S_1.. Linear Transformations التحويلات الخطية

Def :- let V and U be vector spaces over the field K . if $L : V \Longrightarrow U$ is function from V to U , then we say that L is ***linear transformation*** if
(1) for all $U, V \in V \Longrightarrow L(U+V) = L(U) + L(V)$
(2) for $k \in K$, $U \in V \Longrightarrow L(kU) = kL(U)$

ملاحظة :-
ان عملية الجمع U + V خاصة بالفضاء V بينما عملية الجمع L (U) + L (V) هي خاصة بالفضاءU وكذلك عملية الضرب .

Ex:- show that $L : R^3 \Longrightarrow R^2$ be defined by $L(U_1, U_2, U_3) = (U_1, U_2)$ is ***linear transformation***

Sol :-
(1) let $U, V \in R^3 \Longrightarrow U = (U_1, U_2, U_3)$ and $V = (V_1, V_2, V_3)$ by def

$$
\begin{aligned}
L(U+V) &= L((U_1, U_2, U_3) + (V_1, V_2, V_3)) \\
&= L((U_1+V_1, U_2+V_2, U_3+V_3)) \quad \text{(by sum of vector)} \\
&= ((U_1+V_1), (U_2+V_2)) \quad \text{(by def of L)} \\
&= (U_1, U_2) + (V_1, V_2) \\
&= L(U_1, U_2, U_3) + L(V_1, V_2, V_3) \\
&= L(U) + L(V)
\end{aligned}
$$

(2) let $k \in R$

$$
\begin{aligned}
L(kU) &= L(k(U_1, U_2, U_3)) \\
&= L(kU_1, kU_2, kU_3) \\
&= (kU_1, kU_2) \\
&= k(U_1, U_2) \\
&= kL(U_1, U_2, U_3) \\
&= kL(U)
\end{aligned}
$$

by (1) and (2)
therefore , **L** is ***linear transformation***

Ex :- (2) Let $L : R^3 \Longrightarrow R^2$ be defined by

$L(U_1, U_2, U_3) = (U_1, U_1+U_2+U_3)$. show that L is linear transformation
Sol :- (1) let $U, V \in R^3$

$$
\begin{aligned}
L(U+V) &= L((U_1, U_2, U_3) + (V_1, V_2, V_3)) \\
&= L(U_1+V_1, U_2+V_2, U_3+V_3) \\
&= ((U_1+V_1), (U_1+V_1) + (U_2+V_2) + (U_3+V_3)) \\
&= (U_1+V_1, (U_1+U_2+U_3) + (V_1+V_2+V_3)) \\
&= (U_1, U_1+U_2+U_3) + (V_1, V_1+V_2+V_3) \\
&= L(U_1, U_2, U_3) + L(V_1, V_2, V_3) \\
&= L(U) + L(V)
\end{aligned}
$$

(2) Let $K \in R$ and $U \in R^3$

$L(K(U)) = L(KU_1, KU_2, KU_3)$
$= KU_1, KU_1 + KU_2 + KU_3$
$= K(U_1, U_1 + U_2 + U_3)$
$= KL(U)$

by (1) and (2) there for L is linear transformation

Ex :- (3) let $T: R^2 \Longrightarrow R^2$ be defined by $T(x, y) = (x, y+1)$. determine whether T is linear transformation

Sol :-

(1) let $V, U \in R^2 \Longrightarrow T(V) = (V_1, V_2 + 1)$
$T(U) = (U_1, U_2 + 1)$

$T(V+U) = T((V_1, V_2) + (U_1, U_2))$
$= T((V_1 + U_1, V_2 + U_2))$
$= (V_1 + U_1, V_2 + U_2 + 1)$

but $T(V) + T(U) = (V_1 + U_1, V_2 + U_2 + 2)$

$T(V+U) \neq T(V) + T(U)$

Then T is not linear transformation

S_2.. The Kernal and Rang Of Linear Transformation

نواة ومدى التحويلات الخطية

Def :- let $L: V \longrightarrow W$ be a linear transformation the ***kernel*** of L is the sub set of V consisting of all vectors V such that $L(v) = O_W$ and denoted by ker(L)

