# 2 Vectors and the Geometry of Space

## 2.1 Three-Dimensional Coordinate Systems

The Cartesian coordinates (x, y, z) of a point P in space are the numbers at which the planes through ***P*** perpendicular to the axes cut the axes (Figure 1.1). Cartesian coordinates for space are also called rectangular coordinates because the axes that define them meet at right angles. Points on the x-axis have y- and z-coordinates equal to zero. That is, they have coordinates of the form (x, 0, 0). Similarly, points on the y-axis have coordinates of the form (0, y, 0), and points on the z-axis have coordinates of the form (0, 0, z).

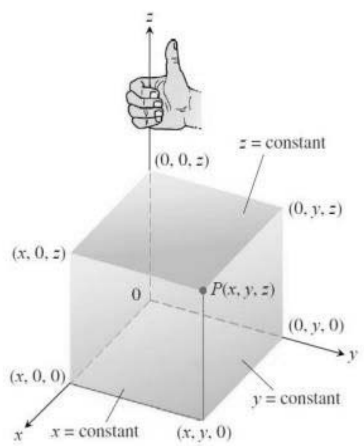
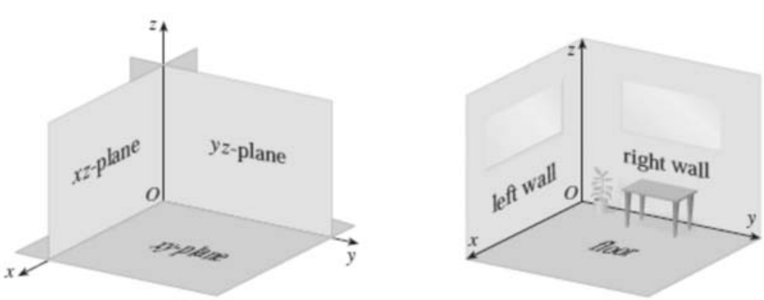



Figure 1.1: The Cartesian coordinate system is right-handed




### Distance in Space

The formula for the distance between two points in the xy-plane extends to points in space.

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

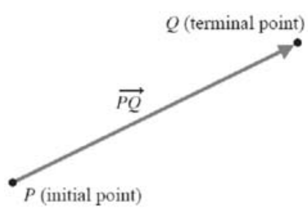
$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

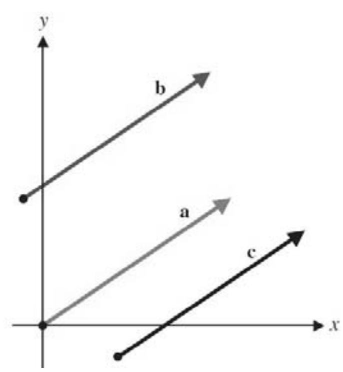
**Example 1**: Finding the Distance Between Two Points The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$\begin{aligned} |P_1P_2| &= \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2} \\ &= \sqrt{16+4+25} \\ &= \sqrt{45} \end{aligned}$$

## Vectors

A **Vector** is an object with length (magnitude) and direction. The direction of a vector is determined by the vector's start (initial) point and end (terminal) point. Two vectors are equal if they have the same length and direction. Changing the position of the vector's start point does not change the vector, therefore, vectors are typically drown with their start point at origin. Vectors with start points at origin are called **Position Vectors**.




We can specify v by writing the coordinates of its terminal point $(v_1, v_2, v_3)$ when $\boldsymbol{v}$ is in the standard position. If $\boldsymbol{v}$ is a vector in the plane its terminal point $(v_1, v_2)$ has two coordinates.

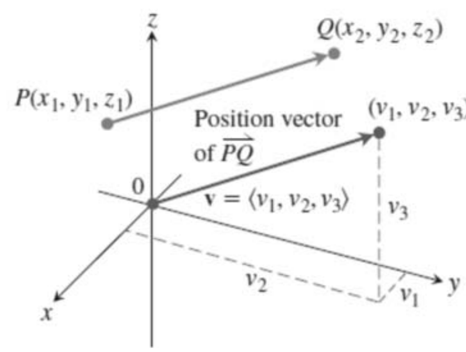
## Component Form and Length of a Vector

The Component Form of the vector in a three-dimensional equal to the vector with the initial point at the origin and terminal point ($v_1$, $v_2$, $v_3$), then the component form of ***v*** is

$$\boldsymbol{v} = (v_1, v_2, v_3)$$

Given the points ***P*** ($x_1$, $y_1$, $z_1$) and ***Q*** ($x_2$, $y_2$, $z_2$) the standard position vector ***v*** = ($v_1$, $v_2$, $v_3$) equal to $\overrightarrow{PQ}$ is

$$\boldsymbol{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$




If v is two-dimensional with ***P*** ($x_1$, $y_1$) and ***Q*** ($x_2$, $y_2$) as points in the plane, then ***v***$= (x_2 - x_1, y_2 - y_1)$. There is no third component for planar vectors.

