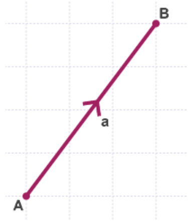
2. Vectors and Matrices

(2.1) Vector

A vector is an object that has both a magnitude and a direction. Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction. The direction of the vector is from its tail to its head.



Vector in two dimensions:

A vector between two points A and B is described as: $\overrightarrow{AB}$.

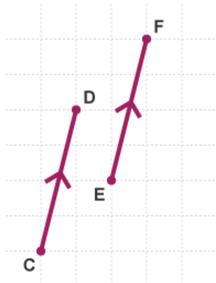
The vector can also be represented by the column vector $\binom{x}{y}$ or row vector (x, y). The top (left) number tells you how many spaces or units to move in the positive x-direction and the bottom (right) number is how many to move in the positive y-direction.

Vectors are equal if they have the same magnitude and direction regardless of where they are.

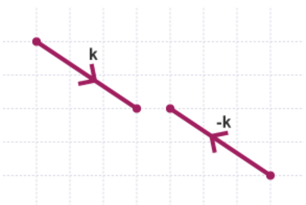
$\overrightarrow{CD} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\overrightarrow{EF} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

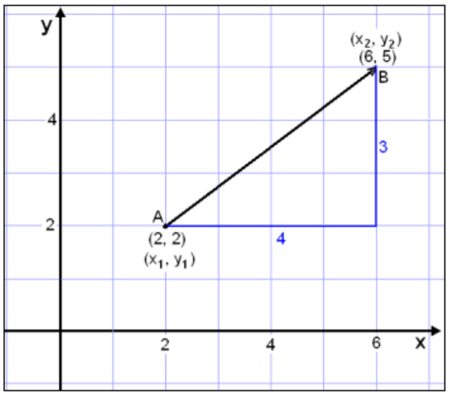
So $\overrightarrow{CD} = \overrightarrow{EF}$



A negative (invers) vector has the same magnitude but the opposite direction. Vector $-k$ is the same as travelling backwards down the vector k.



Example: The point A has coordinates (2, 2) and the point B coordinates (6, 5). The coordinates of the vector $\overrightarrow{AB}$ are $\overrightarrow{AB} = \begin{pmatrix} 6-2 \\ 5-2 \end{pmatrix} = \underline{\begin{pmatrix} 4 \\ 3 \end{pmatrix}}$



We can use the formula for the distance between two points to find the distance between A and B, that is the length of the vector $\overrightarrow{AB}$. The formula is as follows:

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Putting the given coordinates into the formula we get:

$$|\overrightarrow{AB}| = \sqrt{(6-2)^2 + (5-2)^2}$$

$$= \sqrt{4^2 + 3^2} = \underline{5}$$

We see that the numbers under the square root are simply the coordinates of the vector. This is, of course, because the length of the vector is simply the hypotenuse in a right angled triangle with shorter sides 3 and 4.

Vectors in more than two dimensions:

Vectors also work perfectly well in 3 or more dimensions:

Three dimension such as vectors in space $k = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $h = (x, y, z)$

A column or row vectors of n-dimensions such as: $v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$ or

$u = (x_1, x_2, \dots, x_n)$ x_i are called the components of the vector.

Equal vectors: Two vectors u and v are equal, written u = v, if they have the same number of components and if the corresponding components are equal.

The vectors (1,2,3) and (1,3,2) are not equal since corresponding elements are not equal.

Example:

Let

$$\begin{pmatrix} x-y \\ x+y \\ z-1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

Then, by definition of equality of column vectors,

$$\begin{aligned} x-y &= 4 \\ x+y &= 2 \\ z-1 &= 3 \end{aligned}$$

Solving the above system of equations gives $x = 3$, $y = -1$, and $z = 4$.

(2.2) VECTOR ADDITION

The sum of u and v, denoted by u+v, is the vector obtained by adding corresponding components:

Let u and v be column vectors with the same number of components.

$$u + v = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n + v_n \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix} = \begin{pmatrix} 1+4 \\ -2+5 \\ 3-6 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$$

Zero vector: A column vector whose components are all zero is called a zero vector and is also denoted by 0.

