تحليل دالي

الفصل الاول / المحاضرة الاولى

Functional Analysis

What is Functional Analysis?

Functional analysis represents one of the most important branches of mathematical sciences. Together with abstract algebra and mathematical logics it serves as a foundation of many other branches of mathematics.

Functional analysis is, in particular, widely used in probability theory and random functions theory and their numerous applications. Functional analysis serves also as a powerful tool in modern control and information sciences. The main subject of mathematical analysis represents scalar and finite-dimensional vector functions of scalar or finite-dimensional vector variables. Functional analysis is studying more general functions whose arguments and values may be the elements of any sets. While studying functions in mathematical analysis and linear algebra geometrical presentations are widely used; a function is considered as the mapping of one finite-dimensional space into another finite-dimensional space.

For instance, the scalar function of one scalar variable represents the mapping of the real axis R into the real axis R. The scalar function of two (three) scalar variables represents the mapping of the plane R^2 (the three-dimensional space R^3 respectively) into R. While studying more general functions whose arguments and values may be the elements of any sets wonderful analogies appear between many properties of functions and the visual geometric properties of more simple functions.

You meet such analogies in linear algebra where the spaces of any finite dimensions are considered (the n-dimensional spaces R^n at any finite n). In particular, the properties of linear functions in R^n are absolutely identical with the properties of linear functions in one-, two-- and three-dimensional spaces. These properties of functions caused the generalization of the notion of a space and wide application of intuitive geometrical presentations and geometrical terminology while studying any functions.

Functional analysis was born in the works of Italian mathematician Vito Volterra (Volterra 1913, Volterra and Peres 1935). He was the first who considered functions as the points of some space. The spaces whose points are functions are called function spaces.

Volterra defined also a real function whose argument represents the set of all the values of a continuous function in the interval [a, b]. Such a function he called a functional. This was the reason to call the branch of mathematics studying functionals a functional analysis.

It is worthwhile to recall that long before Volterra some functionals where considered by great Euler who created calculus of variations, though he did not use the term "functional".

Primarily functional served as the main object of study in functional analysis. In further development the notion of a function was essentially generalized. Respectively the range of interests of functional analysis was considerably extended. So, the object of functional analysis represents now the study of functions whose arguments and values may be the elements of any sets which are usually called spaces.

In this course we studied the following subjects:

1- Vector Spaces: Finite and Infinite Dimentional, Metric Spaces, Norms & Normed Spaces.

2- Banach Spaces: Some Important Inequalities(Cauchy, Holder and Minkowski's inequalities), Examples of Banach Spaces, Quotient Space of a Normed Linear Space, Continuous and Bounded Linear Transformations, Norm of Bounded Linear Transformations, Linear Operator on a Normed Space. Equivalent Norms, Continuous Linear Functional, Dual Spaces, The Hahan-Banach Theorem.

3- Hilbert Spaces: Definitions, Pre-Hilbert Spaces, Chauchy- Schwarz Inequality, orthogonal, Gram- Schmidt Theorem.

Chapter One: Vector Space

Definition 1.1.

A ***vector space*** *over* F is a non-empty set V together with two functions, one from V x V to V, and the other from F x V to V, denoted by $x + y$ and αx respectively, for all $x, y \in V$ and $\alpha \in$F, such that, for any $\alpha, \beta \in$ F and any $x, y, z \in V$,

(a) $x+y=y+x,\ x+(y+z)=(x+y)+z;$

(b) there exists a unique $0 \in V$ (independent of x) such that $x + 0 = x;$

(c) there exists a unique $-x \in V$ such that $x + (-x) = 0$;

(d) $1x = x,\ \alpha(\beta x) = (\alpha\beta)x;$

(e) $\alpha(x + y) = \alpha x + \alpha y,\ (\alpha + \beta)x = \alpha x + \beta x.$

If F = R (respectively, F = C) then V is a *real* (respectively, *complex)* vector space. Elements of F are called *scalars,* while elements of V are called *vectors*. The operation $x + y$ is called *vector addition,* while the operation ax is called *scalar multiplication*.

Some important inequalities

- ***Holder's inequality*** : if $p,\, q \in \mathrm{IR}$ such that $\frac{1}{p}+\frac{1}{q}=1$, then

$$\sum_{i=1} |x_i y_i| \leq (\sum_{i=1} |x_i|^p)^{1/p} (\sum_{i=1} |y_i|^q)^{1/q}$$

- If $p=2$ then $q=2$ and:

$$\sum_{i=1} |x_i y_i| \leq (\sum_{i=1} |x_i|^2)^{1/2} (\sum_{i=1} |y_i|^2)^{1/2}$$

and is called ***Cauchy - Schwar's inquality***.

