_{j=1}^{Ni} a_j^4 G_j^4(t) - \\
&\Big(\frac{h_{11}g_{12}-h_{12}g_{11}}{\Delta}\Big)\Big(a_0^1(t)+\sum_{j=1}^{Ni} a_j^1 G_j^1(t)\Big) - \Big(\frac{h_{11}g_{12}-h_{12}g_{12}}{\Delta}\Big)\Big(a_0^2(t)+\sum_{j=1}^{Ni} a_j^2 G_j^2(t)\Big) - \Big(\frac{(-h_{11}\times h_{12})}{\Delta}\Big) \\
&\Big(b_0^3(t)+\sum_{s=1}^{Mk} b_s^3 H_s^3(t)\Big) - \Big(\frac{(-h_{11}\times h_{11})}{\Delta}\Big)\Big(b_0^4(t)+\sum_{s=1}^{Mk} b_s^4 H_s^4(t)\Big)\Big)^2 + \left(\begin{aligned}&\sum_{s=1}^{Mk} b_s^1 H_{st}^1(t) + \\ &q_{11}\Big(a_0^1(t)+\sum_{j=1}^{Ni} a_j^1 G_j^1(t)\end{aligned}\right) + \\
&\frac{h_{22}g_{11}-h_{12}g_{12}}{\Delta}\Big(b_0^3(t)+\sum_{s=1}^{Mk} b_s^3 H_s^3(t)\Big) + \Big(\frac{h_{22}g_{12}-h_{12}g_{12}}{\Delta}\Big)\Big(b_0^4(t)+\sum_{s=1}^{Mk} b_s^4 H_s^4(t)\Big)\Big)^2 + \\
&\Big(\sum_{s=1}^{Mk} b_s^2 H_{st}^2(t) + q_{22}\Big(a_0^2(t)+\sum_{j=1}^{Ni} a_j^2 G_j^2(t)\Big) + \frac{h_{11}g_{12}-h_{12}g_{11}}{\Delta}\Big(b_0^3(t)+\sum_{s=1}^{Mk} b_s^3 H_s^3(t)\Big) + \\
&\frac{h_{11}g_{12}-h_{12}g_{12}}{\Delta}\Big(b_0^4(t)+\sum_{s=1}^{Mk} b_s^4 H_s^4(t)\Big)\Big)^2 + \Big(\sum_{s=1}^{Mk} b_s^3 H_{st}^3(t) + q_{33}\Big(a_0^3(t)+\sum_{j=1}^{Ni} a_j^3 G_j^3(t)\Big) + 3 \\
&\Big(b_0^1(t)+\sum_{s=1}^{Mk} b_s^1 H_s^1(t)\Big)\Big)^2 + \Big(\sum_{s=1}^{Mk} b_s^4 H_{st}^4(t) + q_{44}\Big(a_0^4(t)+\sum_{j=1}^{Ni} a_j^4 G_j^4(t)\Big) + \left(\begin{aligned}&b_0^2(t) + \\ &\sum_{s=1}^{Mk} b_s^2 H_s^2(t)\Big)\end{aligned}\right)^2
\end{aligned}\right] \qquad (34)$$

Hence, the critical pointe of this function is then equivalents:

$$\frac{\partial F}{\partial a_j^1} = 0,\ \frac{\partial F}{\partial a_j^2} = 0,\ \frac{\partial F}{\partial a_j^3} = 0,\ \frac{\partial F}{\partial a_j^4} = 0,$$

$$\frac{\partial F}{\partial b_s^1} = 0,\ \frac{\partial F}{\partial b_s^2} = 0,\ \frac{\partial F}{\partial b_s^3} = 0 \text{and} \frac{\partial F}{\partial b_s^4} = 0$$

By choosing a set of linear independent function G^i_j (t) with condition Eq. 29 and 30 and H^1_s with the condition Eq. 31 and 32. And N_i M_k number of selected base. On simple calculate, a linear algebraic solvable system obtained as:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \text{ where } Z_{11} = \left[a^1_j, a^2_j, a^3_j, a^4_j\right]^T \text{ and } Z_{21} = \left[b^1_s, b^2_s, b^3_s, b^4_s\right]^T$$

To clarifying these selection one can see the details in the following illustration. Another direction is possible, section (Eq. 8).

