Differential operator "D operator"

$D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$, … … … … . . , $D^n = \frac{d^n}{dx^n}$

$D^2 = D.D = \frac{d}{dx}(\frac{d}{dx})$

$Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2}$, ……………….., $D^n y = \frac{d^n y}{dx^n}$

$Df(x) = \frac{df(x)}{dx}$, $D^2 f(x) = \frac{d^2 f(x)}{dx^2}$, ……………….., $D^n f(x) = \frac{d^n f(x)}{dx^n}$

Ex.:-

1- Find $D(cos4x)$

Solution:- $D(\cos 4x) = \frac{d}{dx}(\cos 4x) = -4\sin 4x$

2- Find $D(5x^3 - 6x^2)$

Solution:- $D(5x^3 - 6x^2) = \frac{d}{dx}(5x^3 - 6x^2) = 15x^2 - 12x$

3- Find $D(2x^3 - 3x^2 + 6)$ (H.W)

Remark:-

the general form of non-homo. linear equation of n order is:

$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots \dots \dots \dots \dots . . a_{n-1}(x)\frac{dy}{dx} + a_n y = f(x)$

By using D operator ,obtain:

$D^n y + a_1(x)\, D^{n-1}y + \cdots \dots \dots \dots + a_{n-1}(x)Dy + a_n y = f(x)$

$(D^n + a_1(x)\, D^{n-1} + \cdots \dots \dots \dots + a_{n-1}(x)D + a_n)y = f(x)$

Let $F(D) = D^n + a_1(x)\, D^{n-1} + \cdots \dots \dots \dots + a_{n-1}(x)D + a_n$

It is called polynomial's operator

∴ F(D)y = f(x) non-homo. linear O.D.E.

We can write $F(D) = (D - m_1)(D - m_2) \dots \dots \dots . (D - m_n)$

Where $m_1, m_2 \dots \dots . . m_n$ are factors.

<u>Ex.:-</u>write the following equations by using D operator:-

1- $\boldsymbol{y'' + 5y' + 6y = 5x - 3}$

<u>Solution</u>:-

$D^2y + 5Dy + 6y = 5x - 3$

$(D^2 + 5D + 6)y = 5x - 3$

$F(D) = D^2 + 5D + 6$ Polynomial's operator

Or (D+3)(D+2)y=5x-3

3, 2 are factors

2- $\boldsymbol{y'' - 4y' + 3y = e^x}$

<u>Solutions</u>:-

$D^2y - 4Dy + 3y = e^x$

$(D^2 - 4D + 3)y = e^x$

(D-3)(D-1)y=e^x or (D-1)(D-3)y=e^x

Prove that (D-3)(D-1)y= (D-1)(D-3)y

(D-3)(D-1)y=(D-3)(Dy-y)= $D^2y - Dy - 3Dy + 3y$

$= D^2y - 4Dy + 3y$

(D-1)(D-3)y=(D-1)(Dy-3y)= $D^2y - 3Dy - Dy + 3y$

$= D^2y - 4Dy + 3y$

3- $\boldsymbol{y'' + y' - 12y = sinx}$

<u>Solutions</u>:-

$D^2y + Dy - 12y = sinx$

$(D^2 + D - 12)y = sinx$

(D+4)(D-3)y=$sinx$ or (D-3)(D+4)y=$sinx$

Prove that (D-3)(D+4)y= (D+4)(D-3)y (H.W.)

4-Is$(D+3)(D+2x)y=(D+2x)(D+3)y$?

Solutions:-

$(D+3)(D+2x)y=(D+3)(Dy+2xy)$
$=D^2y+2xDy+2y+3Dy+6xy$

$(D+2x)(D+3)y=(D+2x)(Dy+3y)$
$=D^2y+3Dy+2xDy+6xy$
$\therefore (D+3)(D+2x)y \neq (D+2x)(D+3)$

5- $y''-4y=0$
6- $y''-y=0$
7-Is$(D+x^2)(D-2x)y=(D-2x)(D+x^2)y$?

Properties of D
If f,g are derivative functions and c is constant; then:
1- $D^m f(x)+D^n f(x)=D^n f(x)+D^m f(x)$
2-$D^m f(x).D^n f(x)=D^n f(x).D^m f(x)=D^{n+m} f(x)$
3-D(f+g)=Df+Dg
4-D(cf(x))=cDf(x)

Remark:-
In general ,we define an n-th order diff. operator or polynomial operator to be
L=$a_n(x)D^n+a_{n-1}(x)\,D^{n-1}+\cdots\ldots\ldots\ldots+a_1(x)D+a_o(x)$
∴linear n-th order O.D.E. homo. and non-homo. Respectively is:-
L(y)=0 and L(y)=g(x)
L has a linearity property:-
$L[\alpha f(x)+\beta g(x)]=\alpha L[f(x)]+\beta L[g(x)]$
Where f,g are functions and α,β are constant.

Properties of F(D):
If y is derivative function and b is constant ;then:
1-$F(D)e^x=F(b)\,e^x$
2-$F(D)\{e^{bx}y\}=e^{bx}F(D+b)y$
3-$F(D^2)sinbx=F(-b^2)sinbx$
4-$F(D^2)cosbx=F(-b^2)cosbx$

Ex.:-

1-$(\boldsymbol{D^2-1})\boldsymbol{e^{3x}}$

Solution:-

$(D^2-1)e^{3x}=e^{3x}(3^2-1)=8e^{3x}$

2-$(\boldsymbol{D^2-4D+1})\{\boldsymbol{e^{2x}y}\}$

Solution:-

$F(D)=(D^2-4D+1)\ ,\qquad b=2$

$F(D+b)=F(D+2)=((D+2)^2-4(D+2)+1)$

$=(D^2-3)$

$\therefore (D^2-4D+1)\{e^{2x}y\}=e^{2x}(D^2-3)y$

3-$(\boldsymbol{D^4+3D^2-1})\boldsymbol{sin2x}$

Solution:-

$F(D^2)=(D^4+3D^2-1)\ ,b=2$

$F(-b^2)=F(-4)=((-4)^4+3(-4)-1)=3$

$\therefore (D^4+3D^2-1)sin2x=3sin2x$

$4-(\mathbf{D^4-2D^2})\mathbf{cos2x}$

Solution:-

$F(D^2)=(D^4-2D^2)\ ,b=2$

$F(-b^2)=F(-4)=((-4)^2-2(-4))=24$

$\therefore (D^4-2D^2)cos2x=24cos2x$

Exercises:-

1- prove that:

1-$(\boldsymbol{D+1})(\boldsymbol{D^2+2})\boldsymbol{sin2x}=(\boldsymbol{D^2+2})(\boldsymbol{D+1})\boldsymbol{sin2x}$

2-$(\boldsymbol{D-1})(\boldsymbol{D-2})(\boldsymbol{D-3})\boldsymbol{y}=(\boldsymbol{D-2})(\boldsymbol{D-3})(\boldsymbol{D-1})\boldsymbol{y}$

2-find the following by using the properties of F(D):

1-$(\boldsymbol{D^4+2D^2+1})\boldsymbol{cos3x}$

2-$(\boldsymbol{4D^4-4})\boldsymbol{sinx}$

3-$(\boldsymbol{D^2-5D+6})\boldsymbol{e^xy}$