Chapter Tow

"Methods to solve O.D.E. of the first order "

Introduction

In this chapter we will studied a solution for O.D.E. of the first order.

The general form of O.D.E. of the first order and degree is: $M(x,y) + N(x,y)dydx = 0$

or

$M(x,y)dx + N(x,y)dy = 0$

or

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} = f(x,y)$$

Ex.

1−$(y-x)+x^2\frac{dy}{dx}=0$

2−$(y - x) + xy\frac{dy}{dx} = 0$

3−$(x^2y + 2x)dx + (3x - cosx)dy = 0$

4−$\frac{dy}{dx} = yx + 2sinx$

Remark

1- There is no rule to solve all O.D.E.

2- A derivative $\frac{dy}{dx}$ of a D.E. y=y(x) gives slopes of tangent lines at points on its graph.

We will be solving the following equations:

1- Equations with separated and separable variables

2- Equations of homo. type

3- Equations with linear coefficients

4- Exact D.E.-integrating factors

5- Linear D.E.-Bernoulli's Equation

1-Equations with separated and separable variables
Definition

A first order O.D.E. of the form:

$$\frac{dy}{dx} = f_1(x)f_2(y)$$

Is said to be separable or to have separable variables

.Ex

1- $\frac{dy}{dx} = y^2xe^{3x+4y}$ Separable variables

2- $\frac{dy}{dx} = y + sinx\ not$ Separable variables

Ex:

Solve the following equations.

1- $(y+1)dx+y2(x-1)dy=0$

Solution

$$x^2(y+1)dx + y^2(x-1)dy = 0] \div (y+1)(x-1)$$

$$\frac{x^2-1+1}{x-1}dx + \frac{y^2-1+1}{y+1}dy = 0$$

$$\frac{x^2}{x-1}dx + \frac{y^2}{y+1}dy = 0$$

$$\frac{(x-1)(x+1)+1}{x-1}dx + \frac{(y-1)(y+1)+1}{y+1}dy = 0$$

$$\left[x+1+\frac{1}{x-1}\right]dx + [y-1+\frac{1}{y+1}]dy = 0$$

$$\frac{x^2}{2} + x + \ln(x-1) + \frac{y^2}{2} - y + \ln(y+1) + c_1 = 0$$

$$x^2 + 2x + 2\ln(x-1) + y^2 - 2y + 2\ln(y+1) + 2c_1 = 0$$

$$x^2 + y^2 + 2x - 2y + 2\ln(x-1)(y+1) + c = 0$$

2- $xydy + (2x^2-1)(y+2)dx = 0$

Solution

$$\frac{y}{y+2}dy + \frac{2x^2-1}{x}dx = 0$$

$$\frac{y+2-2}{y+2}dy + [2x - \frac{1}{x}]dx = 0$$

$$[1 - \frac{2}{y+2}]dy + [2x - \frac{1}{x}]dx = 0$$

$$\int [1 - \frac{2}{y+2}]dy + \int [2x - \frac{1}{x}]dx = 0$$

$y - 2ln(y+2) + x^2 - lnx + c = 0$

3- $xdx + ydy = 0$

Solution

$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2}$

$x^2 + y^2 = c_1$(Circles with center at the origin)

4- $sin\theta cos\varphi d\theta - cos\theta sin\varphi d\varphi = 0$

Solution

$\frac{sin\theta}{cos\theta} d\theta - \frac{sin\varphi}{cos\varphi} d\varphi = 0$

$-\ln|cos\theta| + \ln|cos\varphi| = \ln c$

$ln \frac{cos\varphi}{cos\theta} = lnc$

$\therefore \frac{cos\varphi}{cos\theta} = c$

$cos\varphi = c\ cos\theta$

5 − $\frac{dy}{dx} = -xy\ , y(4) = -3$

Solution

$ydy = -xdx$

$\frac{y^2}{2} = \frac{x^2}{2} + \frac{c_1}{2}$

$x^2 + y^2 = c_1$

$16 + 9 = c\ \ \rightarrow \therefore\ c = 25$

$x^2 + y^2 = 25$

$y = \mp\sqrt{25} - x^{\,2}$

$y = \varphi 1(x) = \sqrt{25} - x^2$,

$y = \varphi 2(x) = -\sqrt{25} - x^2$

Exercises

Solve the following:

1- $(1 + x)dy - ydx = 0\ (H.W)$

2- $(e^{2y} - y)cosxdydx = e^y sin2x\ , y(0) = 0\ (H.W.)$

2-Equations of homo. type

:Definition

The function f(x,y) is said to be homo. function of degree n if $f(tx, ty) = t^n f(x, y)$

Where t is a function of x or y or constant.

example

Are the functions of homo. Type

1$-\ (\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x^2} + \boldsymbol{3xy} + \boldsymbol{y^2}$

Solution

$$f(tx,ty) = t^2x^2 + 3t^2xy + t^2y^2$$

$$= t^2(x^2 + 3xy + y^2\ )$$

$$= \ t^2 f(x,y)$$

F is homo. of two degree

2− $\boldsymbol{(x,y) = x + x^2y}$

Solution

$(tx,ty) = tx + t^3x^2y\ \neq$

$t^n(x + x^ny)$

F isn't homo.

