كلية العلوم للبنات

قسسم الرياضيات

المرحلة الرابعة

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المحاضرة الرابعة

Definition 1.23. A sequence (x_n) in a normed space X is a ***Cauchy convergent sequence*** if:

$\forall \varepsilon > 0 \ \exists \ k(\varepsilon) \in Z^+$ such that $\| x_n - x_m \| < \varepsilon \quad \forall n,m > k(\varepsilon)$

Theorem 1.24.: Every convergent sequence is a Cauchy convergent sequence.

proof:

Suppose that (x_n) is a convergent sequence in the normed space X, then $\exists \ x \in X$ s.t. $x_n \to x$

Let $\varepsilon>0$, since $x_n \to x \Rightarrow \exists \ k \in Z^+$ s.t. $\| x_n - x \| < \varepsilon/2 \quad \forall n > k$

If $n,m \geq k$,then $\| x_n - x_m \| = \| (x_n - x)+(x - x_m) \| \leq \| x_n - x \| + \| x - x_m \| < \varepsilon/2+\varepsilon/2=\varepsilon$

Then (x_n) is a Cauchy sequence.

Remark: The converse to above theorem may not be true. For example:

Let X= IR-{0}, $(x_n) = (1/n)$

(x_n) Cauchy convergent sequence in IR

Since IR complete $\Rightarrow (x_n) = (1/n) \to 0$ convergent in IR

But (x_n) not convergent in IR-{0} , since $0 \notin$ IR-{0}.

Definition 1.25.: Let X be a normed space, $x_0 \in X$, a function f is said to be ***continuous*** at x_0 if:

$\forall\ \varepsilon > 0, \exists\ \delta(x_0, \varepsilon) > 0$ s.t. $\| f(x) - f(x_0) \| < \varepsilon$ whenever $\| x - x_0 \| < \delta$.

Theorem 1.26. : Let X, Y be two Normed space, a function $f : X \to Y$ continuous at $x_0 \in X$ iff for each sequence (x_n) in X such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Definition 1.27.: Let X be a normed space, a function $f : X \to IR$ is called ***bounded*** if:

$\exists\ M > 0$ s.t. $\| f(x) \| \leq M, \forall x \in X$.

Definition 1.28.: Let (x_n) be a sequence in a normed space X, say (x_n) is ***bounded sequence*** in X if : $\exists\ M > 0$ s.t. $\| x_n \| \leq M, \forall n \in Z^+$.

Theorem 1.29.: If (x_n) is Cauchy convergent sequence in a normed space X then it is bounded.

proof:

Let (x_n) be a Cauchy sequence in X

Given $\varepsilon=1$, $\exists\, k\in Z^+$ s.t. $\| x_n - x_m \| < 1, \forall n, m > k$.

Let $m = k+1 \Rightarrow \| x_n - x_{k+1} \| < 1$

Since $| \| x_n\| - \| x_{k+1} \| | \leq \| x_n - x_{k+1} \| < 1$

$\Rightarrow | \| x_n\| - \| x_{k+1} \| | < 1 \Rightarrow \| x_n\| < 1 + \| x_{k+1} \| , \forall n>k$

Put $M = \max \{ \| x_1\|, \| x_2\|, \ldots , \| x_k\|, \| x_{k+1}\|\} \Rightarrow \| x_n \| \leq M , \forall n\in Z^+$.

Theorem1.30. : Every convergent sequence in the normed space X is bounded.

proof:

Let (x_n) be a convergent sequence in X $\Rightarrow (x_n)$ a Cauchy convergent sequence in X

$\Rightarrow (x_n)$ bounded .

Definition 1.31. : Let X is a normed space, $x_0 \in X$, $r > 0$, define:

1) $B_r(x_0) = \{ x \in X : \| x - x_0 \| < r \}$ is called ***open ball*** of center x_0 and radius r.
2) $D_r(x_0) = \{ x \in X : \| x - x_0 \| \le r \}$ is called ***closed ball*** of center x_0 and radius r.
3) $B_1(0) = \{ x \in X : \| x \| < 1 \}$ is called ***open unite*** of center 0 and radius 1.
4) $D_1(0) = \{ x \in X : \| x \| \le 1 \}$ is called ***closed unite*** of center 0 and radius 1.

Definition 1.32.: Let $\| \cdot \|_1$, $\| \cdot \|_2$ be two norms on vector space X, $\| \cdot \|_1$ is said to be ***equivalent*** to $\| \cdot \|_2$ ($\| \cdot \|_1 \sim \| \cdot \|_2$) if there exist a and b positive real numbers such that:

$$a \| \cdot \|_2 \le \| \cdot \|_1 \le b \| \cdot \|_2$$

Example: Let $X = IR^n$,

$$\| x \| = \sum_{i=1}^{n} | x_i | , \forall x \in IR^n \quad \text{and} \quad \| x \|_e = \left(\sum_{i=1}^{n} | x_i^2 | \right)^{\frac{1}{2}} , \forall x \in IR^n$$

Then $\| x \| \sim \| x \|_e$

proof:

$\sum_{i=1}^{n} |x_i y_i| \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} (\sum_{i=1}^{n} y_i^2)^{\frac{1}{2}}, \forall x_i, y_i \in IR^n$ (by using Cauchy – Schwars inquality)

Put y_i=1, $\forall$ i = 1, 2, ..., n. $\Rightarrow \sum_{i=1}^{n} |x_i| \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} (\sum_{i=1}^{n} 1)^{\frac{1}{2}}$

$\| x\| \leq \| x \|_e \cdot \sqrt{n}$

$\frac{1}{\sqrt{n}} \| x\| \leq \| x \|_e$ (i.e. $a=\frac{1}{\sqrt{n}}$) ..... (1)

But $\| x \|_e \leq \| x\|$ (i.e. b=1) – — (2)

From (1) & (2), we have:

$\frac{1}{\sqrt{n}} \| x\| \leq \| x \|_e \leq \| x\|$ Then $\| x \| \sim \| x \|_e$

Theorem 1.33.: On a finite dimensional normed space, all norms are equivalent.

Examples:

1. X=IR^2, $\| . \|_e$, $\| . \|_2$, $\| . \|_3$ are equivalent.

X= IR^n, $\| . \|_e$, $\| . \|_2$, $\| . \|_3$ are equivalent

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