Chapter4: Curve Fitting and Interpolation

Limits processes are the basis of calculus. For example, the derivative

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

is the limit of the difference quotient where both the numerator and the denominator go to zero. A Taylor series illustrates another type of limit process. In this case an infinite number of terms is added together by taking the limit of certain partial sums. An important application is their use to represent the elementary functions: sin(x), cos(x), e^x, ln(x),etc. Table(4.1) gives several of the common Taylor series expansions.

Table(4.1): Taylor Series Expansions for Some Common Function

$Sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$	*for all x*
$Cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$	*for all x*
$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$	*for all x*
$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$	$-1\le x\le 1$
$\tan^{-1}(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$	$-1\le x\le 1$
$(1+x)^p=1+px+\frac{p(p-1)}{2!}x^2+\frac{p(p-1)(p-2)}{3!}x^3+\cdots$	*for* $\lvert x\rvert<1$

We want to learn how a finite sum can be used to obtain a good approximations to an infinite sum. For illustration we shall use the exponential series in table(4.1) to compute the number $e=e^1$. Here we choose $x=1$ and use the series:

$$e^1=1+\frac{1}{1!}+\frac{1^2}{2!}+\cdots+\frac{1^k}{k!}+\cdots$$

Table(4.2): Partial Sums S_n Used to Determine e

n	$s_n = 1 + \frac{1}{1!} + \frac{1^2}{2!} + \cdots + \frac{1^n}{n!} + \cdots$
0	1
1	2
2	2.5
3	2.666 666 666
4	2.708 333 333
5	2.716 666 666
6	2.718 055 555
7	2.718 253 968
8	2.718 278 769
9	2.718 281 525
10	2.718 281 180
11	2.718 281 826
12	2.718 281 182
13	2.718 281828
14	2.718 281 828
15	2.718 281 828

<u>**Theorem(4.1)**</u>**:** (Taylor Polynomial Approximation)

Assume that $f\epsilon C^{N+1}[a,b]$and $x_0 \in [a,b]$is a fixed value. If $x \in [a,b]$,then:

$$f(x)=P_N(x)+E_N(x) \qquad (4.1)$$

where $P_N(x)$ is a polynomial that can be used to approximate $f(x)$:

$$f(x) \approx P_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \qquad (4.2)$$

The error term $E_N(x)$ has the form:

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x - x_0)^{N+1} \qquad (4.3)$$

for some value c=c(x) that lies between x and x_0.

Example(4.1): Show why 15 terms are all that are needed to obtain the 13-digit approximation

e=2.718 281 828 459 in table(4.2).

$$P_{15}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{15}}{15!} \qquad (4.4)$$

setting $x=1$ in (4.4) gives the partial sum $S_{15}=P_{15}(1)$

$$E_{15}(x) = \frac{f^{(16)}(c)x^{16}}{16!}$$

Since $x_0=0$ and $x=1$ then $0<c<1$

which implies that $e^c<e^1$

$$|E_{15}(x)| = \left|\frac{f^{(16)}(c)x^{16}}{16!}\right| \le \frac{e^c}{16!} < \frac{3}{16!} < 1.433\,844 \times 10^{-13}$$

Exercises:

1. Let f(x)=sin(x) and apply theorem(4.1)
   a. Use $x_0=0$ and find $P_5(x)$, $P_7(x)$, and $P_9(x)$.
   b. Show that if $|x|\le 1$ then the approximation

      $$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

      has the error bound $|E_9(x)| < \frac{1}{10!} \le 2.755\,74 \times 10^{-7}$.
   c. Use $x_0 = \frac{\pi}{4}$ and find $P_5(x)$, which involves powers of $(x-\frac{\pi}{4})$.
2. (a) Find a Taylor polynomial of degree N=5 for $f(x) = \frac{1}{1+x}$ expanded about $x_0=0$.

   (b) Find the error term $E_5(x)$ for the polynomial part(a).