$Ker(L) = \{ v \in V : L(v) = O_W \}$

(النواة:- مجموعة كل العناصر في V بحيث صورتها تساوي المتجه الصفري في W تحت تاثير الدالة الخطية L)

Ex :- if $L: R^3 \longrightarrow R^2$ be defined by $L(U_1, U_2, U_3) = (U_1, U_3)$ find ker.(L)

Sol :-

$Ker(L) = \{ U \in R^3 : L(U) = O_{R^2} \}$
$= \{ (U_1, U_2, U_3) : T(U_1, U_2, U_3) = (0, 0) \}$
$= \{ (U_1, U_2, U_3) : U_1 = 0, U_2 = 0 \}$
$= \{ ((0, 0, U_3) \in R^3, U_3 \in R \}$

Ex :- (2) let $T: R \longrightarrow R^2$ be defined by $T(U) = (U, 2U)$, find ker (T)

Sol :-

$Ker(T) = \{ U \in R : T(U) = O_{R^2} \}$
$= \{ U \in R : (U, 2U) = (0, 0) \}$
$= \{ U \in R : U = 0 \}$
$= \{ 0 \}$

Ex :- (3) let $T : R^3 \longrightarrow R^3$ be defined by
$T (U_1 , U_2 , U_3) = (U_1 + U_2 , U_2 , U_1 - U_3)$, find ker (T)

Sol :-

$$\begin{aligned}
Ker (T) &= \{ U \in R^3 : T (U) = O_{R^3} \} \\
&= \{ (U_1 , U_2 , U_3) \in R^3 : T (U_1 , U_2 , U_3) = (0 , 0 , 0) \} \\
&= \{ (U_1 , U_2 , U_3) \in R^3 : (U_1 + U_2 , U_2 , U_1 - U_3) = (0 , 0 , 0) \} \\
&= \{ (U_1 , U_2 , U_3) \in R^3 : U_1 + U_2 = 0 , U_2 = 0 , U_1 - U_3 = 0 \} \\
&= \{(U_1 , U_2 , U_3) \in R^3 , U_1 = 0 , U_2 = 0 , U_3 = 0 \} \\
&= \{ (0 , 0 , 0) \}
\end{aligned}$$

<u>Ex :-</u> (4) let $T : R^4 \longrightarrow R^2$ be defined by
$T (x , y , z , w) = (x + y , z + w)$, find ker (T)

<u>Def:-</u> let $L : V \longrightarrow W$ is a linear transformation the ***Range*** of L is the set of all vectors in W that are images under L . of vectors in V .

Range (T) = $\{ w \in W : v \in V \ \ s.t \ \ T(v) = w \}$

المدى :- مجموعة المتجهات في W والتي تكون صور لمتجهات من V تحت تأثير الدالة الخطية L

<u>Theorm :-</u> if $T : V \longrightarrow W$ is linear transformation then
(1) Ker (T) is ***subspace*** of V .
(2) Range (T) is ***subspace*** of W .

<u>Proof :- (1)</u>

(a) let $U , V \in$ Ker (T)

$T (U) = 0 \ , T (V) = 0$

$$\begin{aligned}
T (U + V) &= T (U) + T (V) \qquad (T. \text{ linear tr.}) \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

Then , $U + V \in$ Ker (T)

(2) let $K \in R \ , \ U \in V$

$T (U) = 0$

$$\begin{aligned}
T (K U) &= K T (U) \\
&= K . 0 \\
&= 0
\end{aligned}$$

Thus , $K U \in$ Ker (T)

By (1) and (2) Ker (T) is subspace .

