Graphical approach

Before seeking any actual solutions, however, it can be noted that the ODE itself contains a lot of information about the nature of its solutions. This is because the equation y ′ = f(x, y) gives the slope of the function y(x). In the plane of x and y it provides the direction in which the solution must be changing at any point (x, y). This is called the direction field of the ODE.

• ***The direction*** field of the ODE y ′ = f(x, y) is the set of all direction vectors having the same direction as the vector (1, y′), at each point (x, y), in the plane of x and y.

• ***Integral curves*** are curves which are everywhere tangent to the direction field. Each integral curve represents a solution of the ODE.

Example

$\mathbf{y'=2x}$.

Solution:-

$$\frac{dy}{dx} = 2x$$

$$dy = 2xdx$$

By integrating; obtain the general solution:

$y = x^2 + c$, c is arbitrary constant

$c=0 \rightarrow y=x^2$

$c=1 \rightarrow y=x^2+1$

$c=2 \rightarrow y=x^2+2$

$c=-1 \rightarrow y=x^2-1$

$c=-2 \rightarrow y=x^2-2$

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. Are particular solution

Value problems

1-Initial value problem (I.V.P.)

The problem of finding a function y of x when we know its derivative and its value y_0 particular point x_0 at a is called an initial value problem. This problem can be solved in two steps.

1.

$$\int dy = \int f(x)\,dx \quad \rightarrow \quad y = F(x) + c \quad \leftarrow \quad \text{general solution}$$

2. Using the initial data, plug it into the general solution and solve for c.

EXAMPLE 1: Solve the initial value problem.

$$\frac{dy}{dx} = 10 - x, \qquad y(0) = -1$$

SOLUTION:

STEP 1:

$$\frac{dy}{dx} = 10 - x \quad \rightarrow \quad dy = (10 - x)\,dx$$

$$\int dy = \int (10 - x)\,dx \quad \rightarrow \quad y = 10x - \frac{x^2}{2} + c$$

STEP 2: When x = 0, y = 1

$$-1 = 10(0) - \frac{0}{2} + c \quad \rightarrow \quad c = -1$$

$$\text{SOLUTION}: \quad y = 10x - \frac{x^2}{2} - 1$$

EXAMPLE 2: Solve the initial value problem.

$$\frac{dy}{dx} = 9x^2 - 4x + 5, \qquad y(-1) = 0$$

SOLUTION:

STEP 1:

$$\frac{dy}{dx} = 9x^2 - 4x + 5 \quad \rightarrow \quad dy = (9x^2 - 4x + 5)\,dx$$

$$\int dy = \int (9x^2 - 4x + 5)\,dx \quad \rightarrow \quad y = \frac{9x^3}{3} - \frac{4x^2}{2} + 5x + c$$

STEP 2: When x = -1, y = 0.

$$0 = 3(-1)^3 - 2(-1)^2 + 5(-1) + c \quad \rightarrow \quad 0 = -3 - 2 - 5 + c \quad \rightarrow \quad c = 10$$

EXAMPLE 3: Solve the initial value problem.

$$\frac{ds}{dt} = \cos t + \sin t, \qquad s(\pi) = 1$$

SOLUTION:

STEP 1:

$$\frac{ds}{dt} = \cos t + \sin t \quad \rightarrow \quad ds = (\cos t + \sin t)\,dt$$

$$\int ds = \int (\cos t + \sin t)\,dt \quad \rightarrow \quad s = \sin t - \cos t + c$$

STEP 2: When t = π , s = 1.

$$1 = \sin \pi - \cos \pi + c \quad \rightarrow \quad 1 = 0 - (-1) + c \quad \rightarrow \quad c = 0$$

$$\text{SOLUTION :} \quad s = \sin t - \cos t$$

O.D.E. with initial conditions it is called initial value problem (I.V.P.)

$F(x,y,y')=0$,$y(x_0)=y_0$ $F(x,y,y',y'')=0$,$y(x_0)=y_0,y'(x_0)=y_1$

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$F(x,y,y',y'',\ldots\ldots,y^n)=0$,$y(xo)=yo$,$y'(x_0)=y_1\ldots,y_{n-1}(x_0)=y^{n-1}$

Ex.1

If $y=x^2+c$ ***is the solution of I.V.P.*** $y'=2x$, $(1)=5$ ***then find c***

Solution

5=1+c

C=4

$\therefore y=x^2+4$ is the solution of given I.V.P.

EX.2

$y=c_1cos4x+c_2sin4x$ ***is a solution of the equation*** $\boldsymbol{y''+16y=0}$. ***find a solution of the I.V.P.***

$y''+16y=0$, $y(\pi\backslash 2)=-2$, $y'(\pi\backslash 2)=1$

Solution

$y(\pi\backslash 2)=-2$

$y=c_1cos2\pi+c_2sin2\pi=-2$

$c1=-2$

$y'(x)=-4c_1sin4x+4c_2cos4x$

$y'(\pi\backslash 2)=1$

$4c_2=1\rightarrow c_2=1\backslash 4$

The solution of I.V.P. is $y=-2cos4x+(1\backslash 4)sin4x$

Exercises

1- Find c in $y = \frac{1}{3}x^3 + c$ *is a solution of I.V.P.* $y' = x^2$, $x = 1$, $y = 2$.

2- Find c in $y = \frac{1}{4}x^4 + c$ *is a solution of I.V.P.* $y' = x^3$, $y(1) = \frac{1}{2}$.