4.1 Introduction to Interpolation

We saw how a Taylor polynomial can be used to approximate the function f(x). The information needed to construct the Taylor polynomial is the value of f and its derivatives at x_0. A short coming is that the higher-order derivatives must be known, and often they are either not available or they are hard to compute.

Suppose that the function y=f(x) is known at N+1 points $(x_0,y_0),(x_1,y_1),\ldots,(x_N,y_N)$, where the values x_k are spread out over the interval [a,b] and satisfy. In the construction, only the numerical values x_k and y_k are needed.

$$a \le x_0 < x_1 < \cdots < x_N \le b \quad \text{and} \quad y_k=f(x_k)$$

A polynomial p(x) of degree N will be constructed that passes through these N+1 points.

4.2 Lagrange Approximation

Interpolation means to estimate a missing function value by taking a weighted average of known function values at neighboring points. Linear interpolation uses a line segment that passes through two points. The slope between (x_0,y_0) and (x_1,y_1) is $m=(y_1-y_0)/(x_1-x_0)$, and the point-slope formula for the line $y=m(x-x_0)+y_0$ can be rearranged as:

$$y=P(x)=y_0+(y_1-y_0)\frac{x-x_0}{x_1-x_0} \qquad (4.5)$$

when formula (4.5) is expanded, the result is a polynomial of degree≤1. Evaluation of P(x) at x_0 and x_1, respectively:

$$P(x_0)=y_0+(y_1-y_0)(0)=y_0$$

$$P(x_1)=y_0+(y_1-y_0)(1)=y_1$$

The French mathematician Joseph Louis Lagrange used a slightly different method to find this polynomial. He noticed that it could be written as:

$$y=P_1(x)=y_0\frac{x-x_1}{x_0-x_1}+y_1\frac{x-x_0}{x_1-x_0} \qquad (4.6)$$

Each term on the right side of (4.6) involves a linear factor; hence the sum is a polynomial of degree≤1. The quotient in (4.6) are denoted by

$L_{1,0}(x) = \frac{x-x_1}{x_0-x_1}$ and $L_{1,1}(x) = \frac{x-x_0}{x_1-x_0}$ (4.7)

Computation reveals that $L_{1,0}(x_0) = 1$, $L_{1,0}(x_1) = 0$, $L_{1,1}(x_0) = 0$, and $L_{1,1}(x_1) = 1$ so that the polynomial $P_1(x)$ in (4.6) also passes through the two given pints. The terms $L_{1,0}(x)$ and $L_{1,1}(x)$ are called ***Lagrange coefficient polynomials***. Using this notation, (4.6) can be written in summation form:

$P_1(x)=\sum_{k=0}^{1} y_k L_{1,k}$ (4.8)

Example(4.2): Consider the graph y=f(x)=cos(x) over [0,1.2].

a. Use the nodes $x_0=0$, and $x_1=1.2$ to construct a linear interpolation polynomial $P_1(x)$.
b. Use the nodes $x_0=0.2$, and $x_1=1$ to construct a linear interpolation polynomial$Q_1(x)$.

Using (4.6) with the abscissas $x_0=0$, and $x_1=1.2$ and the ordinates $y_0=\cos(0)=1$ and $y_1=\cos(1.2)=0.362\ 358$

$P_1(x)=1\frac{x-1.2}{0-1.2} + 0.362\ 358\frac{x-0}{1.2-0}$

$=-0.833\ 333(x-1.2)+0.301\ 965(x-0)$

When the nodes $x_0=0.2$, and $x_1=1$ with $y_0=\cos(0.2)=0.980\ 067$ and $y_1=\cos(1)=0.540\ 302$ are used, the results is:

$Q_1(x)=0.980\ 067\frac{x-1}{0.2-1} + 0.540\ 302\frac{x-0.2}{1-0.2}$

$=-1.225\ 083(x-1)+0.675\ 378(x-0.2)$

The generalization of(1.8) is the construction of a polynomial $P_N(x)$ of degree at most N that passes through the N+1 points $(x_0,y_0),(x_1,y_1),\ldots,(x_N,y_N)$ and has the form:

