جامعة بغداد

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قسم الرياضيات/ المرحلة الاولى

مادة التفاضل والتكامل

الفصل السادس

الدوال الاسية و اللوغارتمية

Logarithm and Exponential Functions

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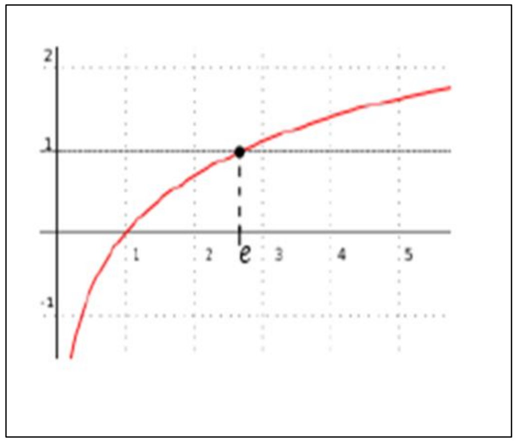
CHAPTER 6:
Logarithm and Exponential Fun.

The Natural Logarithm Function:
We denote it by "ln"

- $y = \ln(x); x > 0$
- $\ln: (0, \infty) \longrightarrow \mathbb{R}$
- $\ln e = 1$
- $\ln 1 = 0$
- $\ln 2 = 0.69$
- $\ln 10 = 2.3$



Properties of ln:

Let $x > 0$ and $y > 0$, then:

1 $\ln(x.y) = \ln x + \ln y$

2 $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

3 $\ln(x^a) = a \ln x$

Examples: Evaluate the following if you know that ($\ln 2 = 0.69$) :

1 $\ln 16 = \ln 2^4 = 4 \ln 2 = 4.(0.69) = 2.76$

2 $\ln 8 = \ln 2^3 = 3.\ln 2 = 3.(0.69) = 2.07$

3 $\ln \frac{1}{2} = \ln 1 - \ln 2 = 0 - (0.69) = -0.69$

4 $\ln \sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2} \cdot \ln 2 = \frac{1}{2} \cdot (0.69) = 0.345$

The Derivative of ln

$$\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}$$

Examples: Find y' for the following functions:

1 $y = \ln(x^2 + 2x)$

$\Longrightarrow y' = \frac{2x+2}{x^2+2x}$

2 $y = \ln(\sin x * x^2)$

$$\Longrightarrow y' = \frac{1}{\sin x * x^2} \cdot [\sin x * 2x + x^2 \cos x]$$

3 $y = \ln(\tan x + \sec x)$

$$\Longrightarrow y' = \frac{\sec^2 x + \sec x \cdot \tan x}{\tan x + \sec x}$$

4 $y = (\ln x)^3$

$$\Longrightarrow y' = 3(\ln x)^2 \cdot \frac{1}{x}$$

5 $y = (\ln \tan x)^5 \cdot \cos x^2$

$$\Longrightarrow y' = (\ln \tan x)^5 \cdot (-\sin x^2 \cdot 2x) + \cos x^2 \cdot 5(\ln \tan x)^4 \frac{1}{\tan x} \cdot \sec^2 x$$

6 $y = \ln(\cos x)$

$$\Longrightarrow y' = \frac{-\sin}{\cos x} = -\tan x$$

7 $y = \ln(\ln x)$

$$\Longrightarrow y' = \frac{\frac{1}{x}}{\ln x}$$

8 $y = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1} x}{x}$

$$\Longrightarrow y' = \frac{1}{x} - \frac{1}{2} \cdot \frac{2x}{1 + x^2} - \frac{x \cdot \frac{1}{1 + x^2} - \tan^{-1} x}{x^2}$$