Two vectors are equal if and only if their standard position vectors are identical. Thus ($u_1$, $u_2$, $u_3$) and ($v_1$, $v_2$, $v_3$) are equal if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$. The **magnitude** or **length** of the vector $\overrightarrow{PQ}$ is the length of any of its equivalent directed line segment representations. In particular, if ***v***$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ is the standard position vector for $\overrightarrow{PQ}$ then the distance formula gives the magnitude or length of v, denoted by the symbol |v| or ||v||.

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The only vector with length 0 is the **zero vector 0** = (0,0,0). This vector is also the only vector with no specific direction.

**Example 2**: Find the (a) component form and (b) length of the vector with initial point ***P*** (-3, 4, 1) and terminal point ***Q*** (-5, 2, 2).

**Solution:**

(a) The standard position vector ***v*** represents $\overrightarrow{PQ}$ has components $v_1$ $= x_2 - x_1 = -5 - (-3) = -2$, $v_2 = y_2 - y_1 = 2 - 4 = -2$, and $v_3 = z_2 - z_1 = 2 - 1 = 1$.

The component form $\overrightarrow{PQ}$ is $\boldsymbol{v} = (-2, -2, 1)$

(b) The length or magnitude of $v = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

## Vector Algebra Operation

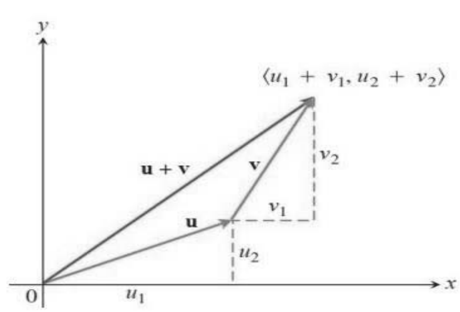
Two principal operations involving vectors are vector addition and scalar multiplication.

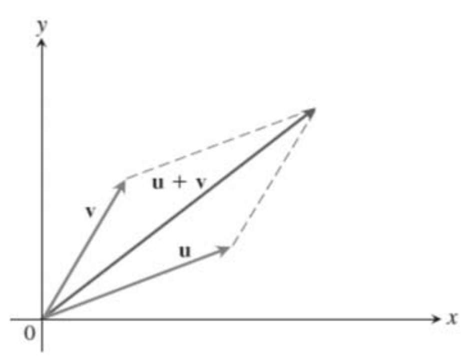
**DEFINITIONS** Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with $k$ a scalar.

**Addition:** $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

**Scalar multiplication:** $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

The definition of vector addition is illustrated geometrically for planar vectors in the following Figures, where the initial point of one vector is placed at the terminal point of the other in the first, while the parallelogram law of addition in the second.

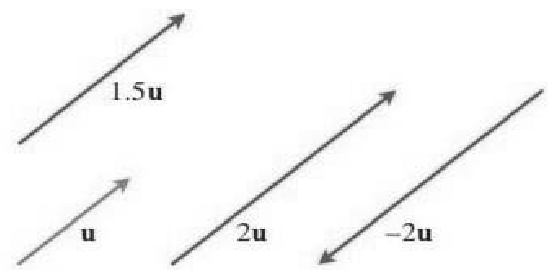


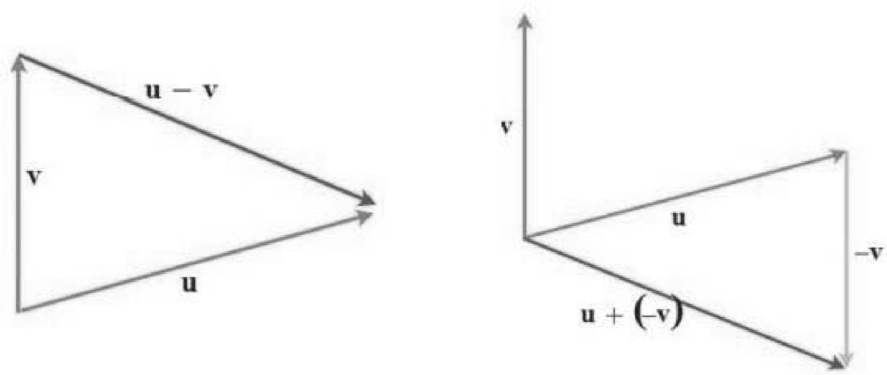
A geometric interpretation of the scalar multiplication ***ku*** of the scalar k and vector ***u*** is illustrated in the following Figure. If k > 0, then ***ku*** has the same direction as ***u***; if k < 0, then the direction of ***ku*** is opposite to that of ***u***. Comparing the lengths of ***u*** and ***ku***, we see that

$$|k\mathbf{u}| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} = |k|\sqrt{u_1^2 + u_2^2 + u_3^2} = |k||\mathbf{u}|$$

The length of ***ku*** is the absolute value of the scalar ***k*** times the length of ***u***. The vector (-1) ***u*** = - u has the same length as ***u*** but points in the opposite direction.