$P_N(x)=\sum_{k=0}^{N} y_k L_{N,k}$ (4.9)

where $L_{N,k}$ is the Lagrange coefficient polynomial based on these nodes

$L_{N,k}=\frac{(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_N)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_N)}$ (4.10)

Example(4.3): Consider y=f(x)=cos(x) over [0,1.2]

a. Use the three nodes $x_0=0, x_1=0.6$ and $x_2=1.2$ to construct a quadratic interpolation polynomial $P_2(x)$.
b. Use the four nodes $x_0=0, x_1=0.4, x_2=0.8$ and $x_3=1.2$ to construct a cubic interpolation polynomial $P_3(x)$.

a.

x_i	0	0.6	1.2
$y_i=\cos(x_i)$	1	0.825 336	0.362 358

$$P_2(x)=1\frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)}+0.825\,336\frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)}$$

$$+0.362\,358\frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)}$$

$$=1.388\,889(x-0.6)(x-1.2)-2.292\,599x(x-1.2)+0.503275x(x-0.6)$$

b.

x_i	0	0.4	0.8	1.2
$y_i=\cos(x_i)$	1	0.921 061	0.696 707	0.362 358

$$P_3(x)=1\frac{(x-0.4)(x-0.8)(x-1.2)}{(0-0.4)(0-0.8)(0-1.2)}+0.921\,061\frac{(x-0)(x-0.8)(x-1.2)}{(0.4-0)(0.4-0.8)(0.4-1.2)}+0.696\,707\frac{(x-0)(x-0.4)(x-1.2)}{(0.8-0)(0.8-0.4)(0.8-1.2)}$$

$$+0.362\,358\frac{(x-0)(x-0.4)(x-0.8)}{(1.2-0)(1.2-0.4)(1.2-0.8)}$$

$$= -2.604\,167(x-0.4)(x-0.8)(x-1.2)+7.195\,789x(x-0.8)(x-1.2)$$

$$-5.443\,021x(x-0.4)(x-1.2)+0.943\,641x(x-0.4)(x-0.8)$$

Exercises: Find Lagrange polynomials that approximate $f(x)=x^3$.

a. Find the linear interpolation polynomial $P_1(x)$ using the nodes $x_0=-1$ and $x_1=0$
b. Find the quadratic interpolation polynomial $P_2(x)$ using $x_0=-1$, $x_1=0$ and $x_2=1$.

c. Find the cubic interpolation polynomial $P_3(x)$ using x_0=-1, x_1=0 x_2=1 and x_3=2.

d. Find the linear interpolation polynomial $P_1(x)$ using the nodes x_0=1 and x_1=2.

4.2.1 Error Terms and Error Bounds:

Theorem(4.2): (Lagrange Polynomial Approximation)

Assume that $f \in C^{N+1}[a,b]$ and that $x_0, x_1, \dots, x_N \in [a,b]$ are N+1 nodes. If $x \in [a,b]$, then :

$$f(x)=P_N(x)+E_N(x) \qquad (4.11)$$

where $P_N(x)$ is a polynomial that can be used to approximate $f(x)$

$$f(x)=P_N(x)=\sum_{k=0}^{N} f(x_k)L_{N,k} \qquad (4.12)$$

The error term $E_N(x)$ has the form:

$$E_N(X)=\frac{(x-x_0)(x-x_1)\dots(x-x_N)f^{(N+1)}(c)}{(N+1)!} \qquad (4.13)$$

for some value c=c(x) that lies in the interval [a,b].