9 $y = x[\sin(\ln x) + \cos(\ln x)]$

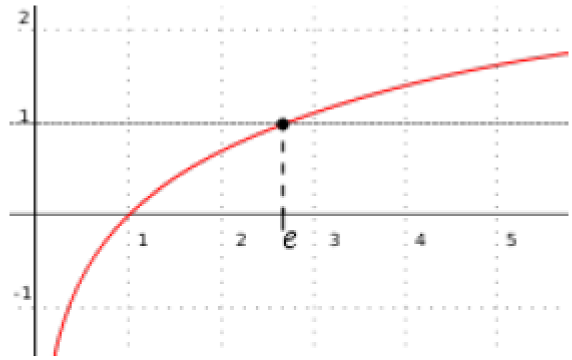
$$\Longrightarrow y' = x\left[\cos(\ln x) \cdot \frac{1}{x} - \sin(\ln x) \cdot \frac{1}{x}\right] + [\sin(\ln x) + \cos(\ln x)] \cdot 1$$

10 $y = x\sec^{-1}x - \ln\left(x + \sqrt{x^2 - 1}\right); x > 1$

$$\Longrightarrow y' = x \cdot \frac{1}{|x|\sqrt{x^2 - 1}} + \sec^{-1} x \cdot 1 - \frac{1 + \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}} \cdot 2x}{x + \sqrt{x^2 - 1}}$$

The Limit of ln

$$\lim_{x\to\infty} \ln x = +\infty$$
$$\lim_{x\to 0+} \ln x = -\infty$$



Examples: Proof the following:

1 $\lim_{x\to1}(x - \ln x) \overset{?}{=} 1$

Proof: $\lim_{x\to1}(x - \ln x) = 1 - \ln 1 = 1 - 0 = 1$

2. $\lim_{x\to1}\cos(\ln x) \overset{?}{=} 1$

Proof: $\lim_{x\to1}\cos(\ln x) = \cos(\ln 1) = \cos(0) = 1$

3. $\lim_{x\to1}\ln x^{(x+1)} \overset{?}{=} 0$

Proof: $\lim_{x\to1}\ln x^{(x+1)} = \lim_{x\to1}(x+1)\ln x$

$$= \lim_{x\to1}(x+1)\lim_{x\to1}\ln x = (1+1)\cdot\ln 1 = 2\cdot 0 = 0$$

4 $\lim_{x\to0}\frac{\ln(x+1)}{x} \overset{?}{=} 1$

Proof: $\lim_{x\to0}\frac{\ln(x+1)}{x} \overset{L'R}{=} \lim_{x\to0}\frac{\frac{1}{x+1}}{1} = \lim_{x\to0}\frac{1}{x+1} = \frac{1}{0+1} = \frac{1}{1} = 1$

5. $\lim_{x\to1}\ln(2-x)^{2\cos x} \overset{?}{=} 0$

Proof: $\lim_{x\to1}\ln(2-x)^{2\cos x} = \lim_{x\to1}[2\cos x\cdot\ln(2-x)]$

$$= 2\lim_{x\to1}\cos x\cdot\lim_{x\to1}\ln(2-x)$$
$$= 2\cdot\cos(1)\cdot\ln(2-1) = 2\cdot\cos(1)\cdot 0 = 0$$

Problems (6.1): Find y' of the following functions:

1 $y = \ln\left(x\sqrt{x^2+1}\right)$

2 $y = -5\ln(3x\sqrt{x+2})$

3 $y = t\ln t - \cos t^2$

4 $y = x^3\ln x - \frac{x}{\ln x}$

5 $y = \frac{1}{2}\ln\frac{1+w}{1-w}$

6 $y = -3\ln\frac{\sin w}{1+w^3}$

7 $y = \ln\frac{\sec x}{2+3x}$

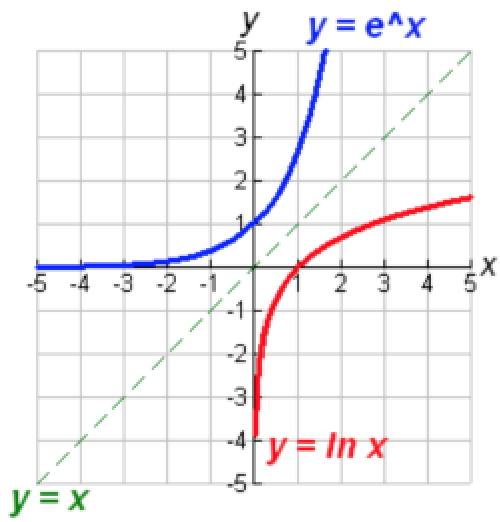
8 $y = \ln(t^2+4) - t\tan^{-1}\frac{t}{2}$

