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مادة التفاضل والتكامل

الفصل السابع

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Indefinite Integration

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CHAPTER 7: Indefinite Integration

7.1 General Indefinite Integration:

In calculus, an antiderivative of a function $f(x)$ is a differentiable function $F(x)$ whose derivative is equal to the original function $f(x)$.

$$F'(x) = f(x) \Rightarrow \frac{dF(x)}{dx} = f(x)$$
$$\Rightarrow \quad dF(x) = f(x)dx$$
$$\Rightarrow \quad \int dF(x) = \int f(x)dx$$
$$\Rightarrow \quad F(x) = \int f(x)dx + C, \text{ where } C \in \mathbb{R}.$$

Indefinite Integrals Properties:

Let $f(x)$ be a function and $x \in \mathbb{R}$, then:

i. $\int af(x) = a\int f(x)dx$, where a is a constant.

ii. $\int (f(x) \mp g(x))dx = \int f(x)dx \mp \int g(x)dx$

Notes:

- $\int (f(x) * g(x))dx \neq \int f(x)dx * \int g(x)dx$
- $\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$

Rules: In general,

i. $\int dx = x + c$

ii. $\int u^n du = \frac{u^{n+1}}{n+1} + C$, where $C \in \mathbb{R}$.

Examples: Evaluate the following integrals:

1) $\int 3dx = 3\int dx = 3x + C$

2. $\int 5x^2dx = 5\int x^2dx = 5\frac{x^3}{3} + C$

3. $\int 6x^{-3}dx = 6\int x^{-3}dx$

$= 6\frac{x^{-2}}{-2} + C = \frac{-3}{x^2} + C$

4. $\int \frac{3\pi}{x^5}dx = \int 3\pi\, x^{-5}dx$

$= 3\pi\int x^{-5}dx = 3\pi\frac{x^{-4}}{-4} + C = \frac{-3\pi}{4x^4} + C$

5. $\int (2x+3)dx = \int 2xdx + \int 3dx$

$= x^2 + C_1 + 3x + C_2 = x^2 + 3x + C,$ *where* $C = C_1 + C_2$.

6. $\int \sqrt{2x+1}dx = \int (2x+1)^{\frac{1}{2}} \cdot \frac{2}{2}\, dx = \frac{1}{2}\int (2x+1)^{\frac{1}{2}} \cdot 2\, dx$

$= \frac{1}{2}\frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3}\sqrt{(2x+1)^3} + C$

7. $\int \left(x^2 - \sqrt{x}\right)dx = \int x^2dx - \int x^{\frac{1}{2}}\, dx = \frac{x^3}{3} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$

8. $\int t^2(1+2t^3)^{\frac{-2}{3}}dt = \frac{1}{6}\int (1+2t^3)^{-\frac{2}{3}} \cdot 6t^2dt$

$$= \frac{1}{6}\frac{(1+2t^3)^{\frac{1}{3}}}{\frac{1}{3}} + C \qquad = \frac{1}{2}\sqrt[3]{1+2t^3} + C$$

9. $\int \frac{x^3-2x^7}{5x^5}dx \quad = \quad \int \frac{x^3}{5x^5} - \frac{2x^7}{5x^5}dx$

$$= \frac{1}{5}\int x^{-2}dx - \frac{2}{5}\int x^2dx \quad = \quad \frac{1}{5}\frac{x^{-1}}{-1} - \frac{2}{5}\frac{x^3}{3} + C$$

$$= \frac{-1}{5x} - \frac{2}{15}x^3 + C$$

10) $\int \frac{(z+1)dz}{\sqrt[3]{z^2+2z+2}} \quad = \int (z^2+2z+2)^{-\frac{1}{3}}(z+1)dz$

$$= \frac{1}{2}\int (z^2+2z+2)^{-\frac{1}{3}}\, 2\,(z+1)dz$$

$$= \frac{1}{2}\frac{(z^2+2z+2)^{\frac{2}{3}}}{\frac{2}{3}} + C \quad = \frac{3}{4}\sqrt[3]{(z^2+2z+2)^2} + C$$

<u>Problems (7.1):</u> Evaluate the following integrals:

1) $\int \frac{dx}{(3x+2)^2}$

2) $\int (x^2+5x)dx$

3) $\int \frac{3r}{\sqrt{1-r^2}}dr$

4) $\int (3x^3+2x^{-5}+4x-9)dx$

5) $\int \frac{-1+5x}{x^{\frac{1}{4}}}dx$

6) $\int \frac{y}{(19-2y^2)^{\frac{1}{3}}}dy$

7) $\int \frac{y}{\sqrt{2y^2+1}}dy$

8) $\int 7w^2\sqrt{2w^3-5}dw$

11) $\int \sqrt{2+5y}\, dy$

12) $\int \sqrt{y^2+2y}\ (y+1)dy$

13) $\int \frac{z^5+z^{-2}-z-\pi}{z^3}dz$

14) $\int (3x-1)^{22}\, dx$

15) $\int \frac{3t}{\sqrt{2-t^2}}dt$

16) $\int \frac{3y^2}{\sqrt{y^3+7}}dy$

17) $\int (2+7t)^{\frac{2}{3}}dt$

18) $\int \frac{-7x}{(x^2-16)^8}dx$

9) $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$

19) $\int \sqrt[3]{r^3+\pi}\ \ r^2 dr$

10) $\int \frac{\theta+1}{\theta^3}d\theta$

20) $\int z\sqrt{2z^2+1}dz$

7.2 Integrals of Trigonometric Functions:

We can derive all the Trigonometric integration forms from the derivative Trigonometric forms as follows:

(1) $\frac{d}{du}\sin(u)=\cos(u)\xrightarrow{\int}\int\cos(u)du=\sin(u)+C$

(2) $\frac{d}{du}\cos(u)=-\sin(u)\xrightarrow{\int}\int\sin(u)du=-\cos(u)+C$

(3) $\frac{d}{du}\tan(u)=\sec^2(u)\xrightarrow{\int}\int\sec^2(u)du=\tan(u)+C$

(4) $\frac{d}{du}\cot(u)=-\csc^2(u)\xrightarrow{\int}\int\csc^2(u)du=-\cot(u)+C$

(5) $\frac{d}{du}\sec(u)=\sec(u)\tan(u)\xrightarrow{\int}\int\sec(u)\tan(u)du=\sec(u)+C$

(6) $\frac{d}{du}\csc(u)=-\csc(u)\cot(u)\xrightarrow{\int}\int\csc(u)\cot(u)du=-\csc(u)+C$

Examples: Evaluate the following integrals:

1) $\int\ \sin(3x)\,dx\ =\frac{1}{3}\int\ sin(3x)\cdot 3\,dx\ \ =-\frac{1}{3}cos(3x)+C$

2) $\int\ \cos(2t)\,dt\ =\frac{1}{2}\int\ cos(2t)\cdot 2\,dt\ \ =\frac{1}{2}sin(2t)+C$

3) $\int\ x\sec^2(x^2)dx\ =\frac{1}{2}\int\ sec^2(x^2)\cdot 2x\,dx\ \ =\frac{1}{2}tan(x^2)+C$

4) $\int\ \cot(5x)\csc(5x)\,dx\ \ =\frac{1}{5}\int\ cot(5x)\,csc(5x)\cdot 5dx\ =-\frac{1}{5}csc(5x)+C$

5) $\int \frac{1}{\sqrt{x}} \csc^2(\sqrt{x})\, dx \; = \; 2\int \frac{1}{2\sqrt{x}} \csc^2(\sqrt{x})\, dx = -2\cot(\sqrt{x}) + c$

Remark (1): Sometimes we should do some algebra to evaluate the integral.

Examples: Evaluate the following integrals:

1) $\int \sec^3(x)\tan(x)dx$

$$= \quad \int \underbrace{sec^2(x)}_{u} \; \underbrace{sec(x)tan(x)dx}_{du} = \frac{sec^3(x)}{3} + C$$

2) $\int \csc^7(x)\cot(x) \; dx = \int \csc^6(x)\csc(x)\cot(x) \; dx = \frac{-\csc^7(x)}{7} + C$

3) $\int \frac{\cos(2x)}{\sin^3(2x)} dx$

$$= \frac{1}{2}\int \underbrace{(sin(2x))^{-3}}_{u} \; \underbrace{\cos(2x) . 2 \, dx}_{du}$$

$$= \frac{1}{2} \; \frac{(\sin(2x))^{-2}}{-2} + c \; = -\frac{1}{4\, sin^2(2x)} + c$$