By the difference ***u – v*** of two vectors, we mean ***u – v*** = u + (– v). Note that (u – v) + v = u, so adding the vector (u – v) to v gives u (Figure 1.6).




**Example :**

Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find (a) $2\mathbf{u} + 3\mathbf{v}$; (b) $\mathbf{u} - \mathbf{v}$; (c) $\left|\frac{1}{2}\mathbf{u}\right|$.

**Solution**: (a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left|\frac{1}{2}\mathbf{u}\right| = \left|\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \rangle\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$

## Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and $a$, $b$ be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

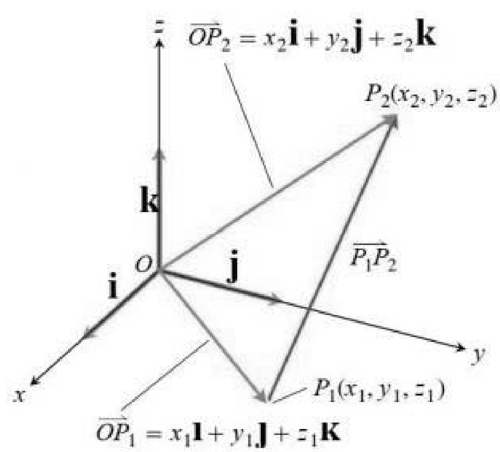
## 2.2 Unit Vectors

A vector v of length 1 is called a unit vector. The standard unit vectors are $\boldsymbol{i} = (1, 0, 0)$, $\boldsymbol{j} = (0, 1, 0)$, and $\boldsymbol{k} = (0, 0, 1)$.

Any vector can be written as follows: $\boldsymbol{v} = (v_1, v_2, v_3) = v_1\boldsymbol{i} + v_2\boldsymbol{j} + 3\boldsymbol{k}$.

We call the scalar (or number) $\mathbf{v_1}$ the i-component of the vector $\boldsymbol{v}$, $\mathbf{v_2}$ the $\boldsymbol{j}$-component, and $\mathbf{v_3}$ the $\boldsymbol{k}$-component.

In component form, the vector from $\boldsymbol{P_1}(x_1, y_1, z_1)$ to $\boldsymbol{P_2}(x_2, y_2, z_2)$



$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

Whenever $\boldsymbol{v} \neq 0$, its length $|v|$ is not zero and

$$\left|\frac{1}{|\mathbf{v}|}\mathbf{v}\right| = \frac{1}{|\mathbf{v}|}|\mathbf{v}| = 1$$

That is, $v/|v|$ is a unit vector in the direction of ***v***, called **the direction** of the nonzero vector **v**.

In summary:

If $\mathbf{v} \neq \mathbf{0}$, then

**1.** $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called the direction of **v**;

**2.** the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses **v** as its length times its direction.

**Example 4**: Find a unit vector ***u*** in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

**Solution**: we divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3-1)\mathbf{i} + (2-0)\mathbf{j} + (0-1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$

The unit vector **u** is the direction of $\overrightarrow{P_1P_2}$.

## Example

A force of 6 newtons is applied in the direction of the vector v = 2i + 2j - k. Express the force F as a product of its magnitude and direction.

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The force vector has magnitude 6 and direction $\frac{\mathbf{v}}{|\mathbf{v}|}$, so

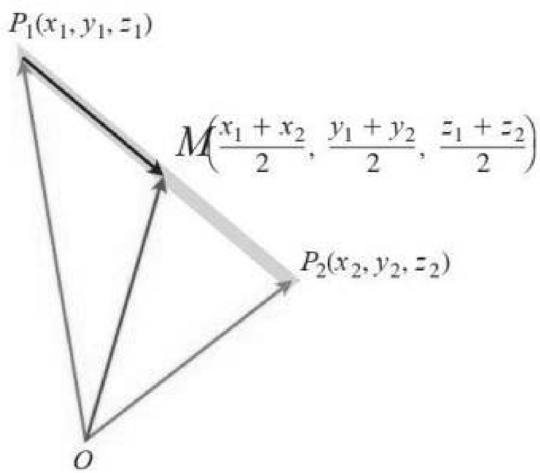
$$\mathbf{F} = 6\frac{\mathbf{v}}{|\mathbf{v}|} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3}$$

$$= 6\left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\right).$$

## 2.3 Midpoint of a Line Segment

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging. The midpoint M of the line segment joining points $\boldsymbol{P_1}(x_1, y_1, z_1)$ and $\boldsymbol{P_2}(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$




**Example**

The midpoint of the segment joining $\boldsymbol{P_1}(3, -2, 0)$ and $\boldsymbol{P_2}(7, 4, 4)$ is

$$\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}\right) = (5, 1, 2)$$

## 2.4 The Dot Product

Dot products are also called inner or scalar products because the product results in a scalar, not a vector. It is used to calculate the angle between two vectors directly from their components; show whether two vectors are orthogonal or not; find the projection vector.

