Vector Analyses

By Dr. Zaynab Anwer and teacher Eman Abdulhadi

1. Vector-Valued Functions and Motion in Space

Lecture 1: Vector Functions and Space Curves

Definition of a Vector Function

A vector function r(t) is typically expressed in terms of its components:

r(t) = x(t)i + y(t)j + z(t)k

- x(t), y(t), z(t): Scalar functions of t (coordinate components).
- i, j, k: Unit vectors in the directions of the x-, y-, and z-axes, respectively.

For instance, $r(t) = t^2 i + sin(t)j + e^t k$ represents a vector function in 3D space, where the component functions t^2, sin(t), and e^t govern the motion along the respective axes.

Definition of Domain

If a vector function is defined as: r(t) = x(t)i + y(t)j + z(t)k

The domain of r(t) is the intersection of the domains of its components, x(t), y(t), and z(t).

Example of Domain

Find the domain of the vector function:

$$r(t) = ln(t)i + \sqrt{t}\, j + (\frac{1}{t})k$$

Solution:

1. The term ln(t) is defined only when $t > 0$.
2. The term sqrt(t) is defined only when $t \geq 0$.
3. The term 1/t is defined only when $t \neq 0$.

The domain of r(t) is the intersection of these constraints:

$t > 0$ (as it satisfies all three conditions)

Thus, the domain of r(t) is the interval $(0, \infty)$.

Limits of Vector Functions

If a vector function is defined as:

$r(t) = x(t)i + y(t)j + z(t)k$

Then the limit as t approaches c is given by:

lim_{t → c} r(t) = lim_{t → c} x(t)i + lim_{t → c} y(t)j + lim_{t → c} z(t)k

The limit exists if and only if the limits of all its components, x(t), y(t), and z(t), exist and are finite.

Examples of Limits

1. Evaluate $\lim_{t \to 2} r(t)$, where $r(t) = t^2 i + \sin(t)j + e^t k$:

$$\lim_{t \to 2} r(t) = 4i + \sin(2)j + e^2 k$$

2. Determine $\lim_{t \to \infty} r(t)$, where $r(t) = (1/t)i + (1/t^2)j + (1/e^t)k$:

$$\lim_{t \to \infty} r(t) = 0i + 0j + 0k$$

Continuity of Vector Functions

A vector function r(t) is continuous at a point t = c if:

1. r(c) is defined.
2. $\lim_{t \to c} r(t)$ exists.
3. $\lim_{t \to c} r(t) = r(c)$.

Example:

$$r(t) = \begin{cases} t^2 i + \sin(t)j + e^t k & t \le 1 \\ 2ti + t^2 j + \ln(t+1)k & t > 1 \end{cases}$$

To check continuity at t = 1:

Left-hand limit: $\lim_{t \to 1^-} r(t) = 1^2 i + \sin(1)j + e^1 k$.

Right-hand limit: $\lim_{t \to 1^+} r(t) = 2(1)i + 1^2 j + \ln(2)k$.

Since the left-hand and right-hand limits are not equal, r(t) is not continuous at t = 1.

Derivatives of Vector Functions

The derivative of a vector function r(t) is defined as:

r'(t) = $\lim_{h \to 0} \frac{[r(t+h) - r(t)]}{h}$

This derivative describes the instantaneous rate of change of r(t) with respect to t. The derivative measures how the vector r(t) changes direction and magnitude as t varies.

In component form:
r'(t) = x'(t)i + y'(t)j + z'(t)k

The derivative of a vector function is also a vector function, describing how each component changes with respect to the parameter t.

Example of Derivative
Find the derivative of the vector function:

$$r(t) = t^2\, i + sin(t)\, j + e^t\, k$$

Solution:

Differentiate each component with respect to t:
1. $dx(t)/dt = 2t$
2. $dy(t)/dt = cos(t)$
3. $dz(t)/dt = e\hat{}t$
Thus, the derivative is: $r'(t) = 2t\, i + cos(t)\, j + e^t\, k$

H.W. (1)

Exercises

1. Find the domain of $r(t) = (t^2)i + ln(t)j + (\frac{1}{t})k$.
2. Determine the domain of $r(t) = \sqrt{t+1}\, i + (\frac{1}{t-2})j + e^t k$.
3. Verify the domain of $r(t) = t^3 i + (ln(t-3))j + \sqrt{(t-4)}k$.

Exercises

1. Evaluate $\lim_{t \to 0} r(t)$, where $r(t) = ln(t+1)i + t^2 j + \sqrt{t}\, k$.
2. Determine if r(t) = (1/t)i + tj + e^tk is continuous at t = 1.
3. Sketch the curve described by r(t) = cos(t)i + sin(t)j + tk.
4. Verify if r(t) = t^3i + t^2j + tk is continuous for all values of t.

