Theorem —The Comparison Test

Let $\sum a_n$, $\sum c_n$, *and* $\sum d_n$ be series with nonnegative terms. Suppose that for some integer *N*

$$d_n \le a_n \le c_n \quad \text{for all} \quad n > N.$$

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

Example:

(a) The series

$$\sum_{n=1}^{\infty} \frac{5}{5n - 1}$$

diverges because its nth term

$$\frac{5}{5n - 1} = \frac{1}{n - \frac{1}{5}} > \frac{1}{n}$$

is greater than the nth term of the divergent harmonic series.

(b) The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

converges because its terms are all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots.$$

The geometric series on the left converges and we have

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1 - (1/2)} = 3.$$

Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then Σa_n and Σb_n both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and Σb_n converges, then Σa_n converges.

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and Σb_n diverges, then Σa_n diverges.

Example:

Which of the following series converge, and which diverge?

(a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

(b) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

(c) $\frac{1 + 2\ln 2}{9} + \frac{1 + 3\ln 3}{14} + \frac{1 + 4\ln 4}{21} + \cdots = \sum_{n=2}^{\infty} \frac{1 + n\ln n}{n^2 + 5}$

Sol: We apply the Limit Comparison Test to each series.

(a) Let $a_n = (2n+1)/(n^2+2n+1)$. For large n, we expect a_n to behave like $2n/n^2 = 2/n$ since the leading terms dominate for large n, so we let $b_n = 1/n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2,$$

Σa_n diverges by Part 1 of the Limit Comparison Test. We could just as well have taken $b_n = 2/n$, but $1/n$ is simpler.

(b) Let $a_n = 1/(2^n - 1)$. For large n, we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

$$= \lim_{n\to\infty} \frac{1}{1 - (1/2^n)}$$

$$= 1,$$

$\sum a_n$ converges by Part 1 of the Limit Comparison Test.

(c) Let $a_n = (1 + n \ln n)/(n^2 + 5)$. For large n, we expect a_n to behave like $(n \ln n)/n^2 = (\ln n)/n$, which is greater than $1/n$ for $n \geq 3$, so we let $b_n = 1/n$. Since

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + n^2 \ln n}{n^2 + 5}$$

$$= \infty,$$

$\sum a_n$ diverges by Part 3 of the Limit Comparison Test. ■

The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converge, the converse is not true

Example:

(a) For $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

The original series converges because it converges absolutely.

(b) For $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, which contains both positive and negative terms, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with $\sum_{n=1}^{\infty} (1/n^2)$ because $|\sin n| \leq 1$ for every n. The original series converges absolutely; therefore it converges. ■

The Ratio Test

Let $\sum a_n$ be any series and suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$

Then (a) the series *converges absolutely* if $\rho < 1$, (b) the series *diverges* if $\rho > 1$ or ρ is infinite, (c) the test is *inconclusive* if $\rho = 1$.

Example

Investigate the convergence of the following series.

(a) $\sum_{n=0}^{\infty}\frac{2^n + 5}{3^n}$ (b) $\sum_{n=1}^{\infty}\frac{(2n)!}{n!n!}$ (c) $\sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}$

Sol:

(a) For the series $\sum_{n=0}^{\infty}(2^n + 5)/3^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3}\cdot\frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3}\cdot\left(\frac{2 + 5\cdot 2^{-n}}{1 + 5\cdot 2^{-n}}\right) \to \frac{1}{3}\cdot\frac{2}{1} = \frac{2}{3}.$$

The series converges absolutely (and thus converges) because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty}\frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty}\frac{5}{3^n} = \frac{1}{1-(2/3)} + \frac{5}{1-(1/3)} = \frac{21}{2}.$$

(b) If $a_n = \frac{(2n)!}{n!n!}$, then $a_{n+1} = \frac{(2n + 2)!}{(n + 1)!(n + 1)!}$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!}$$

$$= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \to 4.$$

The series diverges because $\rho = 4$ is greater than 1.