4) $\int \frac{6-\cos(3x)}{\sin^2(3x)} dx = = \int \frac{6}{\sin^2(3x)} dx - \int \frac{\cos(3x)}{\sin^2(3x)} dx$

$$= 6\int \frac{1}{\sin^2(3x)} dx - \int \frac{1}{\sin(3x)} \; \frac{\cos(3x)}{\sin(3x)} dx$$

$$= 6\int \csc^2(3x)\, dx - \int \csc(3x) \; \cot(3x) dx$$

$$= \frac{6}{3}\int \csc^2(3x) \cdot 3 \, dx - \frac{1}{3}\int \csc(3x)\cot(3x) \cdot 3 \, dx$$

$$= 2(-\cot(3x)) - \frac{1}{3}(-\csc(3x)) + C$$

$$= -2\cot(3x) + \frac{1}{3}\csc(3x) + C$$

5) $\int \frac{\sec^2(2w)}{\tan^3(2w)}dw = \frac{1}{2}\int \frac{\sec^2(2w)}{\tan^3(2w)}dw = \frac{-1}{4\tan^2(2w)} + c$

Remark (2):

A. When the power of $\mathbf{sin}(\boldsymbol{x})$ or $\mathbf{cos}(\boldsymbol{x})$ is **odd**, we use:

$$\sin^2(x) + \cos^2(x) = 1$$

B. When the power of $\mathbf{sin}(\boldsymbol{x})$ or $\mathbf{cos}(\boldsymbol{x})$ is even, we use:

$$\sin^2(x) = \frac{1}{2}(1 - \cos 2x) \quad \text{or} \quad \cos^2(x) = \frac{1}{2}(1 + \cos 2x)$$

Examples: Evaluate the following integrals:

1) $\int \sin^3(x)dx$

$$= \int \sin(x)\sin^2(x)\,dx \quad = \int sin(x)(1 - cos^2(x))dx$$

$$= \int \sin(x)\,dx - \int \underbrace{\cos^2(x)}_{u}\ \underbrace{sin(x)dx}_{du}$$

$$= \cos(x) - \frac{\cos^3(x)}{3} + C$$

2) $\int \cos^2(y)dy$

$$= \int \frac{1}{2}(1 + \cos(2y))dy \qquad = \frac{1}{2}\int dy + \frac{1}{2}\cdot\frac{1}{2}\int cos(2y)\cdot 2dy$$

$$= \frac{1}{2}y + \frac{1}{4}\sin(2y) + C$$

3) $\int \sin^2(y)\, dy =$

$$= \int \frac{1}{2}(1 - cos(2y))dy \qquad = \frac{1}{2}\int dy - \frac{1}{2}\cdot\frac{1}{2}\int cos(2y)\cdot 2dy$$

$$= \frac{1}{2}y - \frac{1}{4}sin(2y) + C$$

4) $\int \cos^4(x)dx$

$$= \int (\cos^2(x))^2 dx \quad = \int \left(\frac{1}{2}(1+\cos(2x))\right)^2 dx$$

$$= \frac{1}{4}\int \left((1 + 2cos(2x) + cos^2(2x)\right)dx$$

$$= \frac{1}{4}\int 1dx + \frac{1}{4}\int 2\cos(2x)\, dx + \frac{1}{4}\int \cos^2(2x)\, dx$$

$$= \frac{1}{4}\int dx + \frac{1}{2}\int cos(2x)dx + \frac{1}{4}\int \frac{1}{2}(1 + cos\, 4x)dx$$

$$= \frac{1}{4}\int dx + \frac{1}{2}\cdot\frac{1}{2}\int \cos(2x)\cdot 2dx + \frac{1}{8}\int dx + \frac{1}{8}\cdot\frac{1}{4}\int \cos 4x\cdot 4dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C$$

Remark (3):

A. If the powers of $\underline{\tan(x)}$ & $\underline{\cot(x)}$ is even, we use:

$$\mathbf{tan^2(x) = sec^2(x) - 1} \qquad \textbf{and} \quad \boldsymbol{cot^2\theta = csc^2\theta - 1}$$