9 $y = z(\ln z)^3$

10 $y = \csc^3 x \ln x^3 + \tan^{-1}\ln x$

The Exponential Function:

We denote it by " e^x" or "exp($\boldsymbol{x}$) "



- $\boldsymbol{y} = e^x$
- $e^x: \mathbb{R} \longrightarrow (0,\infty)$
- $e^x = \ln^{-1}(x)$
- $e^0 = 1$, $\quad e^2 = 7.29$

Properties of e^x:

1 $e^0 = 1$

2 $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$

3 $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$

4 $(e^x)^r = e^{rx}, \forall r \in \mathbb{R}$

5 $e^{-x} = \frac{1}{e^x}$

6 $e^{\ln x} = x = \ln e^x$

Examples: Simplify the following:

1 $\ln\left(e^{-x^2}\right) = -x^2$

2 $\ln\left(e^{\frac{1}{x}}\right) = \frac{1}{x}$

3 $e^{\ln\frac{1}{x}} = \frac{1}{x}$

4 $e^{2\ln x} = e^{\ln x^2} = x^2$

5 $\exp(\ln x - 2\ln y) = \exp(\ln x - \ln y^2) = \exp\left(\ln\frac{x}{y^2}\right) = \frac{x}{y^2}$

6 $e^{x+\ln x} = e^x \cdot e^{\ln x} = e^x \cdot x = x \cdot e^x$

The Derivative of e^x

$$\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$$

Examples: Find y' for the following functions:

1 $y = e^{\tan^{-1} x}$

$$\Longrightarrow y' = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

2 $y = \exp(3) \cdot \exp(\sin 3\theta)$

$$\Longrightarrow y' = \exp(3) \cdot \exp(\sin 3\theta) \cdot \cos 3\theta \cdot 3$$

3 $y = e^{x^3}$

$$\Longrightarrow y' = e^{x^3} \cdot 3x^2$$

4 $y = \ln\frac{e^x}{1+e^x}$

$$\Longrightarrow y' = \frac{1}{\frac{e^x}{1+e^x}} \cdot \frac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2}$$

5 $y = \frac{1}{2}\left(e^{\sin x} - e^{-2x}\right)$

$$\Longrightarrow y' = \frac{1}{2}\left(e^{\sin x} \cdot \cos x - e^{-2x} \cdot (-2)\right)$$

6 $y = \sec^{-1}(e^{2x})$

$$\Longrightarrow y' = \frac{1}{|e^{2x}|\sqrt{(e^{2x})^2 - 1}} \cdot e^{2x} \cdot 2$$

7 $y = e^{\sin 2x}$

$$\Longrightarrow y' = e^{\sin 2x} \cdot \cos 2x. 2$$

8 $y = e^{(\ln x)^3}$

$$\Longrightarrow y' = e^{(\ln x)^3} \cdot 3(\ln x)^2 \cdot \frac{1}{x}$$

9 $y = e^{\cos(x^2)}$

$$\Longrightarrow y' = e^{\cos(x^2)} \cdot -\sin(x^2) \cdot 2x$$

10 $y = e^{\left(3x - e^{-x^2}\right)}$

$$\Longrightarrow y' = e^{\left(3x - e^{-x^2}\right)} \cdot \left(3 - e^{-x^2} \cdot (-2x)\right)$$

Problems (6.2): Find y' of the following functions:

1 $y = x^2 . e^{5x^2}$

2 $y = \exp(\pi) \cdot x + \exp(\cos \pi x)$

3 $y = \frac{e^z - e^{-z}}{e^z + e^{-z}}$

4 $y = e^{\sin^{-1} x} \cdot \sqrt{x}$

5 $y = (9t^2 - 6t + 2)\exp(3t)$

6 $y = \exp(\tan t) \cdot \ln t^5$

7 $y = w^7 \cdot e^{-\sqrt{w}}$

8 $y = e^{\left(\sqrt[3]{x} + \tan x^2\right)}$

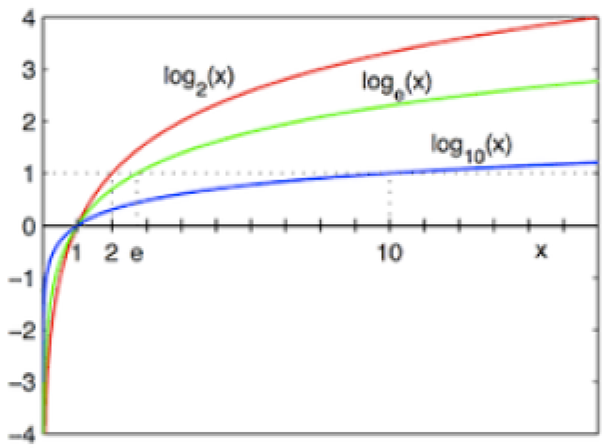
9 $y = \exp\left(\tan^{-1}(t^3)\right)$

10 $y = \ln\left(e^{-\sin\theta} - e^{-\theta^3 + 5}\right)$

The General Logarithm Function:

We denote it by "$\log_a x$"

$$y = \log_a x = \frac{\ln x}{\ln a}; x > 0, a > 0, a \neq 1$$
$$\log_a x: (0, \infty) \longrightarrow \mathbb{R}$$



Properties of $\log_a x$:

1 $\log_a(x.y) = \log_a x + \log_a y$

2 $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

3 $\log_a x^y = y\log_a x$

4 $\log_a a = 1$

5 $\log_a 1 = 0 \Leftrightarrow \ln 1 = 0$

Examples: Simplify the following:

1 $\log_2 16 = \frac{\ln 16}{\ln 2} = \frac{\ln 2^4}{\ln 2} = \frac{4\ln 2}{\ln 2} = 4$

2 $\log_{\frac{1}{7}} 49 = \frac{\ln 49}{\ln\frac{1}{7}} = \frac{\ln 7^2}{-\ln 7} = \frac{2\ln 7}{-\ln 7} = -2$

3 $\log_{10} 10 = 1$

4 $\log_{10} 100 = \log_{10} 10^2 = 2\log_{10} 10 = 2.1 = 2$

5 $\log_{10} \frac{1}{1000} = \log_{10} 10^{-3} = -3\log_{10} 10 = -3.1 = -3$

The Derivative of $\log_a x$

$$\frac{d}{dx}\log_a u = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}, \text{ where } u > 0, a > 0, a \neq 0$$

Examples:

Find y' for the following functions:

1 $y = \log_2(x^2 + 3x) \implies y' = \frac{1}{(x^2+3x)} \cdot \frac{1}{\ln 2} \cdot (2x + 3)$

2 $y = \log_3 x^5 \cdot \cos x^2$

$$\implies y' = \log_3 x^5 \cdot (-\sin x^2 \cdot 2x) + \cos x^2 \cdot \frac{1}{x^5} \cdot \frac{1}{\ln 3} \cdot 5x^4$$

3 $y = \log_7(\tan x + \sin x)$

$$\implies y' = \frac{1}{(\tan x + \sin x)} \cdot \frac{1}{\ln 7} \cdot (\sec^2 x + \cos x)$$

4 $y = \ln x \cdot \log_{10} x \implies y' = \ln x \cdot \frac{1}{x} \cdot \frac{1}{\ln 10} \cdot 1 + \log_{10} x \cdot \frac{1}{x}$

5 $y = \log_a \sin^{-1} x + \frac{x}{e^x}$, where a is a constant.

$$\implies y' = \frac{1}{\sin^{-1}\mathrm{x}\ \cdot} \cdot \frac{1}{\ln \alpha} \cdot \frac{1}{\sqrt{1 - x^2}} + \frac{e^x \cdot 1 - x \cdot e^{\mathrm{x}} \cdot 1}{e^{2x}}$$

Problems (6.3): Find y' of the following functions:

1 $y = x\log_5 x - \cos x^2$

2 $y = e^{-4t}\log_2(1 - t) + \log_a(3t^2)$, where a is a constant.