Ex :- let $T: R^2 \longrightarrow R^2$ such that
$T(0,1) = (1,2)$, $T(1,1) = (2,-2)$, then find $T(3,-2)$ and find $T(a,b)$?
Such that $S = ((0,1),(1,1))$ is spans of R^2

Sol :-

S spans R^2

Then , every vector in R^2 is linear comb . of element of S .

$K_1 (0, 1) + K_2 (1, 1) = (3, -2)$

$(0, K_1) + (K_2, K_2) = (2, -2)$

$(K_2, K_1 + K_2) = (3, -2)$

$K_2 = 3$

$K_1 + K_2 = -3 \Longrightarrow K_1 = -5$

$(3, -2) = -5 (0, 1) + 3 (1, 1)$

$T(3, -2) = T(-5(0, 1) + 3(1, 1))$

$= T(-5(0, 1)) + T(3(1, 1))$

$= -5\,T(0, 1) + 3\,T(1, 1)$

$= -5(1, 2) + 3(2, -3)$

$= (-5, -10) + (6, -9)$

$= (1, -19)$

(2) to find $T(a, b)$

$C_1 (0, 1) + C_2 (1, 1) = (a, b)$

بعد حل هذه المعادلة نحصل على

$C_1 = b - a , \; C_2 = a$

Then

$(a, b) = (b - a)(0, 1) + a(1, 1)$

$T(a, b) = T((b - a)(0, 1) + a(1, 1))$

$= T(b - a)(0, 1) + T(a(1, 1))$

$= (b - a)\,T(0, 1) + a\,T(1, 1))$

$= (b - a)(1, 2) + a(2, -3)$

$= (b - a, 2(b - a) + (2a, -3a)$

$= (b + a, 2b - 5a)$

Exc:- if $T: R^2 \longrightarrow R^2$ is linear trans . and $T(1, 0) = (2, -2), T(0, 1) = (4, 1)$ find $T(3, 2)$ and $T(a, b)$?

S_3-- Matrix Of Linear Transformation مصفوفة التحويل

ملاحظة :- لايجاد مصفوفة التحويل (***Matrix Of Linear Tran*** .) ذات السعة $m * n$ بحيث $T(U) = AU$ حيث

$T: R^n \Longrightarrow R^m$

نفرض ان $e_1, e_2, \ldots\ldots, e_n$ قاعدة اعتيادية فان

$T(e_1), T(e_2), \ldots\ldots, T(e_n)$

تمثل اعمدة المصفوفة A .

Ex : - let $T = R^2 \Longrightarrow R^2$ is L . transformation such that

$$T\begin{pmatrix} x_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} x + 2x_2 \\ x_1 - x_2 \end{pmatrix}$$

Find a matrix A such that $T(x) = A\,x$.

Sol :-

Let $S = ((1,0),(0,1))$ be a basis of R^3 then

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$$

$$A = \begin{pmatrix}1 & 2\\1 & -1\end{pmatrix}$$

<u>Ex :</u>- find (***Matrix Of Linear Tran .***) of $T : R^3 \Longrightarrow R^4$ defined by

$$T\begin{pmatrix}x_1\\x_2\\X_3\end{pmatrix} = \begin{pmatrix}x_1 + 2x_2\\3x_1 - x_2\\x_2 - x_3\\x_1\end{pmatrix}$$

<u>Sol :-</u> let $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ be a basis of R^3
Then

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\3\\0\\1\end{pmatrix},\ T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\\1\\0\end{pmatrix},\ T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\-1\\0\end{pmatrix}$$

اذن حسب تعريف مصفوفة التحويل الخطي تصبح مجموعة الصور لدالة T اعمده للمصفوفة A (مصفوفة التحويل الخطي T)

$$A = \begin{pmatrix}1 & 2 & 0\\3 & -1 & 0\\0 & 1 & -1\\1 & 0 & 0\end{pmatrix}_{4*3}$$