B. If the powers of $\sec(x)$ & $\csc(x)$ is even, we use

$$\mathbf{sec^2(x) = 1 + tan^2(x)} \qquad \textbf{and} \quad \boldsymbol{csc^2\theta = cot^2\theta + 1}$$

Examples: Evaluate the following integrals:

1) $\int \tan^2(3x)\,dx = \int (sec^2(3x) - 1)dx$

$$= \frac{1}{3}\sec^2(3x) \cdot 3\,dx - \int dx$$

$$= \frac{1}{3}\tan(3x) - x + C$$

2) $\int \cot^2(5x)\,dx = \int (csc^2(5x) - 1)dx = \frac{1}{5}\int \csc^2(5x) \cdot 5\,dx - \int dx$

$$= \frac{-1}{5}\cot(5x) - x + C$$

3) $\int \sec^4(x)d\boldsymbol{x}$

$$= \int \sec^2(x)\sec^2(x)\,dx \quad = \int (1 + tan^2(x))sec^2(x)dx$$

$$= \int \sec^2(x)\,dx + \int \tan^2(x)\sec^2(x)\,dx \quad = tan(x) + \frac{tan^3(x)}{3} + C$$

4) $\int \csc^4(x)dx$

$$= \int \csc^2(x)\csc^2(x)\,dx == \int (1 + cot^2(x))csc^2(x)dx$$

$$= \int \csc^2(x)\,dx + \int \cot^2(x)\csc^2(x)\,dx \quad = -cot(x) - \frac{cot^3(x)}{3} + C$$

Problems (7.2): Evaluate the following integrals:

1 $\sin\int(2t)\,dt$

2 $x\sin\int(2x^2)dx$

3 $\cos\int(3\theta - 1)\,d\theta$

4 $4\cos\int(3y)\,dy$

5 $\int(x - 2\sin^2 x)dx$

13. $\int(3\cos(x) - 4x^3 + 2)dx$

14. $\int \sin^2(x)\cos(x)dx$

15. $\int \sin(x)\cos^3(x)dx$

16. $\int \cos^2(2y)\sin(2y)dy$

17. $\int \sin^4\theta d\theta$

6 $\pi \sin^2 \int (3t)\, dt$

7 $\int \cos^3 x\, dx$

8 $\int \frac{1}{\cos^2(3w)} dw$

9 $\frac{2}{3}\cot^2 \int (5y)\, dy$

10 $3\tan^2 \int (2x)\, dx$

11 $\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$

12 $\cos \int (2x+4)\, dx$

18. $\int (1-\sin^2(3t))\cos(3t)dt$

19. $\int \frac{3-\sin x}{\cos^2 x} dx$

20. $\int 2\sin(z)\cos(z)dz$

21. $\int \sqrt{2+\sin(3t)}\, \cos(3t)dt$

22. $\int \frac{\sin\left(\frac{z-1}{3}\right)}{\cos^2\left(\frac{z-1}{3}\right)} dz$

23. $\int (3\sin(2x) + 4\cos(3x))dx$

24. $\int \sin(t)\cos(t)(\sin(t)+\cos(t))dt$

<u>7.3 Integrals of Inverse Trigonometric Functions :</u>

We can derive all the integration forms from our derivatives forms as follows:

(1) $\frac{d}{du}\sin^{-1}(u) = \frac{du}{\sqrt{1-u^2}} \xrightarrow{\int} \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}(u) + C$

(2) $\frac{d}{du}\cos^{-1}(u) = -\frac{du}{\sqrt{1-u^2}} \xrightarrow{\int} \int \frac{1}{\sqrt{1-u^2}} du = -\cos^{-1}(u) + C$

(3) $\frac{d}{du}\tan^{-1}(u) = \frac{\mathrm{du}}{1+u^2} \xrightarrow{\int} \int \frac{1}{1+u^2} du = \tan^{-1}(u) + C$

(4) $\frac{d}{du}\cot^{-1}(u) = -\frac{\mathrm{du}}{1+u^2} \xrightarrow{\int} \int \frac{1}{1+u^2} du = -\cot^{-1}(u) + C$

(5) $\frac{d}{du}\sec^{-1}(u) = \frac{\mathrm{du}}{|u|\sqrt{u^2-1}} \xrightarrow{\int} \int \frac{1}{|u|\sqrt{u^2-1}} du = \sec^{-1}(u) + C$