Exercises

1. Find r'(t) for r(t) = t^3 i + t^2 j + t k.
2. Determine r'(t) for r(t) = ln(t)i + sqrt(t)j + e^(2t)k.
3. Compute r'(t) for r(t) = sin(t)i + cos(t)j + t^4k.

Definition of Velocity Vector, Speed and acceleration vector

If r(t) = x(t)i + y(t)j + z(t)k is the position vector of the particle moving along a smooth curve in space, then

Then the velocity vector is the derivative of r(t): v(t) = r'(t) = dx(t)/dt i + dy(t)/dt j + dz(t)/dt k

At any time t, the direction of **v** is the **direction of motion**, the magnitude of **v** is the particle's **speed,** and the derivative a = dv>dt, when it exists, is the particle's acceleration vector. In summary:

1. Velocity is the derivative of position: $\mathbf{v} = \frac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

Example 1: Straight-Line Motion

Given the position function: r(t) = 5t^2i Find the velocity, speed, and acceleration

Solution:

1. Velocity: v(t) = dr(t)/dt = 10t i
2. Speed: ||v(t)|| = |10t| = 10|t|
3. Acceleration: a(t) = dv(t)/dt = 10 i
4. The direction of motion at time t: $\frac{v}{|v|} = \frac{10\text{t i}}{10|\text{t}|} = \pm i$

Example 2: Circular Motion in 2D

Let the position function be: r(t) = cos(t) i + sin(t) j Find the velocity, speed, and acceleration.

Solution:

1. Velocity: v(t) = dr(t)/dt = -sin(t) i + cos(t) j
2. Speed: $||v(t)|| = \sqrt{(-sin(t))^2 + (cos(t))\text{^}2} = \sqrt{1} = 1$
3. Acceleration: $a(t) = dv(t)/dt = -cos(t)\, i - sin(t)\, j$

Example 3: Motion in 3D

Given: r(t) = t^2 i + e^t j + ln(t) k Find the velocity, speed, and acceleration for t > 0.

Solution:

1. Velocity: v(t) = dr(t)/dt = 2t i + e^t j + 1/t k
2. Speed: ||v(t)|| = sqrt((2t)^2 + (e^t)^2 + (1/t)^2)
3. Acceleration: a(t) = dv(t)/dt = 2 i + e^t j - 1/t^2 k

Example 4: Helical Motion

Given the position: r(t) = cos(t) i + sin(t) j + t k, Find the velocity, speed, and acceleration.

Solution:

1. Velocity:
v(t) = dr(t)/dt = -sin(t) i + cos(t) j + k

2. Speed: $||v(t)|| = \sqrt{(-sin(t)\,)^2 \;+\; (cos(t))^2 + 1^2} \;=\; sqrt(2)$

3. Acceleration: $a(t) \;=\; dv(t)/dt \;=\; -cos(t)\; i \;-\; sin(t)\; j$

Differentiation Rules for Vector Functions

Let u and v be differentiable vector functions of t, C a constant vector, c any scalar, and f any differentiable scalar function

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Example 1: Vector Functions of Constant Length

$$\begin{aligned}
\mathbf{r}(t) \cdot \mathbf{r}(t) &= c^2 \\
\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] &= 0 \\
\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) &= 0 \\
2\mathbf{r}'(t) \cdot \mathbf{r}(t) &= 0.
\end{aligned}$$

The vectors r′(t) and r(t) are orthogonal because their dot product is 0.

Example 2: Derivative of a Dot Product

Let $u(t) \;=\; (t, sin(t), e^t) \;and\; v(t) \;=\; (cos(t), t^2, ln(t+1))$. We aim to compute the derivative of their dot product:

$$\frac{d}{dt}\,(u(t).v(t)) \;=\; (\cos(t) + \;\cos(t)t^2 \;+\; e^t \ln(t+1)) \;+ (\text{-}t\sin(t) + \;2t\sin(t) \;+ \frac{e^t}{t+1})$$

Example 3:

Prove that: $\frac{d}{dt}\left(v \cdot \left(\frac{dv}{dt} \times \frac{d^2v}{dt^2}\right)\right) = v \cdot \left(\frac{dv}{dt} \times \frac{d^3v}{dt^3}\right)$

$$\frac{d}{dt}\left(v \cdot \left(\frac{dv}{dt} \times \frac{d^2v}{dt^2}\right)\right) = \left(\frac{dv}{dt}\right) \cdot \left(\frac{dv}{dt} \times \frac{d^2v}{dt^2}\right) + v \cdot \frac{d}{dt}\left(\frac{dv}{dt} \times \frac{d^2v}{dt^2}\right)$$

The first term becomes:

$$\left(\frac{dv}{dt}\right) \cdot \left(\frac{dv}{dt} \times \frac{d^2v}{dt^2}\right) = 0$$

