2.7 The Cross Product

The Cross products are widely used to describe the effects of forces in studies of electricity, magnetism, fluid flows, and orbital mechanics.

We start with two nonzero vectors ***u*** and ***v*** in space. If ***u*** and ***v*** are not parallel, they determine a plane. We select a unit vector ***n*** perpendicular to the plane by the right-hand rule. This means that we choose ***n*** to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle from ***u*** to ***v*** (following Figure). Then the cross product **u** × **v** ("**u cross v**") is the vector defined as follows:

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\mathbf{n}$$

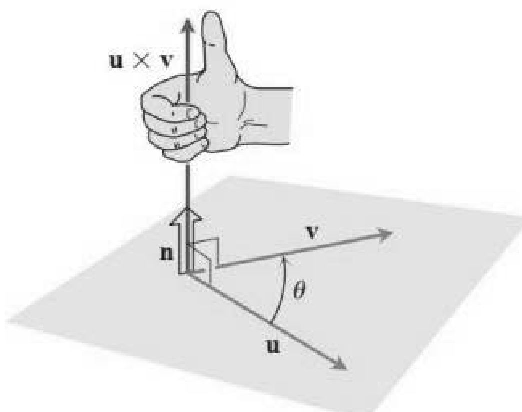


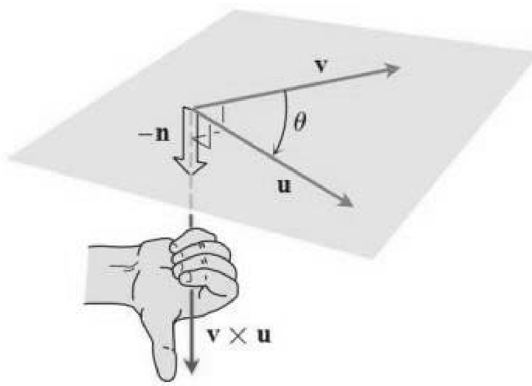
Figure 1.11: The construction **u** × **v**.

Unlike the dot product, the cross product is a vector. For this reason, it is also called the vector product of ***u*** and ***v*** and applies only to vectors in space. The vector is orthogonal to both ***u*** and ***v*** because it is a scalar multiple of ***n***.

Parallel Vectors: Nonzero vectors ***u*** and ***v*** are parallel if and only if ***u*** × ***v*** = 0.

Properties of the Cross Product: If ***u***, ***v***, ***w*** are any vectors and *r, s* are scalars, the

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$



visualizes property 4.

When we apply the definition to calculate the pairwise cross products of ***i***, ***j***, and ***k***, we find

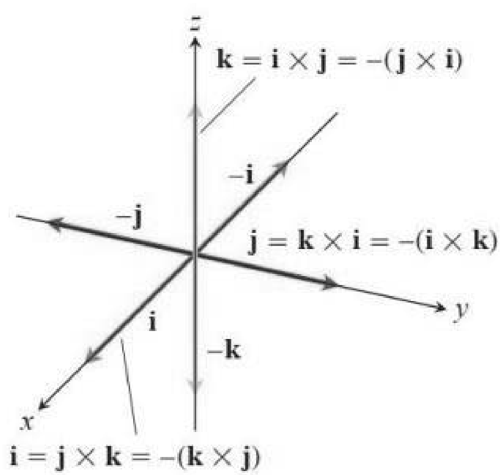
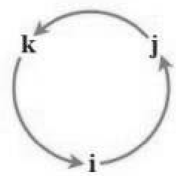


Figure : The pairwise cross product of ***i***, ***j***, and ***k***.

and

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$
$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$



Calculating Cross Product Using Determinants:

If $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

This is the area of the parallelogram determined by $\boldsymbol{u}$ and $\boldsymbol{v}$ (Figure), $|\mathbf{u}|$ being the base of the parallelogram and $|\mathbf{v}||\sin\theta|$ the height. Because $\boldsymbol{n}$ is a unit vector, the area of a parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin\theta||\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta$$

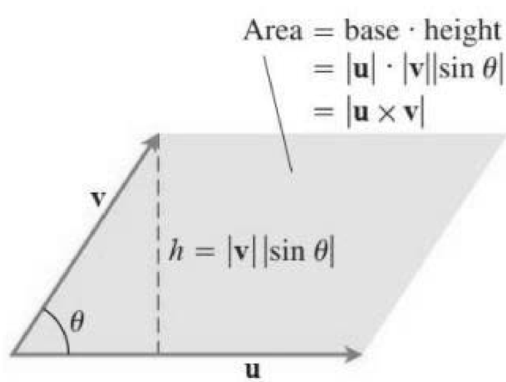


Figure : The parallelogram is determined by $\boldsymbol{u}$ and $\boldsymbol{v}$.

Example : Find ***u*** × ***v*** and ***v*** × ***u*** if ***u*** = 2***i*** + ***j*** + ***k*** and ***v*** = −4i + 3j + k

Solution:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

Example 9: Find unit vector orthogonal to the vectors ***u*** = 3***i*** − ***j*** + ***k*** and ***v*** = 4***j*** + ***k***

Solution:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 4 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix}\mathbf{k} = -5\mathbf{i} - 3\mathbf{j} + 12\mathbf{k}$$

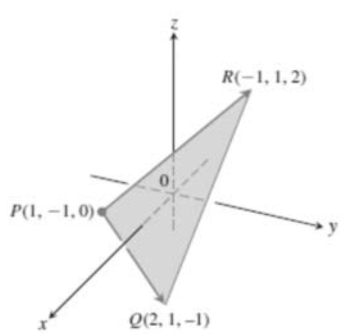
$$|\mathbf{u}\times\mathbf{v}| = \sqrt{(-5)^2 + (-3)^2 + (12)^2} = \sqrt{178}$$

$$\mathbf{z} = \frac{1}{\sqrt{178}}(-5\mathbf{i} - 3\mathbf{j} + 12\mathbf{k})$$

Example

Let P(1, -1, 0), *Q*(2, 1, -1), and *R*(-1, 1, 2) (see the Figure).

a) Find a vector perpendicular to the plane of them
b) Find the area of the triangle with vertices *P*, *Q*, and *R*.
c) Find a unit vector perpendicular to the plane of them



a)

Solution The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$ ■

b)

Solution The area of the parallelogram determined by P, Q, and R is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6\mathbf{i} + 6\mathbf{k}|$$

$$= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}.$$

The triangle's area is half of this, or $3\sqrt{2}$.

c)

Solution Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction $\mathbf{n}$ is a unit vector perpendicular to the plane. Taking values from Examples 2 and 3, we have

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$ ■

2.8 Triple Scalar or Box Product

The product (u × v) .w is called the triple scalar product of u, v, and w (in that order). As you can see from the formula

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}||\mathbf{w}||\cos\theta|,$$

the absolute value of this product is the volume of the parallelogram-sided box determined by u, v, and w. The number |u × v| is the area of the base parallelogram. The number |w| |cos θ| is the parallelogram's height.

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example

Find the volume of the box determined by u = i + 2j - k, v = -2i + 3k, and w = 7j - 4k.

Solution

Using the rule for calculating a 3×3 determinant, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = (1)\begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - (2)\begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed. ■