3 $y = \log_4(\cos 3w)\ \sec\sqrt{w}$

4 $y = \log_{11}(\tan(3\pi t))$

5 $y = \sqrt[3]{1 + \log_3(z^5)}$

6 $y = \log_n(\cos^{-1}\theta) + \sin\theta . \log_3 t^2$, where n is a constant.

7 $y = (\log_6(5z))^3$

8 $y = \sin(\log_2 \theta) + \sec\theta^2$

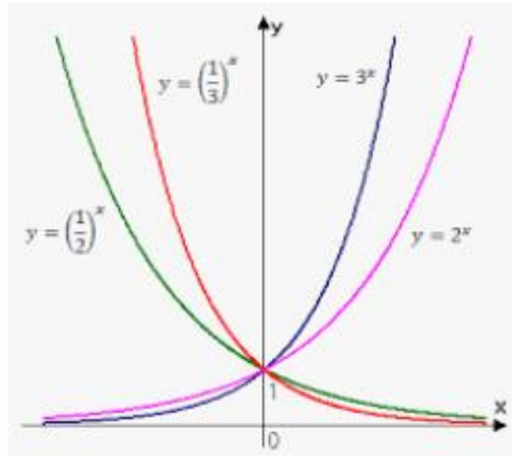
9 $y = \log_b\left(we^{w^2}\right) + \sin^{-1} w$, where b is a constant.

10 $y = \cot(\log_7 t^2) \cdot \frac{1}{t^3}$

The General Exponential Function:

We denote it by " $\boldsymbol{a}^{x}$ "

$$y = a^x = e^{x\ln a}; a > 0$$
$$a^x : \mathbb{R} \longrightarrow (0, \infty)$$



Properties of a^x:

1 $a^1 = a$, where $a > 0$

2 $a^0 = 1$

3 $a^u \cdot a^v = a^{u+v}$

4 $\left(a^{\frac{m}{n}}\right)^n = a^m$

5 $(a.b)^u = a^u.b^u$, where $a > 0$ and $b > 0$

The Derivative of a^x

$$\because \boldsymbol{a^u = e^{u\ln a}}$$

$$\xrightarrow{\frac{d}{dx}} \boldsymbol{\frac{d}{dx}\, a^u = \frac{d}{dx}\left(e^{u\ln a}\right)}$$

$$\longrightarrow \boldsymbol{\frac{d}{dx} a^u = a^u \cdot \ln a \cdot \frac{du}{dx}}$$

Examples: Find y' for the following functions:

1 $y = 2^{x^2+\sec x} \Longrightarrow y' = 2^{x^2+\sec x} \cdot \ln 2 \cdot (2x + \sec x \tan x)$

2 $y = 4^{\sin^{-1} x} \Longrightarrow y' = 4^{\sin^{-1} x} \cdot \ln 4 \cdot \frac{1}{\sqrt{1-x^2}}$

3 $y = x^{\pi} \cdot \pi^{x} \Longrightarrow y' = x^{\pi} \cdot \pi^{x} \cdot \ln \pi \cdot 1 + \pi^{x} \cdot \pi \cdot x^{\pi-1} \cdot 1$

