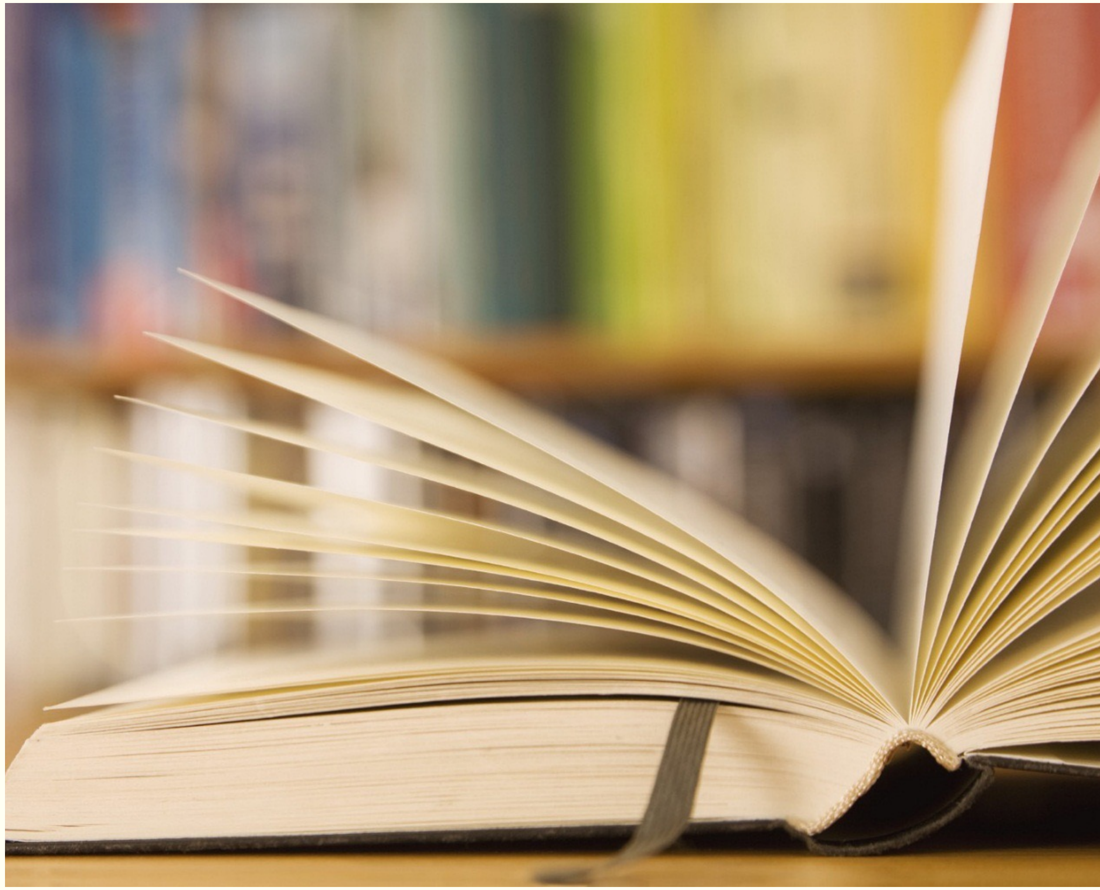
كلية العلوم للبنات
قسم الرياضيات
المرحلة الرابعة



تحليل دالي
المحاضرة الثانية

Definition 1.10 : A ***metric space*** is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on X x X such that for all $x, y, z \in$ X, we have:

(1) d is real-valued, finite and nonnegative function.

(2) $d(x, y)=0$ if and only if $x=y$

(3) $d(x, y) = d(y, x)$ (Symmetry).

(4) $d(x, y) \leq d(x, z)+d(z, y)$ (Triangle inequality).

Examples (H.W. 2-6)

1) **Real line IR**: this is the set of all real numbers, taken with the usual metric defined by:

$$d(x, y) = |x-y| \qquad \forall x,y \in \text{IR}$$

Sol.

