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قسم الرياضيات/ المرحلة الاولى

مادة التفاضل والتكامل

الفصل الخامس

الدوال المثلثية

Trigonometric Functions

اعضاء التدريس

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CHAPTER FIVE: Trigonometric Functions

We will define six trigonometric functions in terms of the central angle θ drawn in the center circle (0,0) and radius r.

In the central angle θ with one of its sides is applied to the x-axis and the other side is drawn from the origin point and cut the circumference of the circle at the point $p(x, y)$, then:

- Sine: $\boldsymbol{\sin\theta = \frac{y}{r}}$
- Cosine: $\boldsymbol{\cos\theta = \frac{x}{r}}$
- Tangent: $\boldsymbol{\tan\theta = \frac{y}{x}}$
- Cotangent: $\boldsymbol{\cot\theta = \frac{x}{y}}$
- Secant: $\boldsymbol{\sec\theta = \frac{r}{x}}$
- Cosecant: $\boldsymbol{\csc\theta = \frac{r}{y}}$

From the previous definition definitions, a relation can be found between trigonometric functions as follows:

$$* \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$* \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{\csc\theta}{\sec\theta}$$

$$* \quad \sec\theta = \frac{1}{\cos\theta}$$

$$* \quad \csc\theta = \frac{1}{\sin\theta}$$

And since the equation of the circle center $(0,0)$ and radius r is:

$$\boldsymbol{x^2 + y^2 = r^2}$$

$$\because x = r\ \cos\theta \quad and \quad y = r\sin\theta$$

Trigonometric Identities

$\Rightarrow\ r^2\, cos^2\theta\ +\ r^2\, sin^2\theta\ =\ r^2$

$\Rightarrow\ r^2\,(cos^2\theta\ +\ sin^2\theta)\ =\ r^2$

$\Rightarrow\ \boldsymbol{cos^2\theta\ +\ sin^2\theta\ =\ 1} \qquad \ldots(\mathbf{1})$

Note: From the above equation, we can derive the following forms:

• If we divide eq.(1) by $\boldsymbol{cos^2\theta}$:

$\Rightarrow\ \boldsymbol{1\ +\ tan^2\theta\ =\ sec^2\theta}$

• If we divide eq.(1) by $\boldsymbol{sin^2\theta}$:

$\Rightarrow\ \boldsymbol{cot^2\theta\ +\ 1\ =\ csc^2\theta}$

Laws of Sum and Subtract Two Angles:

Let A and B be any two angles, then:

$*\ \boldsymbol{\cos(A+B) = \cos A\cos B - \sin A\sin B}$

$*\ \boldsymbol{\cos(A-B) = \cos A\cos B + \sin A\sin B}$

$*\ \boldsymbol{\sin(A+B) = \sin A\cos B + \sin B\cos A}$

$*\ \boldsymbol{\sin(A-B) = \sin A\cos B - \sin B\cos A}$

$*\ \boldsymbol{\tan(A\mp B) = \dfrac{\tan A \mp \tan B}{1 \pm \tan A\tan B}}$

Note: Now, we can use the laws of sum and subtract two angles to derive the following forms:

- $\boldsymbol{\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta}$

  $\Longrightarrow \boldsymbol{\sin(2\theta) = 2\sin\theta\cos\theta} \qquad \ldots(\mathbf{2})$

- $\boldsymbol{\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta}$

  $\Longrightarrow\ \boldsymbol{\cos(2\theta) = \cos^2\theta - \sin^2\theta} \qquad \ldots(\mathbf{3})$

Note: From eq.(1) and eq (3), we can derive the following:

- eq.(1) + eq. (3) $\Rightarrow$ $\mathbf{2\cos^2\theta = 1 + \cos 2\theta}$
- eq.(1) - eq. (3) $\Rightarrow$ $\mathbf{2\sin^2\theta = 1 - \cos 2\theta}$