Theorem (4.3): (Error Bounds for Lagrange Interpolation, Equally Spaced Nodes)

Assume that $f(x)$ is defined on [a,b], which contains equally spaced nodes $x_k=x_0+hk$. Additionally, assume that $f(x)$ and derivatives of $f(x)$, up to order N+1, are continuous and bounded on the special subintervals $[x_0,x_1]$, $[x_0,x_2]$, and $[x_0,x_3]$, respectively; that is:

$$\left|f^{(N+1)}(x)\right| \le M_{N+1} \quad for\ x_0 \le x \le x_N \qquad (4.14)$$

for N=1,2,3. The error terms (4.13) corresponding to the cases N=1,2, and 3 have the following useful bounds on their magnitude:

$$|E_1(x)| \le \frac{h^2 M_2}{8} \quad valid\ for\ x \in [x_0, x_1], \qquad (4.15)$$

$$|E_2(x)| \le \frac{h^3 M_3}{9\sqrt{3}} \quad valid\ for\ x \in [x_0, x_2], \qquad (4.16)$$

$$|E_3(x)| \le \frac{h^4 M_4}{24} \quad valid\ for\ x \in [x_0, x_3], \qquad (4.17)$$

Example(4.4): Consider y=f(x)=cos(x) over [0,1.2]. Use formulas (4.15) through (4.17) and determine the error bounds for the Lagrange polynomial constructed in examples (4.2) and(4.3).

First, determine the bounds M_2, M_3, and M_4 for the derivatives $\left|f^{(2)}(x)\right|, \left|f^{(3)}(x)\right|\ and\ \left|f^{(4)}(x)\right|$, respectively, taken over the interval [0,1.2]:

$$\left|f^{(2)}(x)\right| = |-\cos(x)| \leq |-\cos(0)| = 1 = M_2$$

$$\left|f^{(3)}(x)\right| = |\sin(x)| \leq |\sin(1.2)| = 0.932\ 039 = M_3$$

$$\left|f^{(4)}(x)\right| = |\cos(x)| \leq |\cos(0)| = 1 = M_4$$

For $P_1(x)$ the spacing of the nodes is h=1.2, and its error bound is:

$$|E_1(x)| \leq \frac{h^2 M_2}{8} \leq \frac{(1.2)^2(1)}{8} = 0.180$$

For $P_2(x)$ the spacing of the nodes is h=0.6, and its error bound is:

$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} \leq \frac{(0.6)^3(0.932\ 039)}{9\sqrt{3}} = 0.012\ 915$$

For $P_3(x)$ the spacing of the nodes is h=0.4, and its error bound is:

$$|E_3(x)| \leq \frac{h^4 M_4}{24} \leq \frac{(0.4)^4(1)}{24} = 0.001\ 067$$

Example(4.5): For the data below, obtain the quadratic polynomial and use to estimate f(0.5).

x	1	-1	2
f(x)	0	-2	3

The quadratic Lagrange polynomial are

$$P_2(x)=(0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + (-2)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + 3\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$=(0)\frac{(x-(-1))(x-2)}{(1-(-1))(1-2)} + (-2)\frac{(x-1)(x-2)}{(-1-1)(-1-2)} + 3\frac{(x-1)(x-(-1))}{(2-1)(2-(-1))} \qquad = \frac{2x^2+3x-5}{3}$$

hence $P_2(0.5)$=-1.

Exercises:

1. Consider the Lagrange coefficient polynomial $L_{2,k}(x)$ that are used for quadratic interpolation at the nodes x_0, x_1, and x_2. Define $g(x)=L_{2,0}(x)+L_{2,1}(x)+L_{2,2}(x)-1$.
   a. Show that g is a polynomial of degree ≤ 2.
   b. Show that $g(x_k)=0$ for k=0,1,2.
2. Consider the function f(x)=sin(x) on the interval [0,1]. Use theorem(4.3) to determine the step size h so that:
   a. linear Lagrange interpolation has an accuracy of 10^{-6}.
   b. quadratic Lagrange interpolation has an accuracy of 10^{-6}.
   c. cubic Lagrange interpolation has an accuracy of 10^{-6}.