Numerical Illustration (robotic problem): Consider the mathematical model of one-leg of walking robot of two rigid link, as discussed in section one where the physical parts are given in Table 1 on using the linearization schema (Eq. 14) with selected the first critical points. (The critical point from Table 1).
Hence:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15.1821 & -2.4011 & 0 & 0 \\ 1.4333 & -2.1461 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.2542 \\ -0.83882 \end{bmatrix} u$$

Define the optimization criterion (section 5.1) as follow:

$$\min_u J = \int_{to=0}^{tf=1} \left(x^T Q x + u^T R u\right) dt, \text{where } Q = \begin{bmatrix} q_{11} & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 \\ 0 & 0 & q_{33} & 0 \\ 0 & 0 & 0 & q_{44} \end{bmatrix}, R>0,\ x \in R^4$$

where, $q_{ii} = 3$, i = 1, 2, 3, 4. R = 10, $S_{tf} = 0$, $t_0 = 0$ and $t_f = 1$. We are interesting for the solution on the period 0#2#B from Eq. 16-21 we have the following two point boundary-value problem:

$$\begin{bmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \\ x'_4(t) \\ \lambda'_1(t) \\ \lambda'_2(t) \\ \lambda'_3(t) \\ \lambda'_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -15.1821 & -2.4011 & 0 & 0 & 0 & 0 & -0.0065 & 0.0213 \\ 1.4333 & -2.1461 & 0 & 0 & 0 & 0 & 0.0213 & -0.0703 \\ -3 & 0 & 0 & 0 & 0 & 0 & 15.18211 & 1.4333 \\ 0 & -3 & 0 & 0 & 0 & 0 & 2.4011 & 2.1461 \\ 0 & 0 & -3 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \quad (35)$$

$$x_1(0) = 1,\ x_2(0) = -2,\ x_3(0) = 2,\ x_4(0) = 3 x_1(1) = 0,\ x_2(1) = 0,\ x_3(1) = 0,\ x_4(1) = 0 \text{ and } \lambda(t_f) = x_{tf} S_{tf} = 0 \rightarrow \lambda_i(1) = 0,\ i = 1, 2, 3, 4 \quad (36)$$

Which is equivalent to:

$$F[w] = \frac{1}{2}\int_{to}^{tf} \left[\begin{array}{l} \left(\frac{dx_1}{dt} - x_3\right)^2 + \left(\frac{dx_2}{dt} - x_4\right)^2 + \left(\frac{dx_3}{dt}(-15.1828)x_1 - (-2.4011) x_2 - (-0.0065)\lambda_3 - (0.0213)\lambda_4\right)^2 + \\ \left(\frac{dx_4}{dt}(1.4333)x_1 - (-2.1461)x_2 - (0.0213)\lambda_3 - (-0.0703)\lambda_4\right)^2 + \left(\frac{d\lambda_1}{dt} + 3x_1(15.1821)\lambda_3 + (-1.4333)\lambda_4\right)^2 + \\ \left(\frac{d\lambda_2}{dt} + 3x_2 + 2.4011\lambda_3 + 2.1461\lambda_4\right)^2 + \left(\frac{dx_3}{dt} + 3x_3 + \lambda_1\right)^2 + \left(\frac{dx_4}{dt} + 3x_4 + \lambda_2\right)^2 \end{array} \right] dt \quad (37)$$