This is because the cross product of any vector with itself is perpendicular to that vector, and the dot product of perpendicular vectors is zero.

$$\frac{d}{dt}\left(\frac{dv}{dt} \times \frac{d^2v}{dt^2}\right) = \left(\frac{d^2v}{dt^2}\right) \times \left(\frac{d^2v}{dt^2}\right) + \left(\frac{dv}{dt}\right) \times \left(\frac{d^3v}{dt^3}\right)$$

The first term is zero because the cross product of any vector with itself is zero: $(d^2v/dt^2) \times (d^2v/dt^2) = 0$

Thus, $\frac{d}{dt}\left(v \cdot \left(\frac{dv}{dt} \times \frac{d^2v}{dt^2}\right)\right) = v \cdot \left(\frac{dv}{dt} \times \frac{d^3v}{dt^3}\right)$

Integrals of Vector Functions

For a vector-valued function: $r(t) = x(t)i + y(t)j + z(t)k$

The integral is computed as: $\int r(t)\,dt = (\int x(t)\,dt)\,i + (\int y(t)\,dt)\,j + (\int z(t)\,dt)\,k + C$

Here, C is the constant vector of integration.

Example 1: Integration of a Vector Function

To integrate a vector function, we integrate each of its components. Consider the vector function:

$\int ((\cos t)i + j - 2t\,k)\,dt$

We compute the integral component-wise:

$\int \cos t\,dt\,i + \int dt\,j - \int 2t\,dt\,k$

Evaluating each component:

$(\sin t + C_1)\,i + (t + C_2)\,j - (t^2 + C_3)\,k$

Combining the constants into a single constant vector C :

$\int ((\cos t)i + j - 2t\,k)\,dt = (\sin t)i + t\,j - t^2\,k + C$

Example 2: Definite Integral of a Vector Function

We compute the definite integral of the same vector function over the interval $[0, \pi]$:

$\int_0^{\pi}((\cos t)i + j - 2t\, k)\ dt$

Component-wise integration yields: $\int_0^{\pi} \cos t\, dt\, i + \int_0^{\pi} dt\, j - \int_0^{\pi} 2t\, dt\, k$

Evaluating the integrals: $([\sin t]_0^{\pi})\, i + ([t]_0^{\pi})\, j - ([t^2]_0^{\pi})\, k$

Substituting the limits: $(0 - 0)\, i + (\pi - 0)\, j - (\pi^2 - 0)\, k = \pi\, j - \pi^2\, k$

Example 3: Finding the Position Function

Suppose a hang glider has an acceleration vector given by: a(t) = -(3 cos t)i - (3 sin t)j + 2k

The initial velocity are: v(0) = 3j and r(0) = 4i. We aim to find the position function r(t) .

Step 1: Find the velocity function by integrating acceleration:

$v(t) = \int a(t)\, dt = -(3 \sin t)i + (3 \cos t)j + 2t\, k + C_1$

Using the initial condition (v(0) = 3j): $3j = -(3 \sin 0)i + (3 \cos 0)j + 0k + C_1$

Solving for C_1 , we find $C_1 = 0$.

Step 2: Find the position function by integrating velocity:

$r(t) = \int v(t)\, dt = (3 \cos t)i + (3 \sin t)j + t^2k + C_2$

Using the initial condition r(0) = 4i : $4i = (3 \cos 0)i + (3 \sin 0)j + (0)k + C_2$

Solving for C_2 , we find $C_2 = i$.

The position function is: $r(t) = (1 + 3 \cos t)i + (3 \sin t)j + t^2k$

H.W.(2)

1. Let $F(t) = [t^2, e^t, sin(t)]$. Compute F'(t).

2. Let $A(t) = [t, t^2, e^t]$ and $B(t) = [cos(2t), \sin^{-1}(t), t^3]$. Compute $\frac{d}{dt}(A(t) \cdot B(t))$

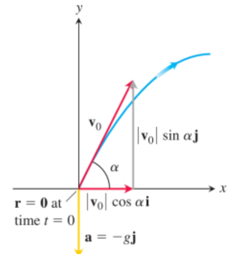
3. Let $A(t) = [t^2, e^t, t]$ and $B(t) = [sin(t), cos(t), t^3]$. Compute $\frac{d}{dt}(A(t) \times B(t))$.

4.A particle's position is given by r(t) = [t^2, ln t, sin(t)]. (a) Compute the velocity vector v(t). (b) Compute the speed (d) Compute the acceleration vector a(t) and find its magnitude.

5. Compute v(t), speed, and a(t) for $r(t) = ln(t)i + \sqrt{(t)}j + e^{2t}k$.

Projectile Motion

Projectile motion describes the motion of an object under the influence of gravity when launched with an initial velocity. It is analyzed in two dimensions: horizontal (x) and vertical (y).