4.3 Divided Difference Interpolation

The Lagrange interpolation polynomial is useful for analysis, but is not the ideal formula for evaluating the polynomial. Here the groundwork is laid for the development of efficient form of the unique interpolating polynomial P_n.

a. by simplifying the construction.
b. by reducing effort required to evaluate the polynomial.

Definition(4.1):

Define $$f[x_i,x_{i+1}]=\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i} \qquad (4.18)$$

is the ***first-order divided difference of f* at *x=x*** $x=x_i$

and $$f[x_i,x_{i+1},x_{i+2}]=\frac{f[x_{i+1},x_{i+2}]-f[x_i,x_{i+1}]}{x_{i+2}-x_i} \qquad (4.19)$$

is the ***second-divided difference of f at*** $x=x_i$.

and the ***recursive rule for constructing k-order divided differences is***

$$f[x_i,x_{i+1},\dots,x_{i+k}]=\frac{f[x_{i+1},\dots,x_{i+k}]-f[x_i,\dots,x_{i+k-1}]}{x_{i+k}-x_i} \qquad (4.20)$$

and is used to construct the divided differences in table (4.3)

Table(4.3): Divided Differences Table

x_i	$f(x_i)$	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3}]$
x_0	f_0			
		$f[x_0,x_1]$		
x_1	f_1		$f[x_0,x_1,x_2]$	
				$f[x_0,x_1,x_2,x_3]$
		$f[x_1,x_2]$		
x_2	f_2		$f[x_1,x_2,x_3]$	
		$f[x_2,x_3]$		
x_3	f_3			

<u>Theorem(4.4):</u> (Newton Polynomial)

Suppose that $x_0,x_1,\ldots,x_N$ are N+1 distinct numbers in [a,b]. There exists a unique polynomial $P_N(x)$ of degree at most N with the property that:

$$f(x_j)=P_N(x_j) \quad \text{for } j=0,1,\ldots,N$$

The Newton form of this polynomial is:

$$P_N(x)=a_0+a_1(x-x_0)+\ldots+a_N(x-x_0)(x-x_1)\ldots(x-x_{N-1}) \qquad (4.21)$$

where $a_k=f[x_0,x_1,\ldots,x_k]$, for k=0,1,…,N.

<u>Example(4.6):</u> Repeating example(4.5) using the polynomial form (4.21) requires a divided difference table.

x_i	f_i	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$
1	0		
		1	
-1	-2		$\frac{2}{3}$
		$\frac{5}{3}$	
2	3		

and Newton polynomial is:

$P_2(x)=f[x_0]+f[x_0,x_1](x-x_0)+ f[x_0,x_1,x_1](x-x_0)(x-x_1)$

$=0+(1)(x-1)+(\frac{2}{3})(x-1)(x-(-1))=\frac{2x^2}{3}+x-\frac{5}{3}$

Corollary(4.1): (Newton Approximation)

Assume that $P_N(x)$ is the Newton polynomial given in theorem(4.4) and is used to approximate the function f(x), that is,

$$f(x)=P_N(x)+E_N(x) \qquad (4.22)$$

If $f\in C^{N+1}[a,b]$, then for each $x\in[a,b]$there corresponds a number c=c(x) in (a,b), so that the error term has the form

$$E_N(x)=\frac{(x-x_0)(x-x_1)\ldots(x-x_N)f^{(N+1)}(c)}{(N+1)!} \qquad (4.23)$$

Exercises:

1. Compute the divided-difference table for the tabulated function.

$\mathbf{x_i}$	4	5	6	7	8
$\mathbf{y_i}$	2	2.236 07	2.449 49	2.645 75	2.828 43

2. Evaluate the Newton polynomial and find f(3)

$\mathbf{x_i}$	-2	0	1	2	5
$\mathbf{f(x_i)}$	-15	1	-3	-7	41

4.4 Equispaced Interpolation:

4.4.1 Difference Operator and Difference Tables:

Differences are similar to divided differences but work with equispaced data. The ***forward difference operator*** Δ is defined by:

$$\Delta^0 f(x)=f(x) \qquad (4.24)$$

$$\Delta f(x)=\Delta^1 f(x)=f(x+h)-f(x) \qquad (4.25)$$

$$\Delta^k f(x)=\Delta(\Delta^{k-1}f(x))=\Delta^{k-1}(\Delta f(x))$$

$$=\Delta^{k-1}f(x+h)-\Delta^{k-1}f(x) \qquad (4.26)$$

The Δ^k are conveniently displayed in a difference table(4.4)

Table(4.4): A Table of Forward Differences

x	**f(x)**	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	$f(x_0)$			
		Δf_0		
x_1	$f(x_1)$		$\Delta^2 f_0$	
		Δf_1		$\Delta^3 f_0$
x_2	$f(x_2)$		$\Delta^2 f_1$	
		Δf_2		
x_3	$f(x_3)$			

Example(4.7): The polynomial $P_3(x)=x^3-6x^2+11x-3$ gives rise to the following difference table at x=2, 4, 6, 8, 10.

x	$P_3(x)$	$\Delta P_3(x)$	$\Delta^2 P_3(x)$	$\Delta^3 P_3(x)$
2	3			
		6		
4	9		48	
		54		48
6	63		96	
		150		48
8	213		144	
		294		
10	507			

4.4.2 Backward Difference Operator ∇:

Define $\nabla^0 f(x)=f(x)$ (4.27)

$\nabla f(x)=\nabla^1 f(x)=f(x)-f(x-h)$ (4.28)

$\nabla^k f(x)=\nabla^{k-1} f(x)-\nabla^{k-1} f(x-h)\ ,\quad k\geq 1$ (4.29)

Table(4.5): A Table of Backward Differences

x	f(x)	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
x_0	y_0			
		∇f_1		
x_1	y_1		$\nabla^2 f_2$	
		∇f_2		$\nabla^3 f_3$
x_2	y_2		$\nabla^2 f_3$	
		∇f_3		
x_3	y_3			

4.4.3 Shift Operator: E

$E^0 f(x)=f(x)$ (4.30)

$Ef(x)=E^1 f(x)=f(x+h)$ (4.31)

$E^{-1} f(x)=f(x-h)$ (4.32)

$E^k f(x)=f(x+kh)=E(E^{k-1} f(x)),\ \ k=\pm 1, \pm 2, \ldots$ (4.33)

E shifts the data point a number of intervals to the left or right.

There are many relationships between the three difference operators, of which two will be useful for the ensuing discussion:

$$\Delta f(x)=f(x+h)-f(x)=Ef(x)-f(x)=(E-1)f(x)$$

$$\rightarrow \Delta \equiv E-1 \quad , E\equiv 1+\Delta \qquad (4.34)$$

and $\nabla f(x)=f(x)-f(x-h)=f(x)-E^{-1}f(x)=(1-E^{-1})f(x)$

$$\rightarrow \nabla \equiv 1 - E^{-1} \quad , E \equiv (1-\nabla)^{-1} \qquad (4.35)$$

4.4.4 Forward Difference Polynomial:

Assume that the nodes $x_0, x_1, \ldots, x_n$ are in ascending order and may be described by an index j and an interval h,

$$x_j = x_0 + jh, \quad j = 0,1,\ldots,n \qquad (4.36)$$

j is the number of intervals between the data point x_j and the origin x_0. For a real number t,

$$x = x_0 + th, \; 0 \le t \le n \qquad (4.37)$$

If $t \in \{0,1,\ldots,n\}$, x corresponds to a data point. Otherwise x corresponds to a point lying between two adjacent data points.