Example:

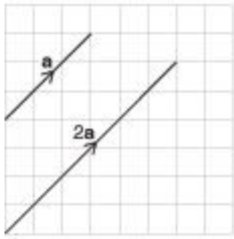
$$u + 0 = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 + 0 \\ u_2 + 0 \\ \cdot \\ \cdot \\ \cdot \\ u_n + 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} = u$$

(2.3) SCALAR MULTIPLICATION

Scalars : are quantities which have magnitude (size) but not direction. We can multiply a scalar by a vector to produce another vector.

The product of a scalar *k* and a column vector u, denoted by *k.u is*

$$k \cdot u = k\begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \\ \cdot \\ \cdot \\ \cdot \\ ku_n \end{pmatrix}$$



Example: If $a = \binom{3}{3}$ then $2a = 2\binom{3}{3} = \binom{6}{6}$

H.W.

Let $u = (2, -7, 1)$, $v = (-3, 0, 4)$ and $w = (0, 5, -8)$.

Find (i) $3u - 4v$, (ii) $2u + 3v - 5w$.

(2.4) Dot Product

The dot product or inner product of vectors u = (u1, u2, . . . , un) and v = (v1, v2, . . . , vn) is denoted by u.v and defined by

$$u.v = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

Multiplication Of A Row Vector And A Column Vector

It is similar to dot product. If a row vector u and a column vector v have the same number of components, then their product, denoted by u.v or simply uv, is the scalar obtained by multiplying corresponding elements and adding the resulting products:

Example:

(i) $(2, -3, 6)\begin{pmatrix} 8 \\ 2 \\ -3 \end{pmatrix}$, (ii) $(1, -1, 0, 5)\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, (iii) $(3, -5, 2, 1)\begin{pmatrix} 4 \\ 1 \\ -2 \\ 5 \end{pmatrix}$.

(i) $(2, -3, 6)\begin{pmatrix} 8 \\ 2 \\ -3 \end{pmatrix} = 2 \cdot 8 + (-3) \cdot 2 + 6 \cdot (-3) = 16 - 6 - 18 = -8$

(ii) & (iii) H.W.

Length of a vector:

The norm or length of a vector, u, is denoted by ||u|| and defined by

$$||u|| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Unit vector:

A Vector has a length of **1** is called unit vector. We can find a unit vector for any vector v by : $u = \frac{v}{||v||}$

H.W.: find the unit vector of $v = (2, 2\sqrt{2}, -2)$

Relation between norm and dot product

$$u.u = ||u||^2$$

H.W

(i) Find $||(3, -4, 12)||$.

(ii) Prove: $||ku|| = |k|\,||u||$, for any real number k.

3. Matrices

A matrix is a rectangular array of numbers; the general form of a matrix with m rows and n columns is:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

and call it an m x n matrix (read (m by n)).

Example:

$\begin{pmatrix} 2 & -1 & 5 \\ -23 & 8 & 4 \end{pmatrix}$ 2x3 matrix and its rows are: (2,-1,5) and (-23,8,4) and its columns are $\begin{pmatrix} 2 \\ -23 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 8 \end{pmatrix}$, *and* $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

Remark

1- Capital letters A, B, . . . denote matrices.
2- The dimension (size) of a matrix is the number of rows and columns it has.
3- Vector is matrix with one row or one column i.e. **A** matrix with one row is simply a row vector, and a matrix with one column is simply a column vector. Hence vectors are a special case of matrices..

Definitions:

(1) **Square matrix:** A matrix that has the same number of rows as columns.

(2) **Zero matrix:** A matrix whose elements are all zero and denoted by 0.

$$\mathbf{0} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}_{\boldsymbol{m\times n}}$$

(3) **Main diagonal:** In a square matrix $A = [a_{ij}]_{n\times}$, the entries for which $i = j$ namely $a_{11}, a_{22}, \ldots, a_{nn}$ $(i = 1,2,\cdots,n)$ are the **diagonal entries** of A which form the **main diagonal** of A.