Remark: Trigonometric function are divided into two types (Odd Functions and Even Functions) as follows:

***Odd Functions:**

$\sin(-\theta) = -\sin\theta$

$\tan(-\theta) = -\tan\theta$

$\cot(-\theta) = -\cot\theta$

$\csc(-\theta) = -\csc\theta$

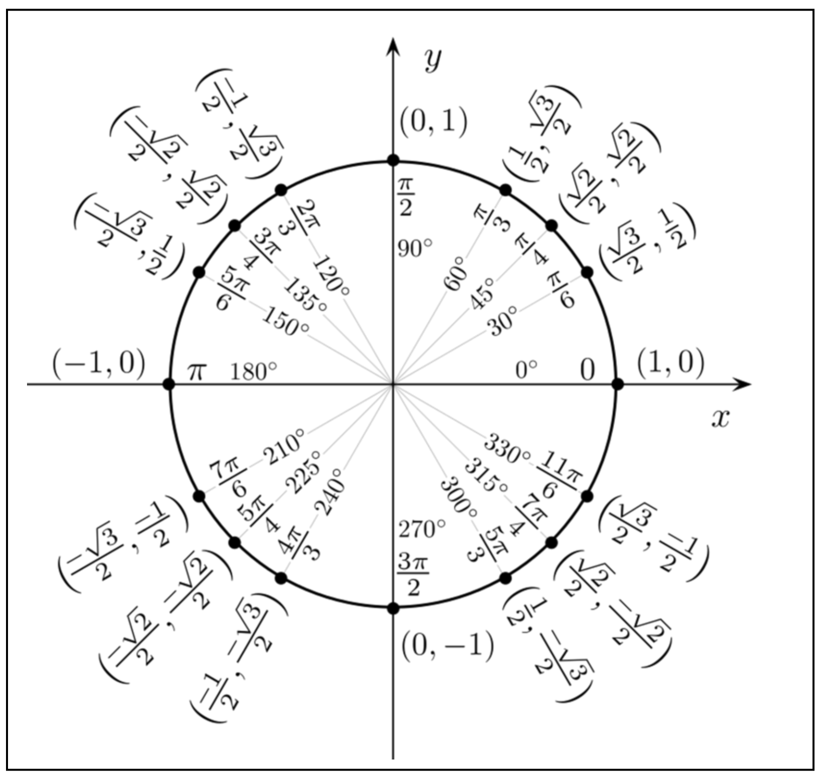
*** Even Functions:**

$\cos(-\theta) = \cos\theta$

$\sec(-\theta) = \sec\theta$

Rules: $\sin\theta, \cos\theta$ and $\tan\theta$ for some standard angles:

θ	$0 = 2\pi$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	1	0	-1
$\cos\theta$	1	0	-1	0
$\tan\theta$	0	∞	0	$-\infty$



θ	$\frac{\pi}{6} = 30^\circ$	$\frac{\pi}{4} = 45^\circ$	$\frac{\pi}{3} = 60^\circ$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

<u>Rules:</u> The $\sin\theta, \cos\theta$, and $\tan\theta$ take positive and negative signs depends on the position in which quarter.

Problems (5.1): Dear student you can use the addition or subtraction laws for two angles for the functions *sin* and *cos* to prove that:

1. $cos(\theta + 2\pi) = cos(\theta)$
2. $sin(\theta + 2\pi) = sin(\theta)$
3. $tan(\theta + 2\pi) = tan(\theta)$
4. $cot(\theta + 2\pi) = cot(\theta)$
5. $sec(\theta + 2\pi) = sec(\theta)$
6. $csc(\theta + 2\pi) = csc(\theta)$
7. $cos(\theta + \pi/2) = -sin(\theta)$
8. $cos(\theta - \pi/2) = sin(\theta)$
9. $sin(\theta + \pi/2) = cos(\theta)$
10. $sin(\theta - \pi/2) = -cos(\theta)$