The basic functions that satisfying the initial and terminal condition be assumed as follows:

$$
\begin{aligned}
&x_1(t) = 1-\left(\frac{t}{t_f}\right)+a_0\left(t^2-tt_f\right)+a_1\left(t^3-t\left(t_f\right)^2\right),\ x_2(t) = -2+\left(\frac{2t}{t_f}\right)+a_2\left(t^2-tt_f\right)+a_3\left(t^3-t\left(t_f\right)^2\right)\\
&x_3(t) = 2-\left(\frac{2t}{t_f}\right)+a_4\left(t^2-tt_f\right)+a_5\left(t^3-t\left(t_f\right)^2\right),\ x_4(t) = 3-\left(\frac{3t}{t_f}\right)+a_6\left(t^2-tt_f\right)+a_7\left(t^3-t\left(t_f\right)^2\right.\\
&\lambda_1(t) = b_0\left(t-t_f\right)+b_1\left(t^2-\left(t_f\right)^2\right),\ \lambda_2(t) = b_2(t-t_f)+b_3(t^2-(t_f)^2)\\
&\lambda_3(t) = b_4\left(t-t_f\right)+b_5\left(t^2-\left(t_f\right)^2\right),\ \lambda_4(t) = b_6(t-t_f)+b_7(t^2-(t_f)^2
\end{aligned}
\tag{38}
$$

Since:

$$\frac{\partial F}{\partial a_j} = 0 \text{ where } j = 0, 1, \ldots, 7 \text{ and } \frac{\partial F}{\partial b_s} = 0, \tag{39}$$

where, $j = 0, 1, \ldots, 7$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \tag{40}$$

Where:

$$A_{11} = \begin{bmatrix}
8.385 & 12.578 & 0 & 0.2697 & 1.1126 & 1.6689 & 0 & -0.0239\\
12.578 & 19.204 & 0.2697 & 0 & 1.6689 & 2.5431 & 0.0239 & 0\\
0 & -0.277 & 0.6667 & 1 & 0 & -0.0400 & 0 & 0\\
0.27 & 0 & 1 & 1.5619 & 0.040 & 0 & 0 & 0\\
1.113 & 1.669 & 0 & 0.0400 & 0.9790 & 1.4686 & 0 & 0.0524\\
1.669 & 2.543 & -0.040 & 0 & 1.4686 & 2.2759 & -0.0524 & 0\\
0 & 0.024 & 0 & 0 & 0 & -0.0524 & 0.6667 & 1\\
-0.024 & 0 & 0 & 0 & 0.0524 & 0 & 1 & 1.5619
\end{bmatrix}
A_{12} = \begin{bmatrix}
-0.5 & -0.5 & 0 & 0 & -3.7848 & -5.2987 & 0.3230 & 0.4522\\
-0.75 & -0.8 & 0 & 0 & -5.2987 & -7.5695 & 0.4522 & 0.6460\\
0.25 & 0.35 & 0 & 0 & -0.4989 & -0.4989 & -0.0036 & -0.0036\\
0.35 & 0.5 & 0 & 0 & -0.7484 & -0.7983 & -0.0053 & -0.0057\\
0 & 0 & -0.5 & -0.5 & -0.6028 & -0.8439 & -0.5282 & -0.7395\\
0 & 0 & -0.75 & -0.8 & -0.8439 & -1.2056 & -0.7395 & -1.0564\\
0 & 0 & 0.25 & 0.35 & -0.0036 & -0.0036 & -0.4883 & -0.4883\\
0 & 0 & 0.35 & 0.5 & -0.0053 & -0.0057 & -0.7324 & -0.7813
\end{bmatrix}$$