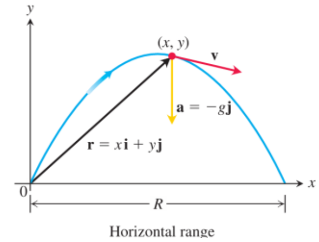


Equations of Motion

1. Initial Velocity (v_0): The initial velocity is split into horizontal (v_{0x}) and vertical (v_{0Y}) components: $v_{0x} = v_0 \cos(\theta)$, $v_{0Y} = v_0 \sin(\theta)$ -----------(1)

2. Forces Acting: The only force acting on the object is gravity, which causes a constant vertical acceleration (g) downward. There are no horizontal forces (ignoring air resistance)

3. Horizontal motion (x):



- The horizontal acceleration is zero ($a_x = 0$).
- The horizontal velocity is constant ($v_x = v_{0x}$).
- The horizontal position is given by integrate v_x : $x(t) = v_{0x}\, t$

$$t = \frac{x}{v_{0x}} - - - - (2)$$

4. Vertical motion (y):

- The vertical acceleration is $a_y = -g$ (downward due to gravity).
- The vertical velocity changes over time: $v_y(t) = v_{0y} - g\,t$.
- The vertical position is given by: $y(t) = v_{0y}t - \frac{1}{2} g\, t^2$. ---------------(3)

5. Trajectory of the Projectile: The path of the projectile is parabolic and described by the equation:

From the equations (1), (2) and (3) we get $y = x\, tan(\theta) - \frac{1}{2} g \left(\frac{x}{v_{0x}}\right)^2 = x\, tan(\theta) - \frac{g x^2}{2\, v_{0x}}$

6. Height, Flight Time, and Range for Ideal Projectile Motion:

Time of Flight: T = $\frac{2v_{0y}}{g} = \frac{2v_0 \sin\theta}{g}$

Proof: since at maximum height $v_y(t) = v_{0y} - gt = 0$ ➔ $t = \frac{v_{0y}}{g}$➔ the time is $2t = \frac{2v_{0y}}{g}$

Maximum Height: H = $\frac{v_{0y}^2}{2g} = \frac{(v_0 sin(\theta))^2}{2g}$

Range: $R = \frac{v_0^2 sin(2\theta)}{g}$

Example

A ball is launched with an initial velocity v_0 = 20 m/s at an angle of θ = 45°. Find: 1) the time of flight, 2) the maximum height, and 3) the range.

Solution

1. Time of Flight: T = $\frac{2v_{0y}}{g} = \frac{2v_0 \sin\theta}{g}$

Substituting v_0 = 20, sin(45°) = √2 / 2, and g = 9.8: T = (2 (20 · √2 / 2)) / 9.8 = (20 √2) / 9.8 ≈ 2.89 seconds

Maximum Height: H = $\frac{v_{0y}^2}{2g} = \frac{(v_0 sin(\theta))^2}{2g}$

Substituting v_0 = 20, sin(45°) = √2 / 2, and g = 9.8: H = ((20 · √2 / 2)²) / (2 · 9.8) = (100) / 19.6 ≈ 5.1 m

3. Range: $R = \frac{v_0^2 sin(2\theta)}{g}$

Using $\sin(2\theta) = \sin(90°) = 1$: $R = (20^2) / 9.8 = 400 / 9.8 \approx 40.8$ m

Example 2:
A projectile is launched with an initial velocity $v_0 = [10, 20]$ m/s. Gravity is $g = 9.8$ m/s^2. Find the position r(t) after t seconds and at t=2.

Solution:

1. Horizontal motion (x-component):

$x(t) = v_{0x}\, t = 10\, t$.

2. Vertical motion (y-component):

$y(t) = v_{0Y}\, t - (1/2)\, g\, t^2 = 20\, t - 4.9\, t^2$.

3. Position vector:

$r(t) = [x(t), y(t)] = [10t, 20t - 4.9t^2]$.

At t = 2 s:

$x(2) = 10(2) = 20$, $y(2) = 20(2) - 4.9(2)^2 = 20.4$.

$r(2) = [20, 20.4]$ m.

Exercises

1. A projectile is launched with an initial velocity v0 = [15, 25] m/s. Assuming $g = 9.8$ m/s^2:
a) Write the parametric equations for the horizontal and vertical positions x(t) and y(t).
b) Find the maximum height of the projectile.
c) Find the time of flight (when the projectile hits the ground).
2. . A projectile is launched from the origin with velocity v0 = [10, 30] m/s.
a) Find the equation of the trajectory y(x).
b) What is the range of the projectile (horizontal distance traveled when it hits the ground)?
3. Find the time of flight and final position:
A ball is thrown from a height of 5 m with an initial velocity v0 = [12, 20] m/s:
a) How long does it take for the ball to hit the ground?
b) Where does it land?