(4) Identity matrix: A square matrix of dimension n×n whose all diagonal elements are all one and all other elements are zero and denoted by $\boldsymbol{I_n}$**.**

$$\boldsymbol{I_n} = \begin{bmatrix} \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \mathbf{1} & \vdots \\ \mathbf{0} & \cdots & \mathbf{1} \end{bmatrix}_{\boldsymbol{n\times n}}$$

(5) **Diagonal Matrix:** A square matrix $A = a_{ij\ n\times n}$ for which every element equal zero except the main diagonal, that is, $a_{ij} = 0$ for $i \neq j$, is called a **diagonal matrix.**

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

(6) **Upper Triangular:** A square matrix $A = a_{ij\ n\times n}$ is called **upper triangular** if $a_{ij} = 0$ for $i > j$.

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

(7) **Lower Triangular:** A square matrix $A = a_{ij\ n\times n}$ is called **lower triangular** if $a_{ij} = 0$ for $i < j$.

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

(3.1) Equal Matrices:

Two matrices A and B are equal, written by $A = B$, if they have the same number of rows and the same number of columns, and if the corresponding elements are equal.

Example:

The statement
$$\begin{pmatrix} x+y & 2z+w \\ x-y & z-w \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$$
is equivalent to the system of equations
$$\begin{cases} x+y = 3 \\ x-y = 1 \\ 2z+w = 5 \\ z-w = 4 \end{cases}$$
The solution of the system of equations is $x = 2,\ y = 1,\ z = 3,\ w = -1$.

H.W.: Let A and B be two matrices given by

$$A = \begin{bmatrix} x+y & 6 \\ 2x-3 & 2-y \end{bmatrix}_{2\times 2} \quad B = \begin{bmatrix} 5 & 5x+2 \\ y & x-y \end{bmatrix}_{2\times 2}$$

Determine if there are values of x and y so that A and B are equal?

Definition:

If $A = a_{ij\, m\times n}$ is a matrix, then the $n \times m$ matrix $A^T = (a_{ij})^T{}_{n\times}$ where $(a_{ij})^T = a_{ji}$ $(1 \le i \le m, 1 \le j \le n)$ is called the **transpose** of A. Thus the transpose of A is obtained by interchanging the rows and columns of A. The first row of A^T is the first column of A; the second row of A^T is the second column of A; and so on.

Example: If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \quad C = [1 \quad 0 \quad -1]$$

then

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 2 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Definition :

A matrix A is called symmetric if $A = A^T$, that is, $(i, j) -$ element of $A = (j, i) -$ element of A^T.

Remark:

(1) A is symmetric if it is a square for which $a_{ij} =$.

(2) If A is symmetric, then the elements of A are symmetric with respect to the main diagonal of A**.**

Example:

$$\text{(a)} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 3 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 4 & 7 \\ 3 & 7 & 5 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 5 & 0 \\ 3 & 5 & 1 & 0 \end{bmatrix}$$

then (a) is not symmetric (b) is symmetric (c) is not symmetric.

(3.2) Matrix Addition:

Let A and B be two matrices with the same shape, i.e. the same number of rows and columns. The sum of A and B, written A +B, is the matrix obtained by adding corresponding elements from A and B:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

Note that A + B has the same shape as A and B. The sum of two matrices with different shapes is not defined.

Example:

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 0 & -6 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1+3 & -2+0 & 3+(-6) \\ 0+2 & 4+(-3) & 5+1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & -3 \\ 2 & 1 & 6 \end{pmatrix}$$

But

$\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 5 & -2 \\ 1 & -3 & -1 \end{pmatrix}$s Not defined since the matrices have different shapes.

Theorem

For matrices A, B and C (with the same shape),

(i) (A + B) + C = A + (B + C), i.e. addition is associative.

(ii) A + B = B + A, i.e. addition is commutative.

(iii) A + 0 = 0 + A = A.

(3.3) SCALAR MULTIPLICATION

The product of a scalar *k* and a matrix *A*, written *kA* or *Ak*, is the matrix obtained by multiplying each element of *A* by *k:*

$$k \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

Note that ***A*** and *kA* have the same shape.

Example:

$$3\begin{pmatrix}1 & -2 & 0\\ 4 & 3 & -5\end{pmatrix} = \begin{pmatrix}3\cdot 1 & 3\cdot(-2) & 3\cdot 0\\ 3\cdot 4 & 3\cdot 3 & 3\cdot(-5)\end{pmatrix} = \begin{pmatrix}3 & -6 & 0\\ 12 & 9 & -15\end{pmatrix}$$

Remark: -A = (-l)A and A - B = A + (-B)

The next theorem follows directly from the above definition of scalar multiplication.