$$A_{21} = \begin{bmatrix}
-0.5 & -0.75 & 0.25 & 0.35 & 0 & 0 & 0 & 0\\
-0.5 & -0.8 & 0.35 & 0.5 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -0.5 & -0.75 & 0.25 & 0.35\\
0 & 0 & 0 & 0 & -0.5 & -0.8 & 0.35 & 0.5\\
-3.785 & -5.2987 & -0.4989 & -0.7484 & -0.6028 & -0.8439 & -0.0036 & -0.0053\\
-5.299 & -7.5695 & -0.4989 & -0.7983 & -0.8439 & -1.2056 & -0.0036 & -0.0057\\
0.3230 & 0.4522 & -0.0036 & -0.0053 & -0.5282 & -0.7395 & -0.4883 & -0.7324\\
0.4522 & 0.6460 & -0.0036 & -0.0057 & -0.7395 & -1.0564 & -0.4883 & -0.7813
\end{bmatrix}
A_{22} = \begin{bmatrix}
1.3333 & 1.4167 & 0 & 0 & 7.0911 & 9.7881 & -0.7167 & -0.9555\\
1.4167 & 1.8667 & 0 & 0 & 4.3940 & 7.0911 & -0.4778 & -0.7167\\
0 & 0 & 1.3333 & 1.4167 & 1.2006 & 1.6007 & 0.5730 & 1.0974\\
0 & 0 & 1.4167 & 1.8667 & 0.8004 & 1.2006 & 0.0487 & 0.5730\\
7.0911 & 4.3940 & 1.2006 & 0.8004 & 7.9754 & 9.94425 & -5.5364 & -6.9205\\
9.7881 & 7.0911 & 1.6007 & 1.2006 & 9.94425 & 12.7339 & -6.9205 & -8.8582\\
-0.7167 & -0.4778 & 0.5730 & 0.0487 & -5.5364 & -6.9205 & 3.2218 & 3.7773\\
-0.9555 & -0.7167 & 1.0974 & 0.5730 & -6.9205 & -8.8582 & 3.7773 & 4.8883
\end{bmatrix}$$

$$Z_{11} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}^T,\ Z_{21} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \end{bmatrix}^T,$$

$$B_{11} = \begin{bmatrix} 9.8889 & 13.3767 & 3.56333 & 5.1783 \\ -2.8204 & -4.1660 & 1.2124 & 1.5686 \end{bmatrix}^T \text{ and } B_{21} = \begin{bmatrix} 0.5 & 1.5 & 6 & 5.75 & -13.2913 \\ -14.8620 & 0.9335 & 3.7848 & & \end{bmatrix}^T$$

Since:

A_{11}, A_{12}, A_{21} and A_{22}(m x m)

$$\det\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12}) =$$

$0.51032 \neq 0$ then

Solving the above system by algebraic equations:

$$\begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$

Hence and the approximate solution to the two point boundary value problem Eq. 35. And have locally to (Eq. 1). Table 3 and 4 numerical rustle of state vector, co state vector and optimal control first system of N.C.V linear algebraic (system) J(u) = 12.411.

The direction two: Since, the linear operator is positive definite with respect to bilinear form:

$$<w_1, w_2> \triangleq \left(w_1, Lw_2\right) \Rightarrow \left(Lw_1, Lw_2\right) > 0$$

$$Lw,\ w = \left(Lw, Lw\right) = \int_{to}^{tf} Lw\left(Lw\right)^T dt³ \geq 0$$

Then by Reiss and Haug (1978) the solution is equivalent to:

$$\min_{\bar{a},\bar{b}\in R^{n\times m}} F\left[\bar{a}, \bar{b}\right] = \frac{1}{2}\int_{to}^{tf}\left[\left[\begin{pmatrix} \frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}, \\ \frac{d\lambda_1}{dt}, \frac{d\lambda_2}{dt}, \frac{d\lambda_3}{dt}, \frac{d\lambda_4}{dt} \end{pmatrix}^T \text{-Aw}\right.\right.$$

$$\left[\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}, \frac{d\lambda_1}{dt}, \frac{d\lambda_2}{dt}, \frac{d\lambda_3}{dt}, \frac{d\lambda_4}{dt}\right)^T \text{- Aw}\right]$$

$$\left.\left[\left[\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}, \frac{d\lambda_1}{dt}, \frac{d\lambda_2}{dt}, \frac{d\lambda_3}{dt}, \frac{d\lambda_4}{dt}\right)^T \text{-Aw}\right]\right]^T\right.$$

Table 3: Nmerical Illustration (robotic problem)