- ***MinKowsk's inquality***: if $p \geq 1$, then:

$$(\sum_{i=1} |x_i + y_i|^p)^{1/p} \leq (\sum_{i=1} |x_i|^p)^{1/p} + (\sum_{i=1} |y_i|^p)^{1/p}$$

Example 1.2. [H.W.2-6]

[1] $S=\{x=(\alpha_n)_{n=1}^{\infty}:\alpha_n\in R \text{ or } C,\forall n\}$ is a vector space over R or C (sequence space).

Sol.

[1] Let $x=(\alpha_n)_{n=1}^{\infty}$ $y=(\beta_n)_{n=1}^{\infty}\in S$, λ is a scalar, then

$$x+y=(\alpha_n)_{n=1}^{\infty}+(\beta_n)_{n=1}^{\infty}=(\alpha_n+\beta_n)_{n=1}^{\infty}\in S$$

$$\lambda(\alpha_n)_{n=1}^{\infty}=(\lambda\alpha_1,\lambda\alpha_2,\ldots,\lambda\alpha_n,\ldots.)=(\lambda\alpha_n)_{n=1}^{\infty}\in S$$

Definition 1.3

Let V be a vector space. A non-empty set $U\subset V$ is a *linear subspace* of V if U is itself a vector space (with the same vector addition and scalar multiplication as in V). This is equivalent to the condition that:

$\alpha x+\beta y\in U$, for all $\alpha,\beta\in F$ and $x, y\in U$

(which is called the *subspace test).*

Example 1.4.

[1]The set of vectors in R^n of the form $(x_1, x_2, x_3, 0,\ldots, 0)$ forms a three-dimensional linear subspace.

[2] The set of polynomials of degree $\leq r$ forms a linear subspace of the set of polynomials of degree $\leq n$ for any $r \leq n$.

Definition 1.5. *Linear independence and dependence* of a given set M of vectors $x_1, \ldots, x_r$ $(r \geq 1)$ in a vector space V are defined by means of the equation:

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_r x_r = 0 \quad \ldots. \; (*)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are scalars. Clearly, equation (*) holds for $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$. If this is the only r-tuple of scalars for which (*) holds, the set M is said to be *linearly independent,* M is said to be *linearly dependent* if M is not linearly independent, that is , if (*) also holds for some r-tuple of scalars, not all zero.

Definition 1.6.: Let V be a vector space over a field F, $x \in V$ is called linear combination of $x_1, x_2, \ldots, x_n \in V$

if $x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = \sum_{i=1}^{n} \lambda_i \alpha_i, \; \lambda_i \in F, \quad 1 \leq i \leq m.$

Definition 1.7.: Let V be a vector space over a field F, and let S={ $x_1, x_2, \ldots, x_n$ }$\subseteq$V, S is said to be ***generated*** V if

$$x = \sum_{i=1}^{n} \lambda_i \alpha_i, \ \forall x_i \in S, \lambda_i \in \mathrm{F}, \ \ 1 \le i \le m.$$

Definition 1.8.: Let V be a vector space over a field F, and A be a non-empty subset of V ($\phi \neq A \subseteq V$),

A is said to be ***basis*** of V if :

- A linearly independent set.
- A generated V.

Definition 1.9. A vector space V is said to be *finite dimensional* if there is a positive integer n such that X contains a linearly independent set of n vectors whereas any set of n+1 or more vectors of X is linearly dependent. n is called the dimension of X, written n=dim X. By definition, X={0} is finite dimensional and dim X=0. If X is not finite dimensional, it is said to be infinite dimensional.

Examples 1.10.: dim R=1, dim R^2=2, dim R^n=n .

Remarks

1. Let V(F) be a finite dimensional V.S. over a field F, and let w subspace of V(F), then dim W $\leq$ dim V, If dim W=dim V then W=V.
2. Let ($\phi \neq S \subseteq V$) then if $0 \in S$ then S is linear dependent subspace.
3. The singleton $\{x\}$ is linear dependent iff $x \neq 0$.
4. Any subset of linear dependent set is linear dependent.
5. Any set containing a linearly dependent subset is linearly dependent too.