Theorem: For any scalars k_1, and k_2, and any matrices A and B (with the same shape):

(i) $(k_1k_2)A = k_1(k_2A)$

(ii) $k_1(A+B) = k_1A + k_1B$

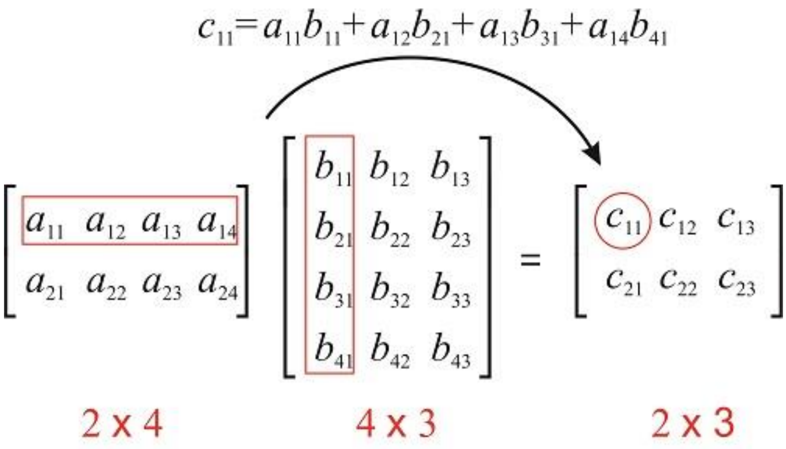
(iii) $(k_1+k_2)A = k_1A + k_2A$

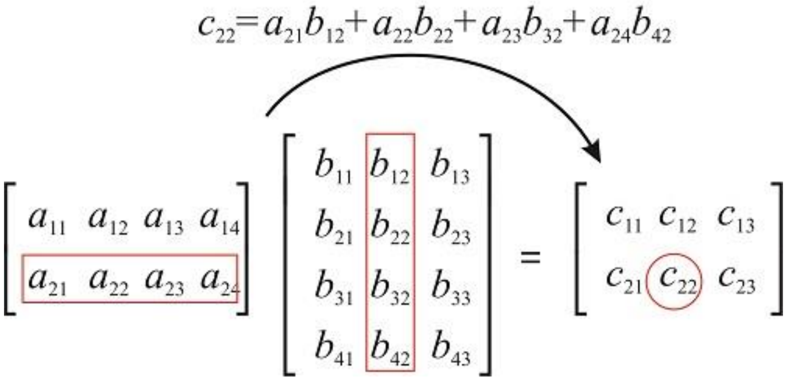
(iv) $1\cdot A = A$, and $0A = 0$

(v) $A + (-A) = (-A) + A = 0.$

(3.4) Matrix Multiplication

Let A and B be matrices such that the number of columns of A is equal to the number of rows of B. Then the product of A and B, written C=AB, is the matrix with the same number of rows as A and of columns as B and whose element in the ith row and jth column is obtained by multiplying the ith row of A by the jth column of B: $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$.





Example: $$AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2\cdot 5 + 3\cdot 6 + 4\cdot 7 & 2\cdot 3 + 3\cdot 4 + 4\cdot 5 \\ 1\cdot 5 + 2\cdot 6 + 3\cdot 7 & 1\cdot 3 + 2\cdot 4 + 3\cdot 5 \end{bmatrix}$$
$$= \begin{bmatrix} 56 & 38 \\ 38 & 26 \end{bmatrix}$$

Remark:

If the number of columns of ***A*** is not equal to the number of rows of ***B,*** say ***A*** is $m \times p$ and B is $q \times n$ where $p \neq q$, then the product A B is not defined.

Theorem:

(i) $(AB)C = A(BC)$

(ii) $A(B+C) = AB + AC$

(iii) $(B+C)A = BA + CA$

(iv) $k(AB) = (kA)B = A(kB)$, where k is a scalar.