Parameters	p^*_1 (given)	p^k_2	T_{eq}	p_3	p_4		
Critical point1	1.501	0.3355	-6.4012	0	0		
α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
0.9786	-3.4572	3.1952	0.1998	35.0545	-19.7437	-1.6427	1.8667
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
-22.8456	10.9089	-11.5163	6.1609	0.3452	1.3123	-0.9799	1.9086

Table 4: Numerical rustle of state vector

	State vector				Co-state				Optimal control u
Time	x_1 (t)	x_2 (t)	x_3 (t)	x_4 (t)	λ_1 (t)	λ_2 (t)	λ_3 (t)	λ_4 (t)	$u = -R^{-1}B^T\lambda$
0	1	-2	2	3	-2.46280	-0.691500	0.330400	-0.294000	0.0491
0.1	1.202710	-1.450100	1.153318	3.040745	-2.02360	-0.219910	0.243315	-0.141670	0.0268
0.2	1.346573	-0.891870	0.568221	2.942512	1.62726	0.162256	0.168240	-0.016660	0.0084
0.3	1.428431	-0.462800	0.200505	2.729024	-1.27380	0.454986	0.105175	0.081039	0.0062
0.4	1.445126	-0.120010	0.005966	2.424000	0.96322	0.658284	0.054120	0.151416	0.0169
0.5	1.393500	0.134250	-0.059600	2.051163	-0.69550	0.772150	0.015075	0.194475	0.0238
0.6	1.203940	0.297774	-0.040400	1.634232	0.47066	0.796584	-0.011960	0.210216	0.0267
0.7	1.072649	0.368665	0.019367	1.196930	0.28868	0.731586	-0.026980	0.198639	0.0259
0.8	0.797107	0.343709	0.075491	0.762976	0.14958	0.577156	-0.030000	0.159744	0.0211
0.9	0.440610	0.221672	0.083770	0.356093	0.05336	0.333294	-0.021000	0.093531	0.0125
1.0	0	0	0	0	0	0	0	0	0

Table 5: Approximate solution of $F[\alpha\cdot, b\cdot]$

α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
8.8752	-3.2141	2.9952	0.2756	34.1545	-17.9537	-1.6427	1.8667
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
-21.9456	10.8764	-11.5163	5.9509	0.3352	1.3123	-0.9799	1.9086

Table 6: Numerical rustle of state vector, co state vector and optimal control first system of NCV by using Hooks and Jeeves (direct optimization technique)

	State vector				Co-state				Optimal control u
Time	x_1	x_2	x_3	x_4	λ_1	λ_2	λ_3	λ_4	$u = -R^{-1}B^T\lambda$
0	1	-2	2	3	0	-1.5224	-5.3692	0.829200	4.337200
0.1	1.174810	-1.414010	1.148818	3.040745	0.1	-1.16978	-4.02444	0.691821	3.782538
0.2	1.295373	-0.907870	0.560221	2.942512	0.2	-0.86168	-2.8592	0.566544	3.254752
0.3	1.359131	-0.483800	0.190005	2.729024	0.3	-0.59812	-1.87348	0.453369	2.753842
0.4	1.363526	-0.144010	-0.006030	2.424000	0.4	-0.37908	-1.06728	0.352296	2.279808
0.5	1.306000	0.109250	-0.072100	2.051163	0.5	-0.20458	-0.4406	0.263325	1.832650
0.6	1.183994	0.273774	-0.052400	1.634232	0.6	-0.0746	0.00656	0.186456	1.412368
0.7	0.994949	0.347335	0.008867	1.196930	0.7	0.010845	0.2742	0.121689	1.018962
0.8	0.736307	0.327709	0.067491	0.762976	0.8	0.05176	0.36232	0.069024	0.652432
0.9	0.405510	0.212672	0.079270	0.356093	0.9	0.048145	0.27092	0.028461	0.312778
1.0	0	0	0	0	1	0	0	0	0