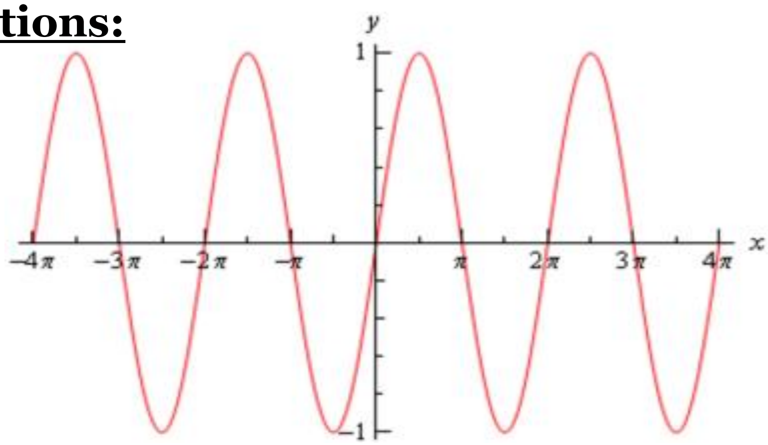
Graphs of Trigonometric Functions:

1. $y = \sin\theta$

Domain: $= \mathbb{R}$

Range: $= [-1,1]$

Period: $= 2\pi$

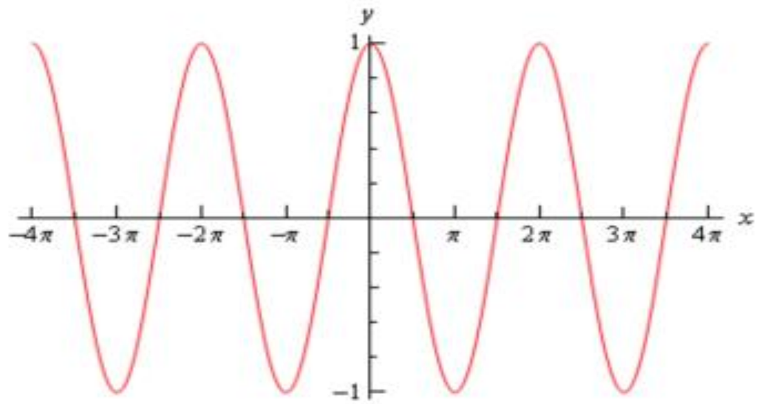


2. $y = \cos\theta$

Domain: $= \mathbb{R}$

Range: $= [-1,1]$

Period : $= 2\pi$

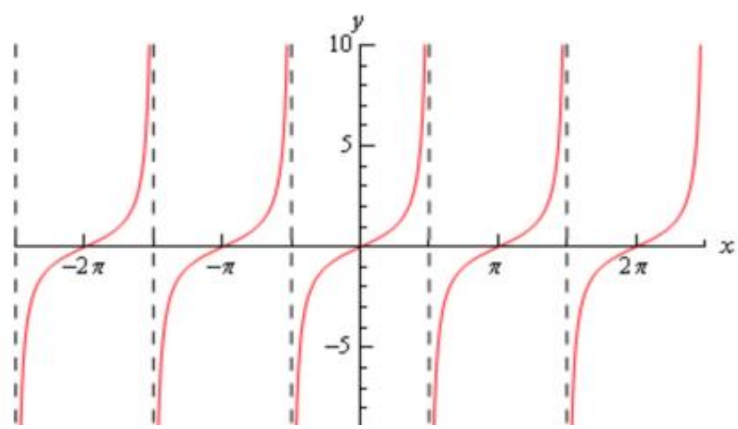


3. $y = \tan\theta$

Domain: $= \mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi : n = 0, \mp 1, \mp 2, \ldots\right\}$