[1] 1- d is real, finite & $d=|x-y| \geq 0$

2) $d(x,y)=0 \leftrightarrow |x-y|=0 \leftrightarrow x-y=0 \leftrightarrow x=y \qquad \forall\ x, y \in \text{IR}$

3) $d(x,y)= |x-y| = |-(y-x)| = |y-x| = d(y, x) \qquad \forall\ x, y \in \text{IR}$

4) $d(x,y)= |x-y| = |x-z+z-y| \leq |x-z| + |z-y| = d(x, z)+d(z, y) \qquad \forall\ x, y, z \in \text{IR}$

Then (IR, d) is a metric space.

A ***norm*** on a vector space is a way of measuring distance between vectors.

Definition 1.11.: A ***norm*** on a linear space V over F is a function $\|\ .\ \| : V \to R$ with the properties that :

(1) $\| x \| \geq 0$ & $\| x \| = 0 \leftrightarrow x = 0$ (positive definite);

(2) $\| x + y \| \leq \| x \| + \| y \|$ for all $x, y \in V$ (triangle inequality);

(3) $\|\alpha x\| = |\alpha|\ \|x\|$ for all $x \in V$ and $\alpha \in F$.

In Definition 1.11(3) we are assuming that F is R or C and $|\ .\ |$ denotes the usual absolute value. If $\|\ .\ \|$ is a function with properties (2) and (3) only it is called a ***semi-norm***.

Definition 1.12. A ***normed linear space*** is a linear space V with a norm $\|\ .\ \|$ (sometimes we write $\|\ .\ \|_V$).

Theorem 1.13. If V is a normed space then:

1) $\| 0 \| = 0$

2) $\| x \| = \| -x \|$ for every $x \in V$.

3) $\| x-y \| = \| y-x \|$ for every $x \in V$.

4) $|\, \| x \| - \| y \| \,| \leq \| x-y \|$ for every $x \in V$.

Proof :

Properties (1), (2) and (3) conclude directly from the definition, to prove property (4):

$x = (x\text{-}y)+y$

$\| x \| = \| (x\text{-}y)+y \| \leq \| x\text{-}y \| + \| y \| \rightarrow \| \mathrm{x} \| - \| \mathrm{y} \| \leq \| \mathrm{x\text{-}y} \|$ ….(1)

Similarly:

$\| y \| - \| x \| \leq \| x\text{-}y \|$

$-(\| x \| - \| y \|) \leq \| x\text{-}y \| \rightarrow (\| x \| - \| y \|) \geq - \| x\text{-}y \|$ ….(2)

From (1) & (2), we get:

$-\| x\text{-}y \| \leq \| x \| - \| y \| \leq \| x\text{-}y \| \rightarrow | \| x \| - \| y \| | \leq \| x\text{-}y \|$

Examples 1.14.:- [H.W.6,7]

[1] The vector space V is normed v.s. with the norm $\| x \| = | x |$ for all $x \in V$.

Proof:

1) Since $| x | \geq 0 \rightarrow \| x \| \geq 0$.

2) $\| x \| = 0 \leftrightarrow | x | = 0 \leftrightarrow x = 0$

3) Let $x \in V$, $\alpha \in F$, then

$$\| \alpha x \| = | \alpha x | = |\alpha| \, | x | = | \alpha | \, \| x \|$$

1) Let $x, y \in V$, then:

$$\| x + y \| = | x + y | \leq | x | + | y | = \| x \| + \| y \|$$

[2] Let $V = R^n$ with the usual Euclidean norm

$$\| x \| = \| x \|_2 = \left(\sum_{j=1}^{n} | x_j |^2\right)^{1/2}$$

proof:

1) Since

$$x_j^2 \geq 0 \text{ for all } j = 1, 2, \ldots, n \rightarrow \| x \| \geq 0$$

2) $\| x \| = 0 \leftrightarrow (\sum_{j=1}^{n} | x_j |^2)^{1/2} = 0 \leftrightarrow \sum_{j=1}^{n} | x_j |^2 = 0$

$$\leftrightarrow x_j^2 = 0 \text{ for all } j = 1,2,\ldots,n \quad \leftrightarrow \quad x_j = 0 \text{ for all } j = 1,2,\ldots,n \quad \leftrightarrow x=0$$

3) Let $x \in \mathrm{IR}^n$, $\alpha \in \mathrm{IR}$:

$$\alpha x = \alpha\,(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$$

$$\| \alpha x \| = (\sum_{j=1}^{n} | \alpha x_j |^2)^{1/2} = | \alpha | (\sum_{j=1}^{n} | x_j |^2)^{1/2} = | \alpha | \, \| x \|.$$