Example: Find the value of x, where $3^x = 2^{x+1}$

Solution: $3^x = 2^{x+1}$

$\overset{\ln}{\Longrightarrow} \ \ln 3^x = ln\, 2^{x+1}$

$\Longrightarrow x \ \ln 3 = (x+1)\ln 2$

$\Longrightarrow x \ \ln 3 = x\, ln\, 2 + \ln 2$

$\Longrightarrow x \ \ln 3 - x\, ln\, 2 = \ln 2$

$\Longrightarrow x\ (ln\, 3 - ln\, 2) = ln\, 2$

$\Longrightarrow x\ \left(ln \frac{3}{2}\right) = ln\, 2 \ \ \Longrightarrow x = \frac{ln\, 2}{ln\frac{3}{2}}$

Problems (6.4): Find y' of the following functions:

1 $y = 5^{x^2+x-1}$

2 $y = 6^{\sin \omega + \ln w + 3}$

3 $y = 2^{\sec \sqrt{t}}$

4 $y = e^{5x} 3^{\tan x}$

5 $y = \ln \frac{x^4}{1+x^3} + 7^{\frac{x^2}{2}}$

6 $y = 2^{-t^2} \cos t^3$

7 $y = \pi^{\cos x} e^{\sqrt{x}} - 5^{-4x^3}$

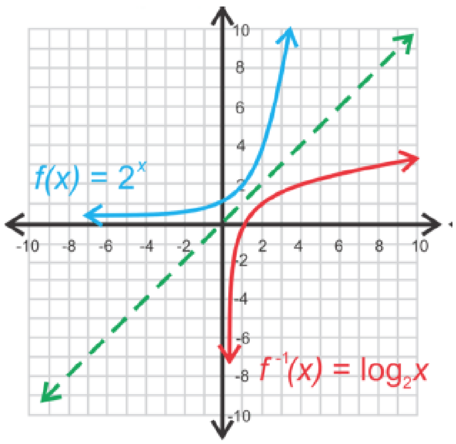
8 $y = \ln(1+e^{2x}) 5x^2$

9 $y = -4\ln w + \frac{4}{w} - 2^{\sin(\sqrt{w})}$

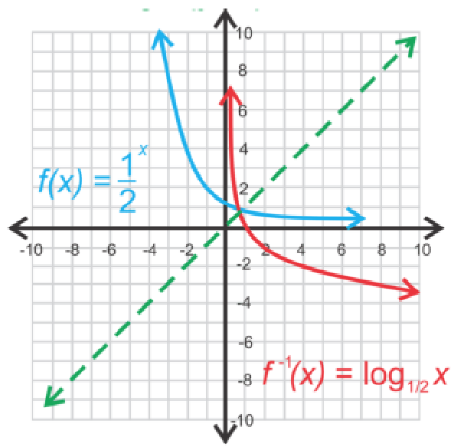
10 $y = 8^{\cos \pi\theta} \ln \sqrt{\theta} + 3^{\sec(5\theta)}$

The Relation between a^x and $\log_a x$

$$\log_a y = x \Leftrightarrow y = a^x; a > 0 \; \& \; a \neq 0$$



$a > 1$



$0 < a < 1$

Logarithmic Differentiation

If in a function where the base and the exponent are variables, we should change the form before we take the derivative.

Examples: Find y' for the following:

1. $y = x^{x^2}$

$$\xrightarrow{ln} \ln y = \ln x^{x^2}$$

$$\longrightarrow \ln y = x^2 \ln x$$

$$\xrightarrow{\frac{d}{dx}} \frac{y'}{y} = x^2 \, \frac{1}{x} + \ln x . 2x$$

$$\longrightarrow y' = y * \left(x^2 \frac{1}{x} + \ln x . 2x \right)$$

$$\xrightarrow{y = x^{x^2}} \; y' = x^{x^2} * \left(x^2 \frac{1}{x} + \ln x . 2x \right)$$

2. $y = (x^2+1)^{\ln x}$

$$\xrightarrow{ln} \ln y = \ln(x^2+1)^{\ln x}$$
$$\longrightarrow \ln y = \ln x \ln(x^2+1)$$
$$\xrightarrow{\frac{d}{dx}} \frac{y'}{y} = \ln x \cdot \frac{2x}{x^2+1} + \ln(x^2+1) \cdot \frac{1}{x}$$
$$\longrightarrow y' = y * \left(\ln x \cdot \frac{2x}{x^2+1} + \ln(x^2+1) \cdot \frac{1}{x}\right)$$
$$\longrightarrow y' = (x^2+1)^{\ln x} * \left(\ln x \cdot \frac{2x}{x^2+1} + \ln(x^2+1) \cdot \frac{1}{x}\right)$$