Range : $= \mathbb{R}$

Period: $= \pi$

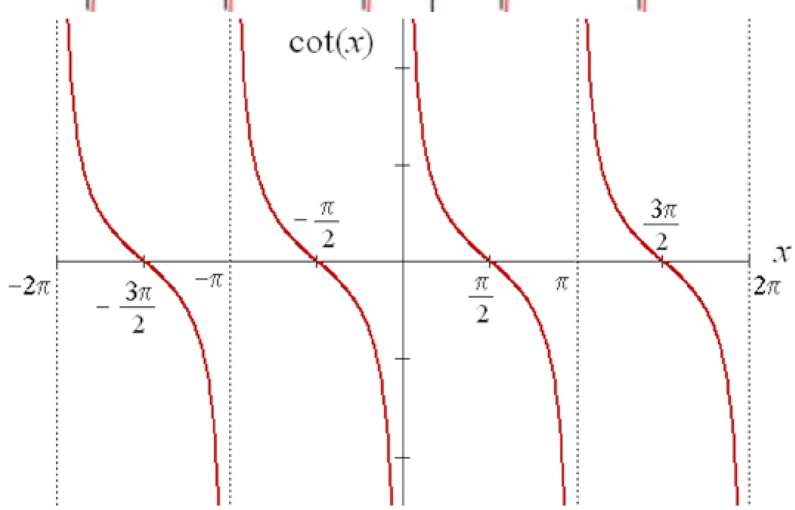


4. $y = \cot\theta$

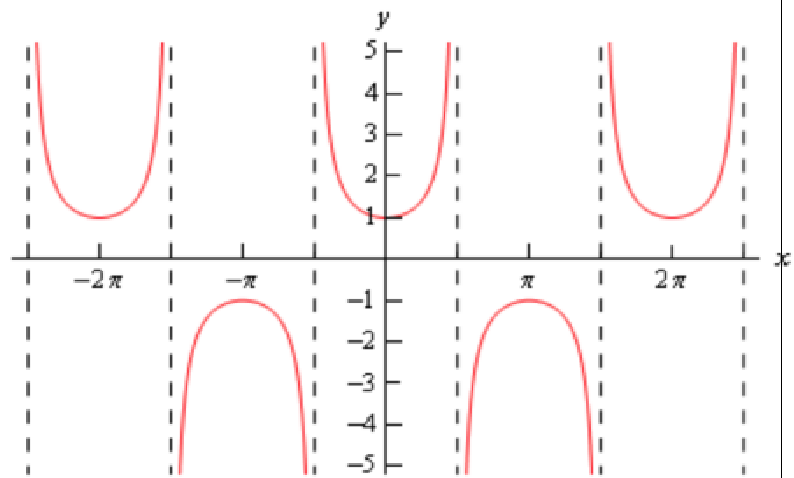
Domain: $= \mathbb{R} \setminus \{n\pi : n = 0, \mp 1, \mp 2, \ldots\}$

Range : $= \mathbb{R}$

Period: $= \pi$



5. $y = \sec\theta$

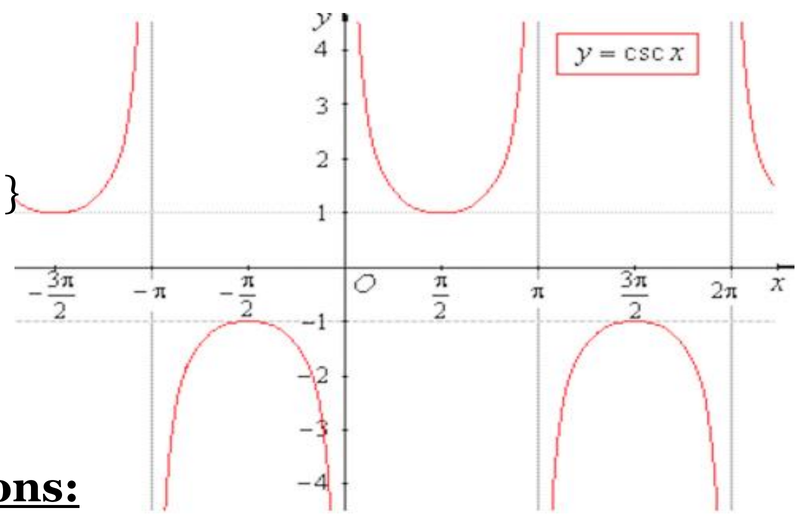


Domain: $= \mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi : n = 0, \mp 1, \mp 2, \ldots\right\}$