(c) If $a_n = 4^n n!n!/(2n)!$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}(n + 1)!(n + 1)!}{(2n + 2)(2n + 1)(2n)!}\cdot\frac{(2n)!}{4^n n!n!}$$

$$= \frac{4(n + 1)(n + 1)}{(2n + 2)(2n + 1)} = \frac{2(n + 1)}{2n + 1} \to 1.$$

Because the limit is $\rho = 1$, we cannot decide from the Ratio Test whether the series converges. When we notice that $a_{n+1}/a_n = (2n + 2)/(2n + 1)$, we conclude that a_{n+1} is always greater than a_n because $(2n + 2)/(2n + 1)$ is always greater than 1. Therefore, all terms are greater than or equal to $a_1 = 2$, and the nth term does not approach zero as $n \to \infty$. The series diverges. ■

The Root Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} = \sqrt[n]{|a_n|} = \rho.$$

Then (a) the series *converges absolutely* if $\rho < 1$, (b) the series *diverges* if $\rho > 1$ or ρ is infinite, (c) the test is *inconclusive* if $\rho = 1$.

Example:

Which of the following series converge, and which diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solution We apply the Root Test to each series, noting that each series has positive terms.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\displaystyle\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1^2}{2} < 1.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges because $\displaystyle\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\left(\sqrt[n]{n}\right)^3} \to \frac{2}{1^3} > 1.$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges because $\displaystyle\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1.$ ■

Alternating Series

A series in which the terms are alternately positive and negative is an alternating series.

Example:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots$$

Theorem

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Example

The series $$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

clearly satisfies the three requirements of Theorem with N = 1; it therefore converges.

1.6 Power Series and Convergence

DEFINITIONS **A power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty}c_nx^n = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots. \qquad (1)$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty}c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Example:

$$\sum_{n=0}^{\infty}x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \qquad (3)$$

Example:

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots \qquad (4)$$

matches Equation (2) with $a = 2, c_0 = 1, c_1 = -1/2, c_2 = 1/4, \ldots, c_n = (-1/2)^n$. This is a geometric series with first term 1 and ratio $r = -\frac{x-2}{2}$. The series converges for $\left|\frac{x-2}{2}\right| < 1$ or $0 < x < 4$. The sum is

$$\frac{1}{1-r} = \frac{1}{1 + \frac{x-2}{2}} = \frac{2}{x},$$

so

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \qquad 0 < x < 4.$$

Example:

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(c) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

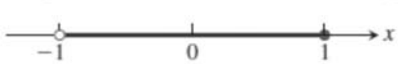
(d) $\sum_{n=0}^{\infty} n! x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Solution Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

(a) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x}\right| = \frac{n}{n+1}|x| \to |x|.$

The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$ because the nth term does not converge to zero. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$, we get $-1 - 1/2 -$

$1/3 - 1/4 - \cdots$, the negative of the harmonic series; it diverges. Series (a) converges for $-1 < x \le 1$ and diverges elsewhere.

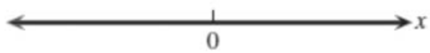


(b) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2 \to x^2.$ $\quad 2(n+1) - 1 = 2n + 1$

The series converges absolutely for $x^2 < 1$. It diverges for $x^2 > 1$ because the nth term does not converge to zero. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \le x \le 1$ and diverges elsewhere.

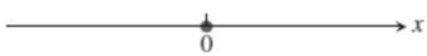
(c) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$ for every x. $\quad \frac{n!}{(n+1)!} = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)}$

The series converges absolutely for all x.



(d) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty$ unless $x = 0$.

The series diverges for all values of x except $x = 0$.



1.7 Taylor and Maclaurin Series

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

Example

Find the Taylor series generated by $f(x) = 1/x$ at a = 2. Where, if anywhere, does the series converge to 1 >x?

Solution We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$f(x) = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = 2!x^{-3}, \quad \cdots, \quad f^{(n)}(x) = (-1)^n n! x^{-(n+1)},$$

so that

$$f(2) = 2^{-1} = \frac{1}{2}, \quad f'(2) = -\frac{1}{2^2}, \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}, \quad \cdots, \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$. ■

Example:

Find the Taylor series generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$