كلية العلوم للبنات

قسم الرياضيات

المرحلة الرابعة

تحليل دالـــــــــــــي

المحاضرة الثالثة

Definition 1.15. A set C in a linear space is *convex* if for any two points $x, y \in C$, $tx + (1 - t)y \in C$ for all $t \in [0; 1]$.

Definition 1.16. A norm $\| \,.\, \|$ is *strictly convex* if $\| x \| = 1$, $\| y \| = 1$, $\| x+y \| = 2$ together imply that $x = y$.

Definition 1.17. :If $(X; \| \,.\, \|_X)$ and $(Y; \| \,.\, \|_Y)$ are normed linear spaces, then the *product*

$$X \times Y = \{ (x, y) \setminus x \in X; y \in Y \}$$

is a linear space which may be made into a normed space in many different ways, a few of which follow.

Example 1.18.

[1] $\|(x, y)\| = \max\{\| x \|_X , \| y \|_Y \}$.

proof:

1) $\| (x,y) \| = 0 \leftrightarrow \max\{\| x \|_X , \| y \|_Y\} = 0 \leftrightarrow \| x \|_X = 0 , \| y \|_Y = 0 \leftrightarrow x = 0, y = 0 \leftrightarrow (x, y) = 0$

2) let $(x_1,y_1) , (x_2,y_2) \in XxY$, then

$(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2)$

$\| (x_1 + x_2, y_1 + y_2)\| = \max \{\| x_1 + x_2\|_X, \| y_1 + y_2\|_Y)\} \leq \max\{\| x_1\|_X + \| x_2\|_X , \| y_1\|_Y + \| y_2\|_Y)$

$\leq \max\{\| x_1\|_X, \| y_1\|_Y \} + \max\{\| x_2\|_X , \| y_2\|_Y\} = \|(x_1, y_1)\| + \|(x_2, y_2)\|$

3) let $(x, y) \in XxY$ and $\alpha \in F$, then

$\| \alpha(x, y)\| = \max\{ \|\alpha x\|_X, \|\alpha y\|_Y\} = \max \{|\alpha| \|x\|_X, |\alpha| \|y\|_Y\} = |\alpha| \max \{\|x\|_X, \|y\|_Y\} = |\alpha|$ $\|(x, y)\|$

[2] ||(x, y)|| = (||x||X + ||y||Y)1/p; (H.W.)

Theorem 1.19. : Every normed linear space is metric space.

proof:

let (X, ||.||) is a normed space. We define the function d: XxX $\rightarrow$ IR as:

$d(x,y)=\|x-y\|$ for all $x,y\in X$, since this function satisfies all the conditions of metric :

1) let $x,y\in X \rightarrow x-y\in X$ (since X is vector space) $\rightarrow \| x-y \| \geq 0 \rightarrow d(x, y) \geq 0$.

2) $d(x, y) = 0 \leftrightarrow \| x-y \| =0 \leftrightarrow x-y = 0 \leftrightarrow x=y$

3) $d(x, y) = \| x-y \| = \| y-x \| = d(y, x)$

4) let $x,y,z\in X$:

$$\| x-y \| = \| (x-z)+(z-y) \| \leq \| x-z \|+ \| z-y \| \rightarrow d(x, y) \leq d(x, z)+ d(z, y)$$

Remark : The converse may be not true, for example:
If X be a v.s., define d: X x X $\to$ IR as:

$$d(x,y)=\begin{cases}0 & x=y\\ 2 & x\neq y\end{cases}$$

And define $\|\,.\,\|$: X $\to$ IR as $\| x \| = d(x, 0)$
(X, $\|\,.\,\|$) fails to be normed space.
Since if $x\neq 0 \to \| x \| = d(x, 0) = 2$
$\| 2x \| = d(2x, 0) \to |2|\,\| x \| = 2 \to 2.2=2 \to 4=2$ C!

Definition 1.20.: Let X = (X; $\|\,.\,\|_X$) be a normed linear space. A sequence of vectors (x_n) in X is said to ***convergent*** if:

$$\exists\, x\in X \quad \text{s.t.}, \ \forall \varepsilon>0 \quad \exists\, k(\varepsilon)\in Z^+ \quad \text{s.t.} \quad \| x_n - x \| < \in \quad \forall\, n>k.$$

And we say x is the convergent point for the sequence (x_n) and write $x_n \to x$ when $n\to\infty$, this means $x_n\to x \ \leftrightarrow \ \| x_n - x \| \to 0$. If (x_n) not convergent is called ***divergent***.