2-boundary value problem (B.V.P.)

A Boundary value problem is **a system of ordinary differential equations with solution and derivative values specified at more than one point**. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem

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Example 1 Solve the following BVP.

$$y'' + 4y = 0 \quad y(0) = -2 \quad y\left(\frac{\pi}{4}\right) = 10$$

Hide Solution ▾

Okay, this is a simple differential equation to solve and so we'll leave it to you to verify that the general solution to this is,

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Now all that we need to do is apply the boundary conditions.

$$-2 = y(0) = c_1$$
$$10 = y\left(\frac{\pi}{4}\right) = c_2$$

The solution is then,

$$y(x) = -2\cos(2x) + 10\sin(2x)$$

When an O.D.E. is to be solved under conditions involving dependent variable and its derivatives at two different values of independent variable then the problem under consideration is said to be a B.V.P.

.Ex

$1 -\ a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \qquad y(a) = y_0,$

$y(b) = y_1$

$2-\ y''+y=0\ ,y(a)=y1\ ,y(b)=y_2$

$3-\ y''+16y=0\ ,y(0)=0\ ,y\left(\frac{\pi}{2}\right)=1$

System of O.D.E

A system of O.D.E. is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.

.Ex

System of O.D.E. of two first order where y, z are dependent variable and x is independent variable:

$\frac{dy}{dx}=f(x,y,z)$

$\frac{dZ}{dx}=g(x,y,z)$

A solution of the system is a pair of functions $y=\varphi_1(x), z=\varphi_2(x)$

Defined on interval I that satisfy each equation of the system on this interval.

Example 1 Write the following 2nd order differential equation as a system of first order, linear differential equations.

$$2y'' - 5y' + y = 0 \quad y(3) = 6 \quad y'(3) = -1$$

Hide Solution ▾

We can write higher order differential equations as a system with a very simple change of variable. We'll start by defining the following two new functions.

$$x_1(t) = y(t)$$
$$x_2(t) = y'(t)$$

Now notice that if we differentiate both sides of these we get,

$$x'_1 = y' = x_2$$
$$x'_2 = y'' = -\frac{1}{2}y + \frac{5}{2}y' = -\frac{1}{2}x_1 + \frac{5}{2}x_2$$

Note the use of the differential equation in the second equation. We can also convert the initial conditions over to the new functions.

$$x_1(3) = y(3) = 6$$
$$x_2(3) = y'(3) = -1$$

Putting all of this together gives the following system of differential equations.

$$x'_1 = x_2 \qquad x_1(3) = 6$$
$$x'_2 = -\frac{1}{2}x_1 + \frac{5}{2}x_2 \qquad x_2(3) = -1$$

Finding O.D.E. from the general solution

We will find O.D.E. from general solution by show the relation between arbitrary constants and order of O.D.E.

.Ex

1- Find O.D.E. from general solution $Y = Ce^x$

$$y' = ce^x \rightarrow y' = y$$

Or $c = \frac{y'}{e^x}$

$$y = \frac{y'}{e^x} e^x$$

$$\therefore y = y'$$

2- **Find O.D.E. from the general solution $y = c_1 x + c_2 x^3$**

$$y' = c_1 + 3c_2x^2$$

$$y'' = 6c_2x \to c_2 = \frac{y''}{6x}$$

$$y' = c_1 + \frac{1}{2}xy''$$

$$c_1 = y' - \frac{1}{2}xy''$$

$$\therefore y = \left(y' - \frac{1}{2}xy''\right)x + \frac{1}{6}x^2y''$$

$$= xy' - \frac{1}{3}x^2y''$$

$$y - xy' + \frac{1}{3}x^2y'' = 0 \quad (O.D.E.)$$

Remark

We can use the determent to finding O.D.E.

Ex.1

Find O.D.E. from the general solution $y=Ax+Bx4$

Solution

$$y = Ax + Bx^4 \quad \rightarrow \; Ax + Bx^4 - y = 0$$

$$y' = A + 4Bx^3 \; \rightarrow A + 4Bx^3 - y' = 0$$

$$y'' = 12Bx^2 \quad \rightarrow 12Bx^2 - y'' = 0$$

$$\begin{bmatrix} x & x^4 & -y \\ 1 & 4x^3 & -y' \\ 0 & 12x^2 & -y'' \end{bmatrix} = 0$$

$$x \begin{bmatrix} 4x^3 & -y' \\ 12x^2 & -y'' \end{bmatrix} - \begin{bmatrix} x^4 & -y \\ 12x^2 & -y'' \end{bmatrix} = 0$$

$$x(-4x^3y'' + 12x^2y') - (-x^4y'' + 12x^2y) = 0$$

$$\text{(O.D.E.)} -3x^4y'' + 12x^3y' - 12x^2y = 0$$

Ex.2

Find O.D.E. from the general solution $y = ce^x$ ***by using determent***

Solution

$$y = ce^x \quad \rightarrow \; ce^x - y = 0$$

$$y' = ce^x \quad \rightarrow ce^x - y' = 0$$

$$\begin{bmatrix} e^x & -y \\ e^x & -y' \end{bmatrix} = 0$$

$$-e^xy' + e^xy = 0$$

$$\therefore -y' + y = 0$$

$$\therefore y' = y \quad (O.D.E.)$$

Ex.3

Find O.D.E. from the general solution $y = c_1 x + c_2 x^2$ $\boldsymbol{by}$ $\boldsymbol{using\ determent}$. ($\boldsymbol{H.W}$.)