Range: $= \mathbb{R} \setminus (-1,1)$

Period: $= 2\pi$

6. $y = \csc\theta$



Domain: $= \mathbb{R} \setminus \{n\pi : n = 0, \mp 1, \mp 2, \ldots\}$

Range: $= \mathbb{R} \setminus (-1,1)$

Period: $= 2\pi$

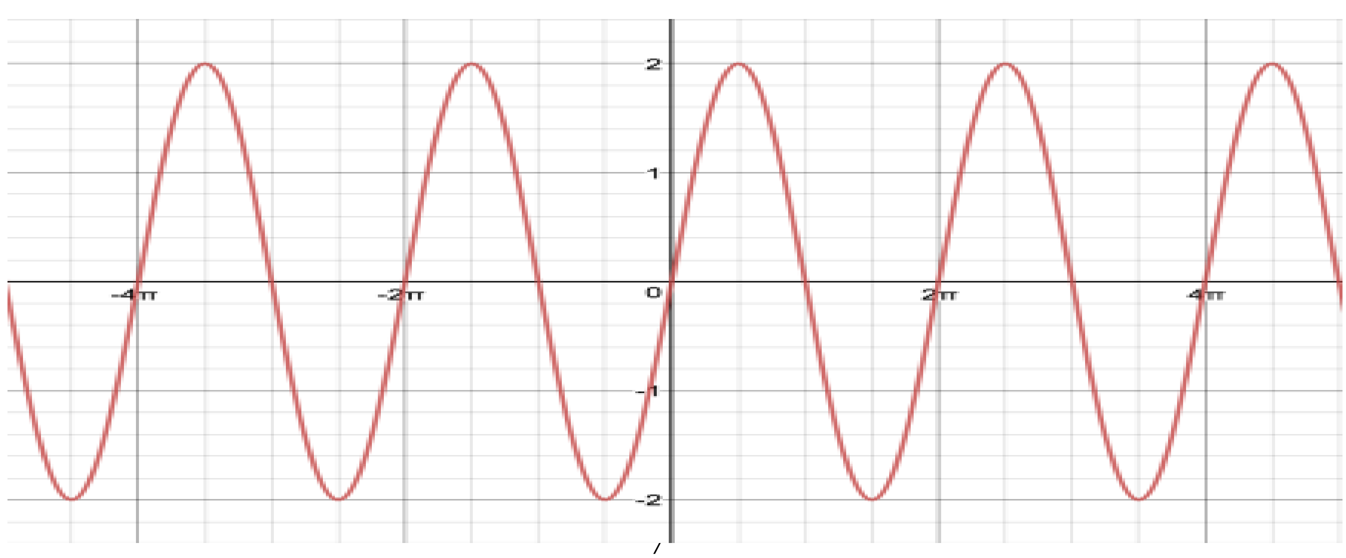
<u>Shifting Trigonometric Functions:</u>

Examples: Plot the following functions:

(1) $y = 2\sin\theta$

$\because D_{\sin\theta} := \mathbb{R} \Longrightarrow D_{2\sin\theta} := \mathbb{R}$