Theorem 1.21.: Let X be a normed space, (x_n), (y_n) be a sequence in X such that $x_n \to x_0$, $y_n \to y$, then:

1. $x_n \pm y_n \to x_0 \pm y_0$
2. $\| x_n \| \to \| x_0 \|$
3. $\| x_n - y_n \| \to \| x_0 - y_0 \|$
4. $\alpha x_n \to \alpha x_0 \quad \forall \alpha \in F$

Proof:

1. Since $x_n \to x_0$, $y_n \to y$, then:

if $\varepsilon > 0$, $\exists\ k_1(\varepsilon) \in Z^+$ s.t. $\| x_n - x_0 \| < \varepsilon / 2$, $\forall\ n > k_1(\varepsilon)$

$\exists\ k_2(\varepsilon) \in Z^+$ s.t. $\| y_n - y_0 \| < \varepsilon / 2$, $\forall\ n > k_2(\varepsilon)$

Define $k_3(\varepsilon) = \max \{ k_1(\varepsilon), k_2(\varepsilon) \}$

$$\| (x_n + y_n) - (x_0 + y_0) \| = \| x_n + y_n - x_0 - y_0 \|$$

$$\leq \| x_n - x_0 \| + \| y_n - y_0 \|$$

$$< \varepsilon / 2 + \varepsilon / 2 = \varepsilon , \ \forall\ n > k_3(\varepsilon) \ \to \ x_n + y_n \to x_0 + y_0$$

2-Since $x_n \to x_0$ T.P. $\| x_n \| \to \| x_0 \|$ i.e. T.P. $| \| x_n \| - \| x_0 \| | \to 0$

By Theorem (1.13.)-4 : $| \| x_n \| - \| x_0 \| | \leq \| x_n - x_0 \|$ ….. (1)

Since $x_n \to x_0 \to \| x_n - x_0 \| \to 0$ …..(2)

By (1) & (2) we get: $| \| x_n \| - \| x_0 \| | \to 0$

Then $\| x_n \| \to \| x_0 \|$

3- T.P. $\| x_n - y_n \| \to \| x_0 - y_0 \|$, i.e. T.P. $| \| x_n - y_n \| - \| x_0 - y_0 \| | \to 0$

Since $x_n \to x_0 \Rightarrow \| x_n - x_0 \| \to 0$

& $y_n \to y_0 \Rightarrow \| y_n - y_0 \| \to 0$

$| \| x_n - y_n \| - \| x_0 - y_0 \| | \leq \| x_n - y_n - x_0 + y_0 \|$

$\leq \| x_n - x_0 \| + \| y_n - y_0 \|$

$\Rightarrow | \| x_n - y_n \| - \| x_0 - y_0 \| | \to 0 \Rightarrow \| x_n - y_n \| \to \| x_0 - y_0 \|$

4- $\|\alpha x_n - \alpha x_0\| = \|\alpha(x_n - x_0)\| = |\alpha| \| x_n - x_0\|$

since $\| x_n - x_0\| \to 0$ where $n \to \infty \Rightarrow \|\alpha x_n - \alpha x_0\|$ where $n \to \infty \Rightarrow \alpha x_n \to \alpha x_0$

Theorem 1.22.: If the sequence (x_n) is convergent in the normed space X then its convergent point is unique.

proof:

suppose that $x_n \to x$ and $x_n \to y$ s.t. $x \neq y$, and let $\| x-y \| = \varepsilon \to \varepsilon > 0$

since $x_n \to x \Rightarrow \exists k_1 \in Z^+$ s.t. $\| x_n - x \| < \varepsilon/2$, $\forall\, n > k_1$

and $x_n \to y \Rightarrow \exists k_2 \in Z^+$ s.t. $\| x_n - y \| < \varepsilon/2$, $\forall\, n > k_2$

put $k = \max\{ k_1, k_2 \}$. Then $\| x_n - x \| < \varepsilon/2$, $\| x_n - y \| < \varepsilon/2 \quad \forall\, n > k$.

$\varepsilon = \| x-y \| = \| (x - x_n) + (x_n - y) \| \leq \| (x_n - x) \| + \| (x_n - y) \| < \varepsilon/2 + \varepsilon/2$

$= \varepsilon$!

and this contradiction then $x=y$.