H.W.:

1- find $(2, -3, 4)\begin{pmatrix} 1 & -3 \\ 5 & 0 \\ -2 & 4 \end{pmatrix}$

2- Given the matrices E, F, G and H, below

$$E = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix} F = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} G = \begin{bmatrix} 4 & 1 \end{bmatrix} H = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Find, if possible.

a. EF
b. FE
c. FH
d. GH

Example:

Given $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, find the product AB.

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [(2a + 3b + 4c)]$$

Note that AB is a 1×1 matrix, and its only entry is $2a + 3b + 4c$.

Theorem: For any square matrix *A, 1) AI=IA=A. (2)* $A^2 = AA$, $A^3 = A^2A$ *and so on.*

Theorem:

(i) $(A+B)^t = A^t + B^t$

(ii) $(A^t)^t = A$

(iii) $(kA)^t = kA^t$, for k a scalar

(iv) $(AB)^t = B^tA^t$.

H.W.

1- Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & -3 \\ -1 & -2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -3 & 0 & 1 \\ 5 & -1 & -4 & 2 \\ -1 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Find: (i) $A+B$, (ii) $A+C$, (iii) $3A-4B$.

Find: (i) AB, (ii) AC, (iii) AD, (iv) BC, (v) BD, (vi) CD.

Find: (i) A^t, (ii) A^tC, (iii) D^tA^t, (iv) B^tA, (v) D^tD, (vi) DD^t.

2- Given the matrices R, S, and T below.

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix} S = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} T = \begin{bmatrix} -2 & 3 & 0 \\ -3 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

Find 2RS - 3ST.

Remark:

1) Note that AB≠BA
2) The cancellation law does not hold for matrices as the following example shows:

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$. Then

$$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix}.$$

But $B \neq C$.

3) AB may be zero with neither A nor B equal to zero; that is, if A and B are two nonzero matrices, it is not necessary $AB \neq \mathbf{0}$. That is, the zero property does not hold for matrix multiplication as the following example shows:

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Definition:

An n×n matrix (square matrix) has an inverse (invertible) if there exists a matrix B such that AB = BA = I_n, where I_n is an n×n identity matrix. B is called the invers of A and denoted by the symbol A^{-1}.

Observe that the above relation is symmetric; That is, if B is the inverse of A then A is also the inverse of B.

Definition:

If A has an inverse we say that A is invertible, otherwise we say that A is **singular matrix (or noninvertible)** .

Example:

Given matrices A and B below, verify that they are inverses.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

Solution:

$$\text{AB} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{and} \quad \text{BA} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example: Fide the inverse of the following matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

Solution: Suppose A has an inverse, and it is:

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then AB=I and hence:

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After multiplying the two matrices on the left side, we get

$$\begin{bmatrix} 3a+c & 3b+6 \\ 5a+2c & 5b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries, we get four equations with four unknowns as follows:

$$3a+c=1 \quad 3b+d=0$$

$$5a+2c=0 \quad 5b+2d=1$$

Using substitution method or elimination method to solve the systems. a=2, b=−1 c=−5, d=3 Therefore,

$$A^{-1} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, find A^{-1} if exist.

Solution:

Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $AA^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\Rightarrow AA^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$a + 2c = 1 - -\!\!\rightarrow E_1$

$2a + 4c = 0 - -\!\!\rightarrow E_2$

$b + 2d = 0 - -\!\!\rightarrow E_3$

$2b + 4d = 1 - -\!\!\rightarrow E_4$

$-2E_1 + E_2 \longrightarrow 0 = -2$ Contradiction (*C!*)

So, the linear systems have no solution. Therefore A has no inverse. That is, A is singular.

Example: Find the inverse, if it exists, of

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating corresponding terms, we see that this is true only if

$$\begin{array}{lll} a - b + c = 1 & d - e + f = 0 & g - h + i = 0 \\ 2b - c = 0 & 2e - f = 1 & 2h - i = 0 \\ 2a + 3b = 0 & 2d + 3e = 0 & 2g + 3h = 1 \end{array}$$

Use substitution or elimination methods to solve these systems.

$$a = 3, b = -2, c = -4$$
$$d = 3, e = -2, f = -5$$
$$g = -1, h = 1, i = 2.$$

Therefore: $A^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$