$\because R_{\sin\theta} := [-1,1] \Longrightarrow R_{2\sin\theta} := [-2,2]$

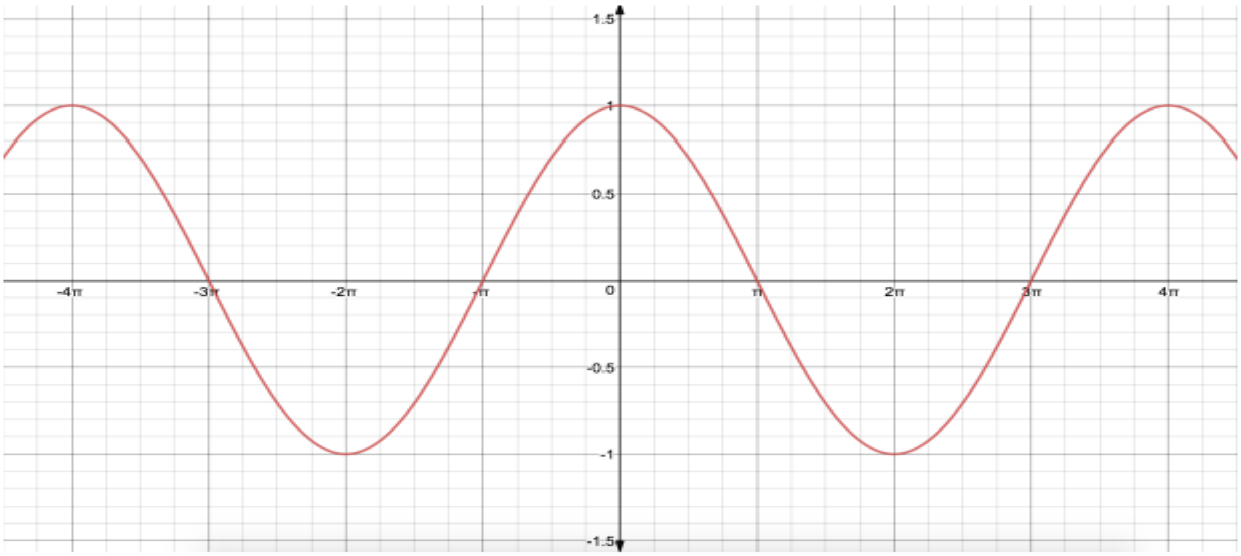


(2) $y = \cos\frac{\theta}{2}$

$\because D_{\cos\theta} := \mathbb{R} \Longrightarrow D_{\cos\frac{\theta}{2}} := \mathbb{R}$

but, $\because -2\pi \leq \dfrac{\theta}{2} \leq 2\pi \quad \Longrightarrow -4\pi \leq \theta \leq 4\pi$

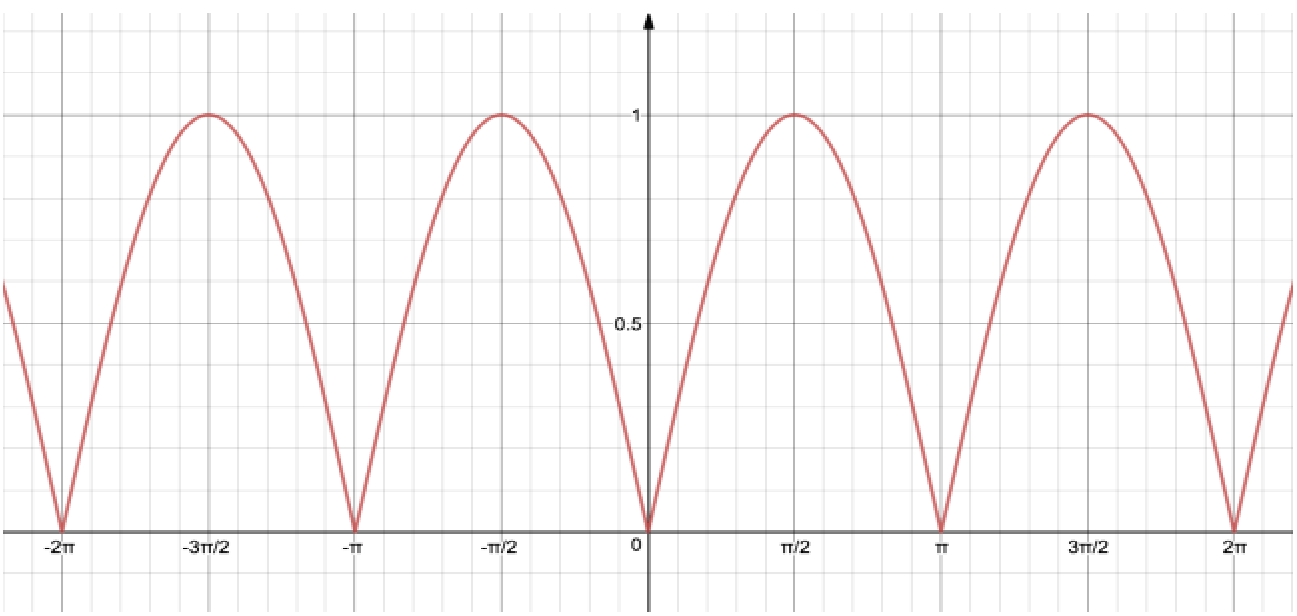
$\because R_{\cos\theta} := [-1,1] \quad \Longrightarrow R_{\cos\frac{\theta}{2}} := [-1,-1]$



