Advanced Calculus

By Dr. Zaynab Anwer Ahmed and Teacher Eman Abduhadi

Weekly Syllabus:

1- Sequences and Series: Sequences, Real Sequences, Convergence, Cauchy sequence, Monotone and Bounded sequences
2- Infinite Convergence, Infinite Series, Geometric Series, Harmonic series, Converging Test, Alternations Series.
3- Absolute Convergence, Rearrangements of Series, Product Infinites Series,
4- Power Series, Calculus of Power Series Taylors and Maclorian series with applications.
5- Vector Functions in Three Dimensional Spaces: Vectors Functions, Velocity.
6- Acceleration and Arc Length, Curvature, The Laws of Planetary Motion.
7- Mid-term Exam
8- Partial Differentiation: Multiple variables Functions, Limits and Continuity.
9- Partial derivatives, Increments and Differentials of Functions of Several Variables.
10- Chain Rule, The Derivative and the Gradient, Tangent and orthogonal planes on the surface
11- Extreme of a Functions of two Variables, Exact Differentials, Line Integrals, Work.
12- Double Integrals: Double integrals, Iterated integrals, Evaluation of Double integrals by Means of Iterated integrals.
13- Other Applications of Double integrals, Green's theorem, Double integrals in Polar Coordinates.
14- Triple integrals, Application in Rectangular Coordinates.
15- Integrals in cylindrical and spherical coordinates.

1- Sequences and Series

1.1 Sequences

A sequence of real numbers $\{a_n\}$ is a function whose domain is the set of natural numbers and its range is real number. Thus, a sequence is a function $f: \mathbb{N} \rightarrow \mathcal{R}$, $f(n) = a_n$. Usually, this sequence will denote by $\{a_1, a_2, a_3, \dots\}$.

Notes:

1- The sequence may be finite or infinite, and defined by a rules.

2- The index n =1,2,3, ..., refers to the term's number, "n^{th} term" is called an index; The first index N is called the initial index.

Sequence	Defining Rule
$0,1,2,\dots,n-1,\dots$	$a_n = n-1$
$1,\frac{-1}{2},\frac{1}{3},\frac{-1}{4},\dots,(-1)^{n+1}\frac{1}{n},\dots$	$a_n = (-1)^{n+1}\frac{1}{n}$
$0,\frac{-1}{2},\frac{2}{3},\frac{-3}{4},\dots,(-1)^{n+1}(\frac{n-1}{n}),\dots$	$a_n = (-1)^{n+1}(\frac{n-1}{n})$
$1,1,2,3,5,8,13,21,34,55,89,\dots,a_{n-1}+a_{n-2},\dots$	$a_n = a_{n-1}+a_{n-2}$

Example:

For example, the sequence 2, 4, 6, 8, 10, 12,c, 2n,… has

first term f(1)=a1 = 2, second term f(2)= a2 = 4, and $f(n) = n^{th}$ term an = 2n.

Example:

The sequence {1, 1/2, 1/3, 1/4, 1/5,...} is written as {1/n}. In this case f(n)=an = 1/n.

$$\{a_n\} = \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\right\}$$

$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1}\frac{1}{n}, \dots\right\}$$

$$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n-1}{n}, \dots\right\}$$

$$\{d_n\} = \{1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}, \dots\}.$$

Arithmetic Sequence

An arithmetic sequence is a sequence where each term differs from the next by the same, fixed quantity.

Example

An example of an arithmetic progression is the sequence $\{a_n\}$ which begins:

$$a_1 = 10, \quad a_2 = 14, \quad a_3 = 18, \quad a_4 = 22, \quad \ldots$$

and which is given by the rule $a_n = 6 + 4n$. Each term differs from the previous by four.

Alternatively, we could describe an arithmetic progression recursively, by giving a starting value a1, and using the rule that $a_n = a_{n-1} + m$.

Example

Does the number 203 belong to the arithmetic sequence $3, 7, 11, \ldots$?

Solution

Here $a = 3$ and $d = 4$, so $a_n = 3 + (n-1) \times 4 = 4n - 1$. We set $4n - 1 = 203$ and find that $n = 51$. Hence, 203 is the 51st term of the sequence.

Geometric sequences

A geometric sequence is a sequence where the ratio between subsequent terms is the same, fixed quantity.

Example

An example of a geometric progression is the sequence $\{a_n\}$ starting:

$$a_1 = 10, \quad a_2 = 30, \quad a_3 = 90, \quad a_4 = 270, \quad \ldots$$

and given by the rule $a_n = 10 \cdot 3^{n-1}$. Each term is three times the preceding term.

Alternatively, we could describe a geometric progression recursively, by giving a starting value a1, and using the rule that $a_n = r * a_{n-1}$.

Example

Does the number 48 belong to the geometric sequence

3072, 1536, 768, …?

Solution

Here $a = 3072$ and $r = \frac{1}{2}$, so $a_n = 3072 \times \left(\frac{1}{2}\right)^{n-1}$.

