الفصل الثاني

Banach Space

Definition 2.1. A normed linear space X is said to be ***complete*** if all Cauchy convergent sequences in X are convergent in X. The complete normed space is called ***Banach space***.

Examples 2.2.

[1] The space F^n with the norm $\| x \| = (\sum_{i=1}^{n} | x_i |^2)^{\frac{1}{2}}$, $\forall x=(x_1,x_2,\dots,x_n) \in F^n$ is a Banach space.

Proof: F^n is a normed space ,

let $\{ x_n\}$ is Cauchy sequence in $F^n \Rightarrow x_m \in F^n \Rightarrow x_m=(x_{1m},x_{2m},\dots,x_{nm})$

let $\varepsilon > 0 \Rightarrow \exists\, k \in Z^+$ s.t. $\| x_m - x_l \| < \varepsilon \quad \forall\, m.l > k$

$\Rightarrow \| x_m - x_l \|^2 < \varepsilon^2 \quad \forall\, m.l > k$ ..... (1)

$x_m - x_l = (x_{1m}-x_{1l}, x_{2m}-x_{2l}, \dots, x_{nm}-x_{nl})$

$\| x_m - x_l \|^2 = \sum_{i=1}^{n} | x_{im} - x_{il} |^2$ ……..(2)

from (1) & (2) , we get:

$$\sum_{i=1}^{n} | x_{im} - x_{il} |^2 \quad < \varepsilon^2 \qquad \forall\ m.l \geq k$$

then

$$| x_{im} - x_{il}|^2 < \varepsilon^2 \qquad \forall\ m.l \geq k \ \Rightarrow \ | x_{im} - x_{il}| < \varepsilon \qquad \forall\ m.l \geq k$$

$\Rightarrow \forall i$, $\{x_{im}\}$ is a Cauchy sequence in F

Since F is complete (because F is IR or C)

$\Rightarrow \forall i, \exists\ x_i \in$ F s.t. $x_{im} \to x_i$

Put $x=(x_1, x_2, \dots ,x_n)$ $\Rightarrow x \in F$, T.P. $x_m \to x.$

Let $\varepsilon > 0$, $\forall\ m > k$, we get:

$$\| x_m - x\|^2 = \sum_{i=1}^{n} | x_{im} - x_i |^2 \ < \varepsilon^2 \ \Rightarrow \| x_m - x\| < \varepsilon \quad \forall m > k$$

$\Rightarrow \{x_m\}$ convergent $\Rightarrow F^n$ is complete

Since F^n is normed space $\Rightarrow F^n$ is a Banach space

[2] H.W. The space l^p $(1 \leq p < \infty)$ with the norm $\| x \| = (\sum_{i=1}^{n} | x_i |^p)^{\frac{1}{p}}, x = (x_1, x_2, \ldots) \in l^p$, is a Banach space.

l^∞

[3] The space l^∞ with the norm $\| x \| = \sup_i | x_i |$ is a Banach space.

Proof:

Let $\{ x_m \}$ is a Cauchy sequence in $l^\infty \Rightarrow x_m \in l^\infty \Rightarrow x_m = (x_{1m}, x_{2m}, \ldots, x_{nm}, \ldots)$

Let $\varepsilon > 0$, $\exists$ k$\in Z^+$ s.t.

$\| x_m - x_l \| < \varepsilon, \ \forall m,l > k$ ....(1)

$x_m - x_l = (x_{1m} - x_{1l}, \ldots, x_{nm} - x_{nl}, \ldots)$

$\| x_m - x_l \| = \sup_i | x_{im} - x_{il} |$ ....(2)

From (1) and (2), we have:

$\sup_i | x_{im} - x_{il} | < \varepsilon \quad , \forall m, l > k$

then for all i, $| x_{im} - x_{il} | < \varepsilon \quad , \forall m, l > k$ ...... (3)

$\Rightarrow \forall i$, then $\{ x_{im} \}$ is Cauchy sequence in F

Since F is complete $\Rightarrow \{ x_{im} \}$ is convergent $\Rightarrow \exists \ x_i \in$ F s.t. $x_{im} \rightarrow x_i$

Put $x = (x_1, x_2, \ldots)$, we must prove that $x \in l^\infty$, $x_m \rightarrow x$

From (3) , we get:

$| x_{im} - x_i | < \varepsilon , \quad \forall\ m > k$ ..... (4)

Since $x_m \in l^\infty$

$\Rightarrow \exists\ k_m \in IR$ s.t.: $| x_{im} | \leq k_m , \quad \forall\ i$

$x_i = (x_i - x_{im}) + x_{im}$

$| x_i | \leq | x_i - x_{im} | + | x_{im} |$

[4] Let X=C[a, b], $\|x\|_1 = \sup\{| f(x) | : a \leq x \leq b\}, \forall x \in [a,b]$ is a Banach space.