4) Let $x,y \in \mathrm{IR}^n$:

$$x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

$$\| x+y \| = (\sum_{j=1}^{n} | x_j + y_j |^2)^{1/2}$$

By using *MinKowski's inquality* where p=2, we have:

$$\| x + y \| = (\sum_{i=1} | x_i + y_i |^2)^{1/2} \leq (\sum_{i=1} | x_i |^2)^{1/2} + (\sum_{i=1} | y_i |^2)^{1/2} = \| x \| + \| y \|$$

[3] There are many other norms on R^n, called the p-norms. For $1 \leq p < \infty$ defined by:

$$\| x \|_p = \left(\sum_{j=1}^{n} | x_j |^p\right)^{1/p}$$

Then $\| \, . \, \|_p$ is a norm on V (to check the triangle inequality use *MinKowski's Inequality*)

$$\left(\sum_{j=1}^{n} | x_j + y_j |^p\right)^{1/p} \leq \left(\sum_{j=1}^{n} | x_j |^p\right)^{1/p} + \left(\sum_{j=1}^{n} | y_j |^p\right)^{1/p}$$

[4] There is another norm corresponding to $p = \infty$, defined by:

$$\| x \|_\infty = \max_{1 \leq j \leq n} \{| x_j |\}$$

where $\| \, . \, \|: IR^n \rightarrow IR$ and $x = (x_1, ..., x_n)$.

proof:

1) Since $| x_i | \geq 0$ for all i=1, … , $n \rightarrow \| x \| \geq 0$.

2) $\| x \| = 0 \leftrightarrow \max \{| x_1|, \ldots , | x_n |\}=0 \leftrightarrow | x_i | = 0$ for all i=1, … , n

$\leftrightarrow x_i = 0$ for all i=1, … , $n \leftrightarrow x=0$

3) Let $x \in \mathrm{IR}^n$ and $\alpha \in \mathrm{IR}$, then

$$\alpha x = \alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$$

$$\| \alpha x \| = \max \{ | \alpha x_1 |, \ldots ,| \alpha x_n |\}$$

$$= \max \{ | \alpha | | x_1 |, \ldots , | \alpha | | x_n |\}$$

$$= | \alpha | \max \{ | x_1 |, \ldots , | x_n |\}$$

$$= | \alpha | \| x \|$$

4) Let $x, y \in \mathrm{IR}^n$

$$x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (, \ldots,)$$

$$\| x + y \| = \max \{ | x_1+y_1|, \ldots , | x_n+y_n | \}$$

$$\leq \max \{ | x_1| + | y_1|, \ldots , | x_n |+|y_n | \}$$

$$\leq \max \{ | x_1|, \ldots , | x_n | \} + \max \{ | y_1|, \ldots , | y_n | \}$$

$$= \| x \| + \| y \|$$

[5] Let X = C[a; b], and put $\| f \| = \sup_{t \in [a,b]} | f(t) |$ This is called the uniform or supremum norm.

proof :

1) Since $| f(t) | \geq 0$ for all $t \in$ [a, b] $\rightarrow \| f \| \geq 0$.

$\| f \| = 0 \leftrightarrow \sup_{t \in [a,b]} | f(t) | = 0 \leftrightarrow | f(t) | = 0$ for all $t \in$ [a, b]

$\leftrightarrow f(t) = 0$ for all $t \in$ [a, b] $\leftrightarrow f = 0$.

2) Let $f \in$ X, $\alpha \in$ IR, then:

$\| \alpha f \| = \sup\{ | \alpha f(t) | : t \in [a, b]\}$

$= \sup\{ | \alpha | | f(t) | : t \in [a, b]\}$

$= | \alpha | \sup\{ | f(t) | : t \in [a, b]\}$

$= | \alpha | \, \| f \|$.

3) $\| f + g \| = \sup\{ | (f + g)(t) | : t \in [a, b]\} = \sup\{ | f(t) + g(t) | : t \in [a, b]\}$

$\leq \sup\{ | f(t) | + | g(t) | : t \in [a, b]\}$

$\leq \sup\{ | f(t) | : t \in [a, b]\} + \sup\{ | g(t) | : t \in [a, b]\} = \| f \| + \| g \|$.

شكرا لاصغائكم