We set $3072 \times \left(\frac{1}{2}\right)^{n-1} = 48$. This gives $\left(\frac{1}{2}\right)^{n-1} = \frac{1}{64}$, that is, $2^{n-1} = 64 = 2^6$, and so $n = 7$.

Hence, 48 is the 7th term of the sequence.

1.2 Convergence and Divergence

DEFINITIONS The sequence $\{a_n\}$ **converges** to the number L if for every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \qquad \Rightarrow \qquad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence

Theorem 1

Suppose that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$ and k is some constant.

$$\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n = kL$$

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - M$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = LM$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}, \text{ if } M \text{ is not } 0$$

Theorem 2

Suppose that $a_n \leq b_n \leq c_n$ for all $n > N$, for some N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem 3

$\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$.

Example:

Determine whether $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit. Since this makes sense for real numbers we consider

$$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} 1 - \frac{1}{x+1} = 1 - 0 = 1.$$

Thus the sequence converges to 1.

Example:

Determine whether $\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ converges or diverges. If it converges, compute the limit.

Sol:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0,$$

using L'Hôpital's Rule. Thus the sequence converges to 0.

Example:

Determine whether $\{(-1)^n\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit.

Sol: Every subsequence has limit the same as the limit of the original sequence. The limit of the even indices is 1 so L =1, while the limit of odd indices is -1 so L must be 1. Since L can't be both -1 and 1,. In other words, the limit does not exist.

Example:

Determine whether $a_n = \{\left(-\frac{1}{2}\right)^n\}_{n\,=\,0}^{\infty}$ converges or diverges. If it converges, compute the limit.

Consider the sequence $c_n \;= |a_n|$ and observe that

$$\lim_{x\to\infty}\left(\frac{1}{2}\right)^x = \lim_{x\to\infty}\frac{1}{2^x} = 0,$$

Then the limit will be 0. Then a_n is converge

Example:

A particularly common and useful sequence is $\{r^n\}$, for various values of r. Some are quite easy to understand: If r = 1 the sequence converges to 1 since every term is 1, and likewise if r = 0 the sequence converges to 0. If r = -1 this is the sequence of previous example and diverges. If r > 1 or r < -1 the terms r^n get large without limit, so the sequence diverges. If 0 < r < 1 then the sequence converges to 0. If -1<r <0 then $|r^n| = |r|^n$ and 0 <|r|<1, so the sequence $\{|r|^n\}$ converges to 0, so also $\{r^n\}$ converges to 0. converges. In summary, $\{r^n\}$ converges precisely when -1 <r < 1 in which case

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Example:

Determine whether {(sinn)/√n} converges or diverges. If it converges, compute the limit.

Sol:

$$\frac{-1}{\sqrt{n}} \le \frac{\sin n}{\sqrt{n}} \le \frac{1}{\sqrt{n}},$$

and can therefore apply the Squeeze Theorem. Since $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$, we get

$$\lim_{n\to\infty} \frac{-1}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0,$$

and so by squeezing, we conclude $\lim_{n\to\infty} a_n = 0$.

1.3 Monotonicity

Definition

A sequence is called increasing if an<an+1 for all n. It is called non-decreasing if an≤ an+1 for all n. Similarly a sequence is decreasing if an> an+1 for all n and non-increasing if an ≥ an+1 for all n.

Definition

If a sequence is increasing, non-decreasing, decreasing, or non-increasing, it is said to be monotonic.

Example:

The sequence $a_n = \frac{2^n - 1}{2^n}$ which starts

$$\frac{1}{2}, \quad \frac{3}{4}, \quad \frac{7}{8}, \quad \frac{15}{16}, \quad \ldots,$$

is increasing. On the other hand, the sequence $b_n = \frac{n+1}{n}$, which starts

$$\frac{2}{1}, \quad \frac{3}{2}, \quad \frac{4}{3}, \quad \frac{5}{4}, \quad \ldots,$$

is decreasing.

1.4 Boundedness

Definition

A sequence {an} is bounded above if there is some number M so that for all n, we have an ≤ M. Likewise, a sequence {an} is bounded below if there is some number M so that for every n, we have an ≥ M.

If a sequence is both bounded above and bounded below, the sequence is said to be bounded.

If a sequence {an} is increasing or non-decreasing it is bounded below (by ao). and if it is decreasing or non-increasing it is bounded above (by ao).

Theorem 4

If a sequence is bounded and monotonic then it converges.

Theorem 5

(i) *If* $\{a_n\}$ *is an unbounded monotonically increasing sequence, then* $\lim a_n = +\infty$.

(ii) *If* $\{a_n\}$ *is an unbounded monotonically decreasing sequence, then* $\lim a_n = -\infty$.

Definition 6.6 *A sequence* $\{a_n\}$ *of real numbers is called a* ***Cauchy sequence*** *if for each* $\epsilon > 0$ *there is a number* $N \in \mathbb{N}$ *so that if* $m, n > N$ *then* $|a_n - a_m| < \epsilon$.

Lemma:

Cauchy sequences are bounded.

Theorem :

A sequence is a convergent sequence if and only if it is a Cauchy sequence.