Table 7: The compression between the solution of both direction

Time	x_1 NCVS	x_1 NCVH&J	Error	x_2 NCVS	x_2 NCVH&J	Error	x_3 NCVS	x_3 NCVH&J	Error	x_4 NCVS	x_4 NCVH&J	Error
0	1	1	0	-2	-2	0	2	2	0	3	3	0
0.1	1.202710	1.174810	0.0279	-1.405010	-1.414010	0.009	1.153318	1.148818	0.0045	3.040745	3.040745	0
0.2	1.346573	1.295373	0.0512	-0.891870	-0.907870	0.016	0.568221	0.560221	0.0080	2.942512	2.942512	0
0.3	1.428431	1.359131	0.0693	-0.462800	-0.483800	0.021	0.200505	0.190005	0.0105	2.729024	2.729024	0
0.4	1.445126	1.363526	0.0816	-0.120010	-0.144010	0.024	0.005966	-0.006030	0.0120	2.424000	2.424000	0
0.5	1.393500	1.306000	0.0875	0.134250	0.109250	0.025	-0.059600	-0.072100	0.0125	2.051163	2.051163	0
0.6	1.270394	1.183994	0.0864	0.297774	0.273774	0.024	-0.040400	-0.052400	0.0120	1.634232	1.634232	0
0.7	1.072649	0.994949	0.0777	0.368335	0.347335	0.021	0.019367	0.008867	0.0105	1.196930	1.196930	0
0.8	0.797107	0.736307	0.0608	0.343709	0.327709	0.016	0.075491	0.067491	0.0080	0.762976	0.762976	0
0.9	0.440610	0.405510	0.0351	0.221672	0.212672	0.009	0.083770	0.079270	0.0045	0.356093	0.356093	0
1.0	0	0	0	0	0	0	0	0	0	0	0	0

By using the same procedure above the problem is transferred into an optimization method when a suitable basis function have been used to approximated the solution (Eq. 28-30).

Since, the problem of quadratic optimization, on using some direct optimization method, the solution may be found. Hooke and Jeeves method (Kirgat and Surde, 2014) of direct search optimality technique have been adapted to find the approximate solution of $F[\bar{a}, \bar{b}]$ Eq. 33.

Table 5-7 numerical Rustle of state vector, co state vector and optimal control first system of NCV by using

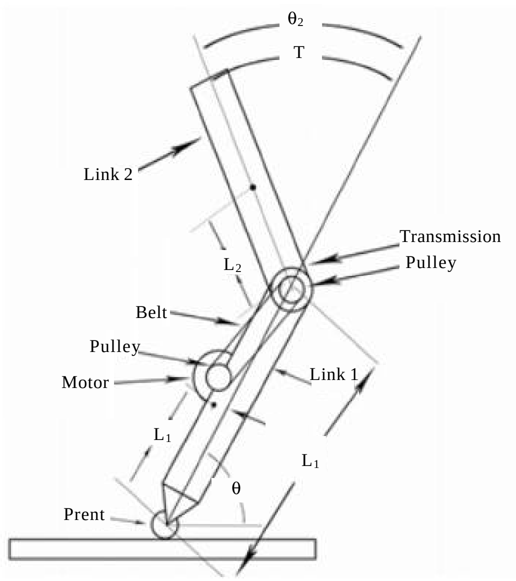

Fig. 1: schematic model of the under actuated leg

Hooks-and Jeeves (direct ptimization technique) J(u) = 11.0825. The compression between the solutions of both direction are below.

The model of one-leg of walking robot of two rigid links control connected by dynamic control tem system based on Pannu *et al*. (1996) and Anderson *et al*. (2010) is adapted as follows in Fig. 1. The basic lows for the derivation of this model may be derived using the following steps:

Step 1: Consider the position of the centers of mass as a function of the generalized coordinates:

- L_{c1y}: distance to center of mass of link 1 along the center line
- L_{c1x}: distance to center of mass of link 1 orthogonal the center line
- L_{c2}: distance to center of mass of link 2 along the center line
- L_1: lengh of link 1
- M_1: mass of link 1
- M_2: mass of link 2
- I_1: is the moment of inertia of link 1 about center of mass
- I_2: is the moment of inertia of link 2 about center of mass
- I_m: is the moment of inertia of trnsmission of pulley and belt
- n: Tranmission reduction ratio
- H: 2×2 inertia matrex
- V: 2×1 corolis vector
- G: 2×1 gravity vector
- h: 2×2 linearized inertia matrix at the operating point
- g: 2×2 linearized gravity vector at the operating point
- 2_1: Angle of link 1 relative to horizontal (+CCW)
- 2_2: Angle of link 2relative to link (+CCW)
- p_1: Equilibrime angle for link 1 relative to horizontal
- P_2: Equilibrime angle for link 1 relative to link 1
- T: Torque provided for tranmission (+CCW)
- T_{eq}: equilibriom Torque
- J: 2×1 input torque vector

$$x_1 = \left(L_{c1x} - L_{c1y}\right)\cos(\theta_1), \quad y_1 = \left(L_{c1x} - L_{c1y}\right)\sin(\theta_1)$$
$$x_2 = L_1 \cos\theta_1 + L_{c2}\cos(\theta_1+\theta_1), \quad y_2 = L_1 \sin\theta_1 + L_{c2}\sin(\theta_1+\theta_1)$$

Step 2: Based on the result of step 1, the following is obtained:

$$\dot{x}_1 = -\left(L_{c1x} - L_{c1y}\right)\sin(\theta_1)\dot{\theta}_1, \quad \dot{y}_1 = \left(L_{c1x} - L_{c1y}\right)\cos(\theta_1)\dot{\theta}_1$$
$$\dot{x}_2 = -L_1\sin(\theta_1)\dot{\theta}_1 - L_{c2}\sin(\theta_1+\theta_2)\dot{\theta}_1 - L_{c2}\sin(\theta_1+\theta_2)\dot{\theta}_2$$
$$\dot{y}_2 = L_1\cos(\theta_1)\dot{\theta}_1 - L_{c2}\cos(\theta_1+\theta_2)\dot{\theta}_1 - L_{c2}\cos(\theta_1+\theta_2)\dot{\theta}_2$$

Step 3: omputing the kinematic energy:

$$K\left(\theta, \dot{\theta}\right) = \frac{1}{2}\left(m_1\left(\dot{x}_1^2+\dot{y}_1^2\right) + m_2\left(\dot{x}_2^2+\dot{y}_2^2\right) + I_1\dot{\theta}_1^2 + I_2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2\right.$$

Where:

$$\left(\dot{x}_1^2+\dot{y}_1^2\right) \triangleq \left[\left(L_{c1x}^2+L_{c1y}^2\right)\sin^2(\theta_1)\dot{\theta}_1^2 + \left(L_{c1x}^2+L_{c1y}^2\right)\cos^2(\theta_1)\dot{\theta}_1^2\right] = \left[\left(L_{c1x}^2+L_{c1y}^2\right)\dot{\theta}_1^2\left(\sin^2(\theta_1)+\cos^2(\theta_1)\right)\right] = \left[\left(L_{c1x}^2+L_{c1y}^2\right)\dot{\theta}_1^2\right]$$

Similarly:

$$\left(\dot{x}_2^2+\dot{y}_2^2\right) = \begin{pmatrix} \begin{pmatrix} -L_1\sin(\theta_1)\dot{\vartheta}_1 - L_{c2}\sin(\theta_1+\theta_2)\dot{\theta}_1 - \\ L_{c2}\sin(\theta_1+\theta_2)\dot{\theta}_2 \end{pmatrix}^2 + \\ \begin{pmatrix} L_1\cos(\theta_1)\dot{\theta}_1 - L_{c2}\cos(\theta_1+\theta_2)\dot{\theta}_1 - \\ L_{c2}\cos(\theta_1+\theta_2)\dot{\theta}_2 \end{pmatrix}^2 \end{pmatrix} =$$
$$\left[\left(L_1^2\right)\dot{\theta}_1^2 + \left(L_{c2}^2\right)\left(\theta_1+\theta_2\right)^2 + 2L_1L_{C2}\left(\dot{\theta}_1^2+\dot{\theta}_1\dot{\theta}_2\right)\cos(\theta_2)\right]$$