3− $(x,y) = e^{\frac{y}{x}} + sin(yx)$

Solution

$$f(tx,ty) = e^{ty/tx} + sin(\frac{ty}{tx})$$

$$f(tx,ty) = t^0(e^{y/x} + sin(\frac{y}{x})$$

F is homo. of zero degree

Definition

O.D.E. $M(x,y)dx + N(x,y)dy = 0$ is called O.D.E. of homo. type in x,y if M,N are homo. functions of equal degree.

1-Ex

Are the following equations of homo. Type

$(x^2 + y^2)dx + (x^2y + x^3)dy = 0$

isn't of homo. type

2 − $(2xy + y^2)x + (x + x^2y)dy = 0$

isn't of homo. type

3 − $(2xy + x^2)x + (x^2 + y^2)dy = 0$

of homo. type

Remark

1- let f(x,y) is homo. function of n degree $f(tx, ty) = t^n f(x, y)$,then:−

Let $t = \frac{1}{x}$

$$\therefore f\left(1, \frac{y}{x}\right) = \frac{1}{x^n} f(x, y) \therefore f(x, y) = x^n f\left(1, \frac{y}{x}\right)$$

$\therefore f(x, y) = x^n \varphi(\frac{y}{x}) where\ \varphi$ is a function of $\frac{y}{x}$.

Every function in y/x is homo. of zero degree

2- we can solve O.D.E.of homo. type by the following steps:

1- write the equation by general form

2- let $y = vx$

$$dy = vdx + xdv$$

3- substituted (1) in given equation obtain equation of separable variables in x,v variables solve this equation

4- Substituted $v = \frac{y}{x}$ obtain the solution of given equation in x,y variables

.Ex

1- Find the general solution of I.V.P.

$$(x^2 + y^2)x - 2xydy = 0 \,, y(1) = 1$$

:Solution

$M(x,y) = \ x^2 + y^2$ homo. of $second\ degree$

$N(x,y) = -2xy$ homo. of second degree

$\therefore$ O.D.E. of homo. type

$y = vx \,, dy = vdx + xdv$

$$(x^2 + x^2v^2)dx - 2x^2v(vdx + xdv) = 0$$

$$(x^2 + x^2v^2)dx - (2x^2v^2dx + 2x^3vdv) = 0$$

$$(x^2 + x^2v^2 - 2x^2v^2)dx - (2x^3vdv) = 0$$

$$\left(x^2(1 - v^2)\right)dx - 2x^3vdv = 0] \div \frac{x^3}{1 - v^2}$$

$$\frac{dx}{x} - \frac{2v}{1 - v^2}dv = 0 \,, x \neq 0 \,, v^2 \neq 1$$

By integrating

$$lnx + \ln(1 - v^2) = lnc$$

$$x(1 - v^2) = c$$

$$x\left(1 - \frac{y^2}{x^2}\right) = 0$$

$$x^2 - y^2 = cx$$

$\therefore c = 0 \ \rightarrow x^2 - y^2 = 0 \ the\ solution\ of\ I.V.P.$

2- $\boldsymbol{Solve\ the\ I.V.P.}\ (2xy + y^2)x - 2x2dy = 0\ y(e) = e$

:Solution

$M(x,y) = \ 2xy + y^2$ homo. of second degree

$N(x,y) = -2x^2$ homo. of second degree

$\therefore$ O.D.E. of homo. type

y=vx , dy=vdx+xdv

$(2vx^2 + x^2v^2)dx - 2x^2(vdx + xdv) = 0$

$$(2vx^2 + x^2v^2)dx - (2x^2vdx + 2x^3dv) = 0$$

$$(2vx^2 + x^2v^2 - 2x^2v)dx - (2x^3dv) = 0$$

$$x^2v^2dx - (2x^3dv) = 0]/x^3v^2$$

$\frac{dx}{x} - 2\frac{dv}{v^2} = 0$

By integrating

$$lnx + \frac{2}{v} = c$$

$$lnx + \frac{2x}{y} = c$$

ylnx+2x=cy

y(e)=e

e+2e=ce

c=3

ylnx+2x=3y the solution of given I.V.P.

Exercises

1- Find the general solution of $xdy - ydx = \sqrt{x2 + y^2}dx$

2- Solve the equation $(\boldsymbol{2x - 3y})\boldsymbol{dx} - (\boldsymbol{2y + 3x})\boldsymbol{dy} = \boldsymbol{0}$