Proof:

T.P. (C[a, b], $\|.\|_1$) is Banach space

1. C[a, b] is v.s. over IR

2. (C[a, b], $\|.\|_1$) is normed space

3. T.P. (C[a, b], $\|.\|_1$) is complete

Let (f_m) be a Cauchy seq. in C[a, b]

Given $\varepsilon>0$, $\exists$ $k\in Z^+$ s.t. $\| f_m - f_n \|_1 < \varepsilon$, $\forall$ $m, n > k$

$\| f_m - f_n \|_1 = \sup\{| (f_m - f_n)(x)| : a \le x \le b\} = \sup\{| f_m(x) - f_n(x)| : a \le x \le b\} < \varepsilon$, $\forall$ $m, n > k$

$\Rightarrow | f_m(x) - f_n(x) | < \varepsilon$, $\forall$ $x \in [a, b]$, $\forall$ $m, n > k$

Since (f_m) is Cauchy seq. in IR, IR is complete

Then (f_m) is convergent

i.e. ∃ $f \in$ IR (f cont's & bounded) s.t. $f_m \to f$

$\Rightarrow$ (C[a, b], $\|.\|_1$) is complete n.s.

$\Rightarrow$ (C[a, b], $\|.\|_1$) is Banach space

[5] Let X=C[0, 1], $\| \,.\, \|_2$: C[0, 1] $\to$ IR defined by

$$\| f\|_2 = \int_0^1 | f(x) | \, dx, \forall f \in C[0,1]$$

Then (C[0, 1], $\| f\|_2$) is not Banach space because it is normed space but not complete

Proof:

Let (f_n) is Cauchy seq. in C[0, 1], where:

$$f_n = \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -nx + \frac{1}{2}n + 1 & \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n} \\ 0 & \frac{1}{2} + \frac{1}{n} < x \le 1 \end{cases}$$

let $m, n > 3$, then:

$$\| f_m - f_n \| = \int_0^1 |(f_m - f_n)(x)|\,dx = \int_0^1 | f_m(x) - f_n(x) |\,dx$$

$$= \int_0^{1/2} |(f_m(x) - f_n(x)|\,dx + \int_{1/2}^1 | f_m(x) - f_n(x) |\,dx$$

$$\le \int_0^{1/2} |1-1|\,dx + \int_{1/2}^1 |(f_m(x)|\,dx + \int_{1/2}^1 | f_n(x) |\,dx$$

$$\le \int_{1/2}^{\frac{1}{2}+\frac{1}{m}} \left|-mx + \frac{1}{2}m + 1\right| dx + \int_{1/2}^{\frac{1}{2}+\frac{1}{n}} \left|-nx + \frac{1}{2}n + 1\right| dx = [-m\frac{1}{2}x^2 + \frac{1}{2}mx + x]_{1/2}^{\frac{1}{2}+\frac{1}{m}} + [-n\frac{1}{2}x^2 + \frac{1}{2}nx + x]_{1/2}^{\frac{1}{2}+\frac{1}{n}}$$

Since $-mx + \frac{1}{2}m + 1 \ge 0$ when $\frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{m}$ $\Rightarrow \| f_m - f_n \| \le \frac{1}{2m} + \frac{1}{2n}$ as $m, n \to \infty$

$\Rightarrow (f_n)$ is Cauchy convergent seq.

T.P. (f_n) is not convergent, Suppose (f_n) is convergent

$\exists\ f \in$ C[0, 1] s.t. $f_n \to f$

i.e. $\lim_{m \to \infty} f_n(x) = f(x), \forall x \in [0,1]$

$$\Rightarrow f(x) = \begin{cases} 1 & , 0 \le x \le 1 \\ 0 & , \frac{1}{2} < x \le 1 \end{cases}$$ C!

Since f is not continuous at $x = ½$

$\Rightarrow$ (C[0, 1], $\| f \|_2$) is not complete $\Rightarrow$ not Banach space.

Thank You