Hence, the kinematic energy:

$$K\left(\theta, \dot{\theta}\right) = \frac{1}{2}\begin{pmatrix} m_1\left(L_{c1x}^2+L_{c1y}^2\right) + m_2L_1^2 + m_2L_{c2} + \\ 2m_2L_1L_{c2}\cos(\theta_2) + I_1 + I_2 \end{pmatrix}\dot{\theta}_1^2 +$$
$$\frac{1}{2}\left(m_2L_{c2}^2 + I_2n^2I_m\right)\dot{\theta}_2^2 + \left(m_2L_{c2}^2 + 2m_2L_1L_{c2}\cos(\theta_2) + I_2\right)\dot{\theta}_1\dot{\theta}_2$$

Step 4 : The potential energy of this system:

$$P(\theta) = m_1gy_1+m_2gy_2 \triangleq m_1g(L_{c1x}-L_{c1y})\cos(\theta_1)+ m_2g(L_1\sin(\theta_1)+L_{c2}\cos(\theta_1+\theta_2)$$

Step 5: Based on Euler-Lagrange equations of motion (one can Eq. 28), the mathematical model is then found as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta}\right)-\frac{dL}{d\theta} = t$$

Where:

$$L = K\text{-}P,\ \theta = (\theta_1,\ \theta_2) \text{and } \tau = (\tau_1,\ \tau_2)$$

Then:

$$M_{11} = I_1+I_2+m_1\left((L_{c1x})^2+(L_{c1y})^2\right)+ m_2\left((L_1)^2+(L_{c2})^2+2(L_1)(L_{c2})\cos(\theta_2)\right)$$

M_{11} is different from that of Eq. 23 which has erroneous in this term:

$$M_{12} = I_2+m_2\left((L_{c2})^2+L_1(L_{c2})\cos(\theta_2)\right)$$
$$M_{22} = I_2+n^2I_m+m_2(L_{c2})^2, V_{11} = -2m_2L_1L_{c2}\sin(\theta_2)\dot{\theta}_1$$
$$V_{12} = -m_2L_1(L_{c2})\sin(\theta_2)\dot{\theta}_2, V_{21} = m_2L_1(L_{c2})\sin(\theta_2)\dot{\theta}_1$$
$$V_{22} = 0;\ G_1 = m_1g\left((L_{c1}y)\cos(\theta_1)+(L_{c1x})\sin(\theta_1)\right)+ m_2g\left((L_1)\cos(\theta_1)+(L_{c2})\cos(\theta_1+\theta_2)\right)$$
$$G_2 = m_2g(L_{c2})\cos(\theta_1+\theta_2),\ \tau_1 = 0;\ \tau_2 = T$$

In this study, the rigid body mechanics robot manipulator motion of one-leg, two links (based on Rao (2009) is formulated with the help of Lagrangian mechanics:

$$M(\theta)\ddot{\theta}+V(\theta,\ \dot{\theta})+G(\theta) = \tau$$

Where:

2, R^2	=	The position coordinates
$\dot{\theta}$ and $\ddot{\theta}$	=	Standing for associated velocities and accelerations
J, R	=	The driving forces (control optimality)
M(2) = M^T(2)>0	=	The (generalized) moment of inertia
$V(\theta, \dot{\theta})\dot{\theta}$	=	The Coriolis, centripetal and frictional forces
G(2)	=	The gravitational forces

all vary along the trajectories.

CONCLUSION

The purpose of this study is extend the previous stud of Magri (1974) and Jawad (2007) and their applicability to the two boundary value problem with non-symmetric linear operator defined a suitable H space which are the resulting of the necessary and sufficient condition of optimality of nonlinear robotic control problem.

RECOMMENDATIONS

As one can see the numerical solution using proposed approach (with both directions) from. Both direction are efficient and the second direction gives J(u) = 11.0825<J(u) = 12.411.

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