Example:

Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find its limit.

Solution The limit leads to the indeterminate form 1^∞. We can apply l'Hôpital's Rule we first change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n\ln\left(\frac{n+1}{n-1}\right).$$

Then,

$$\begin{aligned} \lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} n\ln\left(\frac{n+1}{n-1}\right) && \infty \cdot 0 \text{ form} \\ &= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} && \frac{0}{0} \text{ form} \\ &= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} && \text{L'Hôpital's Rule: differentiate numerator and denominator.} \\ &= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2. \end{aligned}$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 4 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to e^2.

Theorem 6:

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Example:

These are examples of the limits in Theorem 6

(a) $\frac{\ln(n^2)}{n} = \frac{2 \ln n}{n} \to 2 \cdot 0 = 0$ Formula 1

(b) $\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \to (1)^2 = 1$ Formula 2

(c) $\sqrt[n]{3n} = 3^{1/n}(n^{1/n}) \to 1 \cdot 1 = 1$ Formula 3 with $x = 3$ and Formula 2

(d) $\left(-\frac{1}{2}\right)^n \to 0$ Formula 4 with $x = -\frac{1}{2}$

(e) $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \to e^{-2}$ Formula 5 with $x = -2$

(f) $\frac{100^n}{n!} \to 0$ Formula 6 with $x = 100$

1.5 Infinite Series:

DEFINITIONS Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
&\vdots \\
s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\
&\vdots
\end{aligned}
$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Example

The series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$s_1 = 1 \qquad = 1$

$s_2 = 1 + \frac{1}{2} \qquad = \frac{3}{2}$

$s_3 = 1 + \frac{1}{2} + \frac{1}{4} \qquad = \frac{7}{4}$

$\vdots$

$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \qquad = \frac{(2^n - 1)}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$

Then the sequence of partial sum converge to 2, hence the sum of infinite series is 2

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots, \qquad r = 1/2, a = 1$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots. \qquad r = -1/3, a = 1$$

Remark:

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example:

The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

Example:

Show that the folloeing series is converge:

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1-r} = \frac{5}{1 + (1/4)} = 4.$$

The nth-Term Test for a Divergent Series:

Theorem :

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Remark

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Example:

The following are all examples of divergent series.

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$. $\lim_{n\to\infty} a_n \neq 0$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist.

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

Theorem :

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum k a_n = k \sum a_n = kA$ (any number k).

Remark

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

Caution Remember that $\sum(a_n + b_n)$ can converge when $\sum a_n$ and $\sum b_n$ both diverge. For example, $\sum a_n = 1 + 1 + 1 + \cdots$ and $\sum b_n = (-1) + (-1) + (-1) + \cdots$ diverge, whereas $\sum(a_n + b_n) = 0 + 0 + 0 + \cdots$ converges to 0.

Example

Find the sums of the following series.

(a)
$$\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \qquad \text{Difference Rule}$$

$$= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/6)} \qquad \text{Geometric series with } a = 1 \text{ and } r = 1/2, 1/6$$

$$= 2 - \frac{6}{5} = \frac{4}{5}$$

(b)
$$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n} \qquad \text{Constant Multiple Rule}$$

$$= 4\left(\frac{1}{1 - (1/2)}\right) \qquad \text{Geometric series with } a = 1, r = 1/2$$

$$= 8$$

Remark

We can write the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

as

$$\sum_{n=0}^{\infty} \frac{1}{2^n}, \qquad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \qquad \text{or even} \qquad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}.$$

The partial sums remain the same no matter what indexing we choose to use.