3 $y = (\sin x)^{\tan x}$

$$\xrightarrow{ln} \ln y = \ln(\sin x)^{\tan x}$$
$$\longrightarrow \ln y = \tan x \cdot \ln(\sin x)$$
$$\xrightarrow{\frac{d}{dx}} \frac{y'}{y} = \tan x \cdot \frac{\cos x}{\sin x} + \ln(\sin x) \cdot \sec^2 x$$
$$\longrightarrow y' = y * \left(\tan x \frac{\cos x}{\sin x} + \ln(\sin x) \cdot \sec^2 x\right)$$
$$\longrightarrow y' = (\sin x)^{\tan x} * \left(\tan x \cdot \frac{\cos x}{\sin x} + \ln(\sin x) \cdot \sec^2 x\right)$$
$$\longrightarrow y' = (\sin x)^{\tan x} * \left(\frac{\sin x}{\cos x} \cdot \frac{\cos x}{\sin x} + \ln(\sin x) \cdot \sec^2 x\right)$$
$$\longrightarrow y' = (\sin x)^{\tan x} * (1 + \sec^2 x \cdot \ln(\sin x))$$

4. $y = \sqrt[5]{\frac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)}}$

$$\longrightarrow y = \left(\frac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)}\right)^{\frac{1}{5}}$$
$$\xrightarrow{ln} \ln y = \frac{1}{5} * \ln\left(\frac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)}\right)$$

$$\longrightarrow \ln y = \frac{1}{5} * \ln((x+2)^3(x+1)^2) - \ln((x-4)^2(x+3))$$

$$\longrightarrow \ln y = \frac{1}{5} * (\ln(x+2)^3 + \ln(x+1)^2) - (\ln(x-4)^2 + \ln(x+3))$$

$$\longrightarrow \ln y = \frac{1}{5} * (3\ln(x+2) + 2\ln(x+1) - 2\ln(x-4) - \ln(x+3))$$

$$\overset{\frac{d}{dx}}{\rightarrow} \frac{y'}{y} = \frac{1}{5}\left(\frac{3}{x+2} + \frac{2}{x+1} - \frac{2}{x-4} - \frac{1}{x+3}\right)$$

$$\longrightarrow y' = y * \frac{1}{5}\left(\frac{3}{x+2} + \frac{2}{x+1} - \frac{2}{x-4} - \frac{1}{x+3}\right)$$

$$\longrightarrow y' = \sqrt[5]{\frac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)}} * \frac{1}{5}\left(\frac{3}{x+2} + \frac{2}{x+1} - \frac{2}{x-4} - \frac{1}{x+3}\right)$$

<u>Problems (6.5):</u>

1 Find y' for the following:

(a) $y = x^x$

(b) $y = e^{\sqrt{z^2+e^z}}$

(c) $y = x^{\sqrt{5}+\ln x}$

(d) $y = n^{\ln n}$

(e) $y = (\tan t)^{3\sin^{-1} t}$

2 Using L'R. proof the following:

(a) $\lim_{x\to\infty} \frac{3x^2+x+4}{5x^2+8x} = \frac{3}{5}$

(b) $\lim_{x\to\frac{\pi}{2}} \frac{\sin x-1}{\cos x} = 0$

(c) $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

(d) $\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} = 1$

(e) $\lim_{x\to 1} \frac{x\ln x-x+1}{(x-1)\ln x} = 1$

(f) $\lim_{x\to 0} \frac{e^x-1-x-\frac{x^2}{2}}{x^3} = \frac{1}{6}$

(g) $\lim_{x\to-\infty} \frac{3x-2}{e^{x^2}} = 0$