$f(x)=f(x_0+th)=E^t f(x_0)=(1+\Delta)^t f(x_0)$

$$= \left[1 + t\Delta + \frac{t(t-1)}{2!}\Delta^2 + \frac{t(t-1)(t-2)}{3!}\Delta^3 + \cdots\right] f(x_0)$$

then $P_n(x)=f_0+t\Delta f_0+\frac{t(t-1)}{2!}\Delta^2 f_0 + \cdots + \frac{t(t-1)(t-2)\ldots(t-n+1)}{n!}\Delta^n f_0 \qquad (4.38)$

which is the ***Newton-Gregory forward difference polynomial***.

Example(4.8): Construct a difference table for the function f where f(0.5)=1, f(0.6)=2 and f(0.7)=5, and use quadratic interpolation to estimate f(0.53).

The difference table is:

x	f(x)	Δf(x)	Δ^2f(x)
0.5	1		
		1	
0.6	2		2
		3	
0.7	5		

the quadratic polynomial $P_2(x)=f_0+t\Delta f_0+\frac{1}{2}t(t-1)\Delta^2 f_0$

at x=0.53 , $t=\frac{x-x_0}{h}$, h=0.1, we choose $x_0=0.5$

$\rightarrow t=\frac{0.53-0.5}{0.1}=0.3$

and $f(0.53)\approx P_2(0.53)=1+0.3(1)+(0.3)(0.3-1)(2)/2!=1+0.3-0.105=1.195$

An alternative form of P_n uses the backward difference operator ∇

$$P_n(x)=P_n(x_0+th)=f_0+t\nabla f_0+\frac{t(t+1)}{2!}\nabla^2 f_0+\cdots+\frac{t(t+1)\ldots(t+n-1)}{n!}\nabla^n f_0 \qquad (4.39)$$

Example(4.9): Repeat example(4.8) using the backward formula(4.39) to find f(0.63).

The difference table is identical to that of example(4.8)

x	f(x)	$\nabla f(x)$	$\nabla^2 f(x)$
0.5	1		
		1	
0.6	2		<u>2</u>
		<u>3</u>	
0.7	<u>5</u>		

The quadratic polynomial $P_2(x_2+th)=f_2+t\nabla f_2+\frac{1}{2}t(t+1)\nabla^2 f_2$

since $t=\frac{x-x_2}{h}=\frac{0.63-0.7}{0.1}=-0.7$ and $f(0.63)\approx P_2(0.63)=5-3*0.7+\frac{1}{2}(-0.7)(-0.7+1)(2)=2.69$

Exercises:

1. Construct a difference table for the data

x	0	0.2	0.4	0.6	0.8	1
f(x)	0.55	0.82	1.15	1.54	1.99	2.5

and use to find f(0.23) and f(0.995).

4.5 Curve Fitting

4.5.1 Least Squares Approximation:

Let Y_i represent an experimental value, and let y_i be a value from the equation $y_i=ax_i+b$ where x_i is a particular value of the variable assumed free of error. We wish to determine the best values for a and b so that the y's predict the function values that correspond to x-values. Let $e_i=Y_i-y_i$. The least-squares criterion requires that: $S=e_1^2+e_2^2+\cdots+e_N^2=\sum_{i=1}^{N} e_i^2=\sum_{i=1}^{N}(Y_i-ax_i-b)^2$ be a minimum. N is the number of x,Y-pairs. We reach the minimum by proper choice of the parameters a and b, so they are the " variables" of the problem. At a minimum for S, the two partial derivatives ${\partial S}/{\partial a}$ *and* ${\partial S}/{\partial b}$ will be both zero, that is:

$$\frac{\partial S}{\partial a}=0=\sum_{i=1}^{N} 2(Y_i-ax_i-b)(-x_i),$$

$$\frac{\partial S}{\partial b}=0=\sum_{i=1}^{N} 2(Y_i-ax_i-b)(-1),$$

Dividing each of these equations by -2 and expanding the summation, we get:

$$\left.\begin{aligned} a\sum x_i^2+b\sum x_i &= \sum x_i y_i \\ a\sum x_i+bN &= \sum Y_i \end{aligned}\right\} \qquad (4.40)$$