The dot product ***u* ● *v* (*u* dot *v*)** of vectors ***u*** = $\langle u_1, u_2, u_3\rangle$ and ***v*** = $\langle v_1, v_2, v_3\rangle$ is

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

**Example** : Finding Dot Product

(a) $\langle 1, -2, -1\rangle \bullet \langle -6, 2, -3\rangle = (1)(-6) + (-2)(2) + (-1)(-3) = -7$

(b) $\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \bullet (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1.$

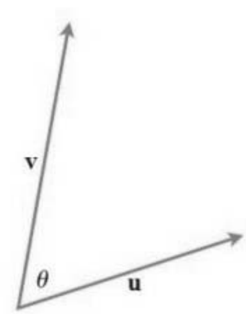
## Properties of the Dot Product

If ***u***, ***v***, and ***w*** are any vectors and ***c*** is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0.$

## Angle Between Vectors

As seen in this Figure



Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathrm{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ ,

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

The angle $\theta$ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

$$0 \le \theta \le \pi$$

**Example**

Find the angle between **u** = **i** - 2**j** - 2**k** and **v** = 6**i** + 3**j** + 2**k**.

**Solution** We use the formula above:

$$\mathbf{u}\cdot\mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

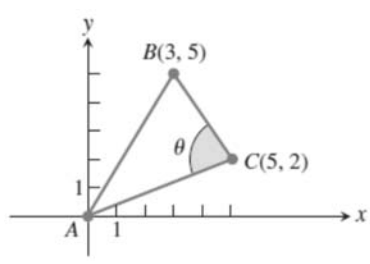
$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians or } 100.98°.$$

**Example**

Find the angle u in the triangle ABC determined by the vertices A = (0, 0), B = (3, 5), and C = (5, 2)




**Solution**

The angle u is the angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. The component forms of these two vectors are:

$$\overrightarrow{CA} = \langle -5, -2 \rangle \quad \text{and} \quad \overrightarrow{CB} = \langle -2, 3 \rangle.$$

First we calculate the dot product and magnitudes of these two vectors.

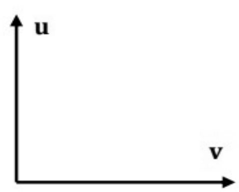
$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$$
$$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$
$$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

Then applying the angle formula, we have

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}\right)$$
$$= \cos^{-1}\left(\frac{4}{(\sqrt{29})(\sqrt{13})}\right)$$
$$\approx 78.1^\circ \text{ or } 1.36 \text{ radians.}$$

## 2.5 Perpendicular (Orthogonal) Vectors

Vectors ***u*** and ***v*** are orthogonal (or perpendicular) if and only if **u** ● **v** = 0.




If we have two vectors ***u*** and ***v***, from **dot product**, we can know:

1. **u** ● **v** = (+), acute angle;
2. **u** ● **v** = (–), obtuse angle;
3. **u** ● **v** = (0), right angle.

**Example**

**(a)** $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.

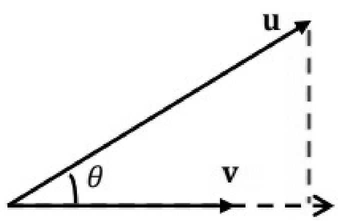
**(b)** $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0$.

**(c)** $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ since

$$\begin{aligned}\mathbf{0}\cdot\mathbf{u} &= \langle 0, 0, 0\rangle \cdot \langle u_1, u_2, u_3\rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) \\ &= 0.\end{aligned}$$

## 2.6 Vector Projections

The vector projection of ***u*** onto a nonzero vector ***v*** is




$$\begin{aligned}\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= (|\mathbf{u}|\cos\theta)\frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\end{aligned}$$

The number $|u|\cos\theta$ is called the **scalar component** of u in the direction of v .

$$|\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}$$

**Example**

Find the vector projection of $\boldsymbol{u} = 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ onto $\boldsymbol{v} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$ and the scalar component of ***u*** in the direction of ***v***.

**Solution**:

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \frac{6-6-4}{1+4+4}(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = \frac{-4}{9}(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = \left(-\frac{4}{9}\mathbf{i}+\frac{8}{9}\mathbf{j}+\frac{8}{9}\mathbf{k}\right)$$

We find the scalar component

$$|\mathbf{u}|\cos\theta = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}\right) = 2-2-\frac{4}{3} = -\frac{4}{3}$$