(3) $y = |\sin\theta|$

$\because D_{\sin\theta} := \mathbb{R} \quad \Longrightarrow D_{|\sin\theta|} := \mathbb{R}$

$\because R_{\sin\theta} := [-1,1] \quad \Longrightarrow R_{|\sin\theta|} := [0,1]$

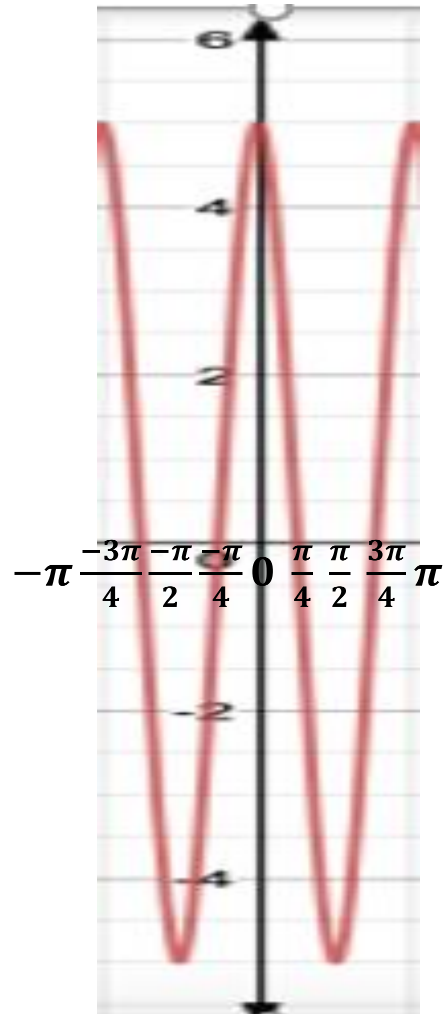


(4) $y = 5\cos(2\theta)$

$\because D_{\cos\theta} := \mathbb{R} \quad \Longrightarrow D_{5\cos(2\theta)} := \mathbb{R}$

but, $\because -2\pi \le 2\theta \le 2\pi \Longrightarrow -\pi \le \theta \le \pi$

$\because R_{\cos} := [-1,1] \Longrightarrow R_{5\cos(2\theta)} := [-5,5]$



Problems (5.2):

Q1/ Sketch the graph the following functions:

1) $y = \sin\left(\frac{\theta}{2}\right)$

2) $y = \cos(3\theta)$

3) $y = 1 + \sin(\theta)$

4) $y = \frac{1+\cos(2\theta)}{2}$

5) $y = |\sin(4\theta)|$

6) $y = 2\sin(\theta + \pi)$

7) $y = 1 + \cos(-x)$

8) $y = \cos\left(x - \frac{\pi}{2}\right) - 1$

9) $y = \cos\left(\frac{x}{2}\right) + 2$

10) $y = |\cos x| - 1$

11) $y = -\sin(-x)$

Q2/Prove that:

$a)$ $\sin^2\theta + \cos^2\theta + \tan^2\theta = \sec^2\theta$ $\qquad b)$ $\tan^2\theta - \sin^2\theta = \tan^2\theta \, . \sin^2\theta$

$c)$ $\sqrt{\frac{1+\tan^2\theta}{1+\cot^2\theta}} = \tan\theta$ $\qquad d)$ $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \csc\theta + \cot\theta$ $\qquad e)$ $\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

Limits of Trigonometric Functions:

Theorems:

1. $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

2. $\lim\limits_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$

Result: $\lim\limits_{\theta \to 0} \dfrac{\tan\theta}{\theta} = 1$

proof:

$$\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \frac{\frac{\sin\theta}{\cos\theta}}{\theta}$$

$$= \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta}\right)$$

$$= \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right) \cdot \left(\lim_{\theta \to 0} \frac{1}{\cos\theta}\right)$$

$$= (1) \cdot \left(\frac{1}{\cos(0)}\right) = 1$$

Examples: Find the following limits?

(1) $\lim\limits_{t \to 0} \dfrac{\sin 3t}{t}$

$$\lim_{t \to 0} \frac{\sin 3t}{t} = \lim_{t \to 0} \frac{3 \cdot \sin 3t}{3.t} = 3 \cdot \lim_{t \to 0} \frac{\sin 3t}{3t} = 3.1 = 3$$

(2) $\lim_{x \to \infty} x \cdot \sin\frac{1}{x}$

Let $y = \frac{1}{x} \Longrightarrow x = \frac{1}{y}$

$\because x \to \infty \Longrightarrow y \to 0$

Hence, $\lim_{x\to\infty} x \cdot \sin\frac{1}{x} = \lim_{y\to 0}\frac{1}{y}\cdot \sin y = \lim_{y\to 0}\frac{\sin y}{y} = 1$

(3) $\lim\limits_{x\to 0}\frac{1-\cos x}{2x}$

$$\lim_{x\to 0}\frac{1-\cos x}{2x} = \lim_{x\to 0}\frac{-1}{2}\cdot\frac{\cos x - 1}{x} = \frac{-1}{2}\cdot\lim_{x\to 0}\frac{\cos x - 1}{x} = \frac{-1}{2}\cdot 0 = 0$$

(4) $\lim\limits_{h\to 2}\frac{\cos\left(\frac{\pi}{h}\right)}{h-2}$

$$\lim_{h\to 2}\frac{\cos\left(\frac{\pi}{h}\right)}{h-2} = \frac{0}{0}$$

$$\therefore \lim_{h\to 2}\frac{\cos\left(\frac{\pi}{h}\right)}{h-2} \overset{L'R}{=} \lim_{h\to 2}\frac{-\sin\left(\frac{\pi}{h}\right)\cdot\frac{h\cdot 0-\pi\cdot 1}{h^2}}{1-0} = -\lim_{h\to 2}\left(-\sin\left(\frac{\pi}{h}\right)\cdot\frac{\pi}{h^2}\right)$$

$$= -\sin\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{4} = -1\cdot\frac{-\pi}{4} = \frac{\pi}{4}$$

(5) $\lim\limits_{\theta\to\frac{\pi}{2}}\frac{1-\sin\theta}{1+\cos(2\theta)}$

$$\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin\theta}{1+\cos(2\theta)} = \frac{0}{0}$$

$$\therefore \lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin\theta}{1+\cos(2\theta)} \overset{L'R}{=} \lim_{\theta\