All the summations in(4.40) are from i=1 to i=N. Solving these equations gives

$$a=\frac{N\sum x_i y_i-\sum x_i \sum y_i}{N\sum x_i^2-(\sum x_i)^2} \qquad (4.41)$$

$$b=\frac{\sum y_i \sum x_i^2-\sum x_i \sum x_i y_i}{N\sum x_i^2-(\sum x_i)^2} \qquad (4.42)$$

<u>Example(4.10):</u> Find the least-squares line for the data point given in the following table:

x	-1	0	1	2	3	4	5	6
y	10	9	7	5	4	3	0	-1

N=8, $\sum x_i$=20, $\sum x_i^2$=92, $\sum y_i$=37, $\sum x_i y_i$=25

from equations(4.41) and (4.42), we get:

a=-1.6071429, b=8.6428571

and y=-1.6071429x+8.6428571

4.5.2 The Power Fit $y=Ax^M$

Some situations involve $f(x)=Ax^M$, where M is a Known constant. In this cases there is only one parameter A to be determined.

Theorem(4.5): (Power Fit)

Suppose that $\{(x_k,y_k)\}$, k=1,…,N are N points, where the abscissas are distinct. The coefficient A of the least-squares power curve $y=Ax^M$ is given by

$$A=\left(\sum_{k=1}^{N} x_k^M y_k\right)\Big/\left(\sum_{k=1}^{N} x_k^{2M}\right) \qquad (4.43)$$

Example(4.11): Find the constant g in the relation $d=\frac{1}{2}gt^2$ using the following table:

t	0.2	0.4	0.6	0.8	1.0
d	0.196	0.785	1.7665	3.1405	4.9075

Here M=2, N=5, $\sum d_k t_k^2=7.6868$, $\sum t_k^4=1.5664$

and the coefficient A=7.6868/1.5664=4.9073, so we get g=2A=9.7146.

4.5.3 Data Linearization Method for $y=Ce^{Ax}$:

Suppose that we are given points $(x_1,y_1),\dots,(x_N,y_N)$ and want to fit an exponential curve of the form

$$y=Ce^{Ax} \qquad (4.44)$$

The first step is to take the logarithm of both sides:

$$\ln(y)=Ax+\ln(C) \qquad (4.45)$$

Then introduce the change of variables:

$$Y=\ln(y),\ X=x\ ,\ \text{and}\ B=\ln(C) \qquad (4.46)$$

This results in a linear relation between the new variables X and Y

$$Y=AX+B \qquad (4.47)$$

Example(4.12): Use the data linearization method and find the exponential fit $y=Ce^{Ax}$ for the five points (0,1.5), (1,2.5), (2,3.5), (3,5), and (4,7.5).

x	0	1	2	3	4
y	1.5	2.5	3.5	5	7.5

$$\sum x_i = 10\ ,\ \sum Y_i = \sum lny_i = 6.19886\ ,\ \sum x_i^2 = 30\ ,\sum x_i lnY_i = \sum x_i lny_i = 16.309743 \text{ and } N=5$$

therefore we have a=0.3912023, b=0.457367

then C is obtained with the calculation $C=e^{0.457367}=1.57991$

and $y=1.57991e^{0.3912023x}$

Exercises:

1. Find the least-squares line for the data

x	-6	-2	0	2	6
y	7	5	3	2	0

2. Find the power fits $y=Ax^2$ and $y=Bx^3$ for the following data:

x	0.5	0.8	1.1	1.8	4
y	7.1	4.4	3.2	1.9	0.9

3. For the given data find the least-squares curve $f(x)=Ce^{Ax}$

x	-1	0	1	2	3
y	6.62	3.94	2.17	1.35	0.89