(6) $\frac{d}{du}\csc^{-1}(u) = -\frac{\mathrm{du}}{|u|\sqrt{u^2-1}} \xrightarrow{\int} \int \frac{1}{|u|\sqrt{u^2-1}} du = -\csc^{-1}(u) + C$

Examples: Evaluate the following integrals:

1) $\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{2dx}{\sqrt{1-(2x)^2}}$

$$= \frac{1}{2} sin^{-1}(2x) + C \quad or \quad \frac{-1}{2} cos^{-1}(2x) + C$$

2) $\int \frac{dt}{1+t^2} = tan^{-1}(t) + C \quad or \quad - cot^{-1}(t) + C$

3) $\int \frac{dx}{x\sqrt{4x^2-1}} = \int \frac{2\,dx}{2x\sqrt{(2x)^2-1}}$

$$= sec^{-1}|2x| + C \quad or \quad - csc^{-1}|2x| + C$$

4) $\int \frac{-dx}{\sqrt{4-25x^2}}$ $\qquad (4-25x^2 = 4\left(1-\frac{25}{4}x^2\right) = 4(1-(\frac{5}{2}x)^2))$

$$= \int \frac{-dx}{\sqrt{4\left(1-\frac{25}{4}x^2\right)}}$$

$$= \int \frac{-dx}{2\sqrt{1-\left(\frac{5}{2}x\right)^2}} = \frac{-1}{2}\cdot\frac{2}{5}\cdot\int \frac{\frac{5}{2}dx}{\sqrt{1-\left(\frac{5}{2}x\right)^2}}$$

$$= -\frac{1}{5} sin^{-1}\left(\frac{5}{2}x\right) + C \quad or \quad \frac{1}{5} cos^{-1}\left(\frac{5}{2}x\right) + C$$

5) $\int \frac{\cos(x)dx}{\sqrt{1-\sin^2(x)}} = \int \frac{\cos(x)\mathrm{dx}}{\sqrt{1-(\sin x)^2}} = sin^{-1}(sin(x)) + C = x + C$

6) $\int \frac{\tan^{-1}(x)}{1+x^2}dx = \int tan^{-1}(x)\cdot\frac{dx}{1+x^2} = \frac{\left(tan^{-1}(x)\right)^2}{2} + C$

7) $\int \frac{\sqrt{\sec^{-1}(x)}}{x\sqrt{x^2-1}}dx = \int (sec^{-1}(x))^{\frac{1}{2}}\cdot\frac{1}{x\sqrt{x^2-1}}dx = \frac{(sec^{-1}(x))^{\frac{3}{2}}}{\frac{3}{2}} + C$

Problems (7.3): Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{1-(2+9z)^2}} dz$
2. $\int \frac{-1}{\sqrt{1-16w^2}} dw$
3. $\int d(\csc^{-1}(t))$
4. $\int d(\sec^{-1}(t))$
5. $\int \frac{x}{1+16x^4} dx$
6. $\int \frac{-3t}{9t^4+9} dt$
7. $\int \left(\cos^{\frac{1}{2}} \sin(x) - \frac{13}{1+x^2}\right) dx$
8. $\int \left(\cos^3(2x)\sin(2x) - \frac{x^3}{\pi}\right) dx$
9. $\int \frac{x}{9x+x^3} dx$
10 $\int \frac{1}{25+t^2} dt$
11. $\int \frac{3}{16z^2+4} dz$
12. $\int \frac{1}{81w^2+9} dw$
13. $\int \frac{\sec^2(x)}{\sqrt{1-\tan^2(x)}} dx$
14. $\int \frac{\sec^2(x)}{\sqrt{1-\tan^2(x)}} dx$
15. $\int \frac{\left(\sin^{-1}(x)\right)^2}{\sqrt{1-x^2}} dx$
16. $\int \frac{\left(\cos^{-1}(2z)\right)^2}{\sqrt{1-4z^2}} dz$
17. $\int \frac{\sin\left(\tan^{-1}(x)\right)}{1+x^2} dx$
18. $\int \frac{\sin\left(\sin^{-1}(3x)\right)}{\sqrt{1-9x^2}} dx$
19. $\int \frac{1}{1+25x^2} dx$
20. $\int \frac{-\pi}{3+27t^2} dt$

7.4 Integrals of Logarithmic Functions:

$$\because \frac{d}{du}\ln(u) = \frac{1}{u}du \xrightarrow{\int} \int \frac{1}{u}du = \ln|u| + C \ , \ \ u \neq 0$$

Examples: Evaluate the following integrals:

1) $\int \frac{2}{x}dx \quad = 2\int \frac{1}{x}dx \quad = 2ln\,|x| + C$

2) $\int \left(\frac{3}{x^2} + \frac{5}{x}\right) dx \quad = \quad 3\int x^{-2}dx + 5\int \frac{1}{x}dx$

$$= 3\,\frac{x^{-1}}{-1} + 5\ln|x| + C \qquad = \quad \frac{-3}{x} + 5ln\,|x| + C$$

3) $\int \frac{x}{(2x^2+3)}\,dx \;= \frac{1}{4}\int \frac{4x}{(2x^2+3)}\,dx \;=\; \frac{1}{4}ln|2x^2+3| + C$

4) $\int \frac{\ln(x)}{x}\,dx \;= \int\; ln(x)\cdot\frac{1}{x}\,dx \;=\; \frac{(ln(x))^2}{2} + C$

5) $\int \frac{1}{x\ln x}\,dx \;=\; \int \frac{\frac{1}{x}}{ln(x)}\,dx \;=\; ln\,|ln(x)| + C$

6) $\int \frac{e^x}{1+2e^x}\,dx \;= \frac{1}{2}\int \frac{2e^x}{1+2e^x}\,dx \;=\; \frac{1}{2}ln|1+2e^x| + C$

7) $\int \frac{\sec^2(x)}{\tan(x)}\,dx \;=\; ln\,|tan(x)| + C$

8) $\int \frac{\sec(2x)\tan(2x)}{\sec(2x)}\,dx = \frac{1}{2}\int \frac{2\,sec(2x)\,tan(2x)}{sec(2x)}\,dx \;= \frac{1}{2}\; ln\,|sec(2x)| + C$

9) $\int \tan(u)\,du \;=\; -\int \frac{-\sin(u)}{\cos(u)}\,du \;=\; -ln\,|cos(u)| + C$

10) $\int \cot(u)\,du \;= \int \frac{cos(u)}{sin(u)}\,du \;=\; ln\,|sin(u)| + C$

11) $\int \sec(u)\,du \;=\; \int\; sec(u)\cdot\frac{(sec(u)+tan(u))}{(sec(u)+tan(u))}\,du$

$$= \int \frac{\sec^2(u)+\sec(u)\tan(u)}{\tan(u)+\sec(u)}\,du \;=\; ln\,|tan(u)+sec(u)| + C$$

12) $\int \csc(u)\,du \;=\; \int\; csc(u)\cdot\frac{(csc(u)+cot(u))}{(csc(u)+cot(u))}\,du$

$$= -\int \csc(u) \cdot \frac{(-\csc(u)-\cot(u))}{(csc(u)+\cot(u))} du \quad = -\int \frac{-\csc^2(u)-csc(u)\cot(u)}{\cot(u)+\csc(u)} du$$

$$= -\ln|\cot(u)+\csc(u)| + C$$

Problems (7.4): Evaluate the following integrals:

1. $\int \frac{1}{x-3} dx$
2. $\int \frac{\cos(x)}{\sin(x)} dx$
3. $\int \frac{dx}{x \ln^5(x)}$
4. $\int \frac{\sin(x)}{2-\cos(x)} dx$
5. $\int \frac{x\ dx}{4x^2+1}$
6. $\int \frac{x}{1-x^2} dx$
7. $\int \frac{2x-5}{x} dx$
8. $\int \frac{\ln^2(x)}{x} dx$
9. $\int \frac{x^{\frac{3}{2}}}{4\ x^{\frac{5}{2}}} + 6\ dx$
10. $\int \frac{5\ dx}{\sqrt{1-9x^2}}$
11. $\int \frac{x+10}{x^2} dx$
12. $\int \frac{\sin(\theta)}{1+7\cos(\theta)} d\theta$
13. $\int \frac{y^2+2y+1}{(y+1)^3} dy$
14. $\int \frac{ds}{\tan^{-1}(s)+s^2\tan^{-1}(s)}$
15. $\int \frac{dx}{2-3x}$
16. $\int \frac{\ln(x)}{4x\ln(2)} dx$
17. $\int \frac{x^2}{4-x^3} dx$
18. $\int \frac{w^2+2w-1}{w+4} dw$
19. $\int \frac{x}{x+1} dx$
20. $\int \frac{\ln(3)\cos(x)}{-5-\sin(x)} dx$

7.5 Integrals of General Logarithmic Function:

$$\because \frac{\boldsymbol{d}}{\boldsymbol{du}} \log_a(u) = \frac{1}{u \cdot \ln(a)} du \overset{\int}{\rightarrow} \int \frac{1}{u \cdot \ln(a)} du = \log_a(u) + C$$

where $a > 0$ and $a \neq 1$ (i.e., $\ln(a) \neq 0$)

Examples: Evaluate the following integrals:

1) $\int \frac{x}{x^2 \ln 5} dx \;= \frac{1}{2}\int \frac{2x}{x^2 ln\, 5} dx \quad = \quad \frac{1}{2} log_5(x^2) + C$

2) $\int \frac{\cos(3t)}{\sin(3t)\ln 4} dt = \frac{1}{3}\int \frac{3cos(3t)}{sin(3t)\ ln\, 4} dt = \frac{1}{3} log_4(sin(3t)) + C$

3) $\int \frac{1}{(\sqrt{1-x^2})\sin^{-1}(x)\ln 3} dx = log_3(sin^{-1}(x)) + C$

4) $\int (\sqrt{w} - \frac{1}{\sqrt{1-4w^2}\ \cos^{-1}(2w)\ \ln 3}) dw = \frac{2}{3}\sqrt{w^3} + \frac{1}{2} log_3(cos^{-1}(x)) + C$

<u>Problems (7.5):</u> Evaluate the following integrals:

1. $\frac{\cos(x)}{\sin(x)\ln 3} dx$
2. $\int \frac{1}{\cos^{-1}(w)\ln 2} \frac{1}{\sqrt{1-w^2}} dw$
3. $\int \frac{1}{e^{2x}\ln 3} \cdot e^{2x} dx$
4. $\int \frac{1}{\cot^{-1}(x)\ln 4} \cdot \frac{1}{1+x^2} dx$
5. $\int \frac{1}{e^{\sin(x)}\ln 7} \cdot e^{\sin(x)} \cos(x) dx$
6. $\int \frac{1}{2t\ln 3} dt$
7. $\int \frac{\frac{1}{1+x^2}}{\tan^{-1}(x)\ln 11} dx$
8. $\int \left(\cos(4z) - \frac{1}{\sqrt{1-z^2}\sin^{-1}(z)\cdot\ln 2}\right) dz$
9. $\int \frac{x^3}{x^4\ln 5} dx$
10. $\int \frac{\sin(5w)}{\cos(5w)\ln 7} dw$

<u>7.6 Integrals of Exponential Function:</u>

$$\because \frac{d}{du} e^{(u)} = e^{(u)} du \xrightarrow{\int} \int e^{(u)} du = e^{(u)} + C$$

<u>Examples:</u> Evaluate the following integrals:

1) $\int e^{2x} dx = \frac{1}{2}\int 2e^{2x} dx = \boldsymbol{\frac{1}{2} e^{2x} + C}$

2) $\int e^{\sin(3x)} \cdot \cos(3x)\, dx$

$= \frac{1}{3}\int 3 \cdot e^{\sin(3x)} \cdot \cos(3x)\, dx = \boldsymbol{\frac{1}{3} e^{sin(3x)} + C}$

3) $\int \frac{4}{e^{3x}} dx = \int \frac{4}{-3} \cdot (-3) \cdot e^{-3x} dx = \boldsymbol{\frac{4}{-3} e^{-3x} + C}$

4) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \boldsymbol{\ln|e^x + e^{-x}| + C}$

5) $\int e^{2x} \sin(e^{2x}) dx = \frac{1}{2} \int 2 \cdot e^{2x} sin(e^{2x}) dx = \boldsymbol{-\cos(e^{2x}) + C}$

6) $\int \frac{e^{\sin^{-1}(2x)}}{\sqrt{1-4x^2}} = \frac{1}{2} \int \frac{2 \cdot e^{\sin^{-1}(2x)}}{\sqrt{1-(2x)^2}} = e^{\sin^{-1}(2x)} + C$

7) $\int \frac{e^{3w}}{5-2e^{3w}} dw = \frac{1}{-6} \int \frac{-6e^{3w}}{5-2e^{3w}} dw = \frac{-1}{6} ln|5 - 2e^{3w}| + C$

8) $\int \frac{e^{4t} - \pi}{e^{2t}} dt = \int (\frac{e^{4t}}{e^{2t}} - \frac{\pi}{e^{2t}}) dt$

$= \int (e^{2t} - \pi e^{-2t}) dt = \frac{e^{2t}}{2} + \frac{\pi e^{-2t}}{2} + c$

9) $\int \tan(e^{2x}) e^{2x} dx = \int \frac{sin(e^{2x})}{cos(e^{2x})} e^{2x} dx = \frac{-1}{2} ln|sin(e^{2x})| + C$

10) $\int x^2 e^{2x^3} dx = \frac{1}{6} \int 6x^2 e^{2x^3} dx = \frac{1}{6} e^{2x^3} + c$

<u>Problems (7.6):</u> Evaluate the following integrals:

1. $\int xe^{x^2} dx$
2. $\frac{e^{\cot^{-1}(3x)}}{1+9x^2} dx$
3. $\int \frac{e^x}{1+2e^x} dx$
4. $\frac{e^{7x}}{3-e^{7x}} dx$
5. $\int (x+1) e^{x^2+2x} dx$
6. $\int e^{\frac{\pi}{3}} \cos\left(e^{\frac{\pi}{3}}\right) dx$
7. $\int \left(xe^{x^2} + e^{-3x} \sin(e^{-3x})\right) dx$
8. $\int \frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}} dx$
9. $\left(e^{5x} \cos(e^{5x}) - e^{\frac{x}{2}}\right) dx$
10. $\int \frac{e^{\tan^{-1}(2t)}}{1+4t^2} dt$

7.7 Integrals of general Exponential Function:

$$\because \frac{d}{du} a^{(u)} = a^{(u)} \ln(a)\, du \xrightarrow{\int} \int \boldsymbol{a}^{(u)} d\boldsymbol{u} = \frac{\boldsymbol{a}^{(u)}}{\ln(\boldsymbol{a})} + C$$

where $a > 0$ and $a \neq 1$ (i.e., $\ln(a) \neq 0$)

Examples: Evaluate the following integrals:

1) $\int 3^x dx \quad = \frac{3^x}{\ln 3} + C$

2) $\int 5^{2t-2} dx \quad = \frac{1}{2}\int 5^{2t-2}\, 2\, dx \quad = \frac{5^{2t-2}}{2\ln 5} + C$

3) $\int \cos(\theta) \cdot 4^{-\sin(\theta)} d\theta \ = -\int -\cos(\theta) \cdot 4^{-\sin(\theta)} d\theta \quad = \frac{4^{-\sin(\theta)}}{\ln 4} + C$

4) $\int 6^{\ln|\cos(x)|} \cdot \tan(x) dx \ = -\int 6^{\ln|\cos(x)|} \cdot \left(\frac{-\sin(x)}{\cos(x)}\right) dx$

$= -\frac{6^{\ln|\cos(x)|}}{\ln 6} + C$

Problems (7.7): Evaluate the following integrals:

1. $\int x 2^{-x^2} dx$
2. $\int 3^{\sqrt{t}} \frac{1}{\sqrt{t}} dt$
3. $\int 3^{\ln|\sin(\theta)|} \cot(\theta) d\theta$
4. $\int 4^{\sin(2x)} \cos(2x) dx$
5. $\int \frac{z^{\ln(z)}}{z} dz$
6. $\int (r+1) 2^{r^2+2r} dr$
7. $\int 6^{\tan^{-1}(t)} \frac{1}{1+t^2} dt$
8. $\int 4^{e^{3x}} e^{3x} dx$
9. $\int 3^{w^2} 2w dw$
10. $\int 9^{\ln|\cos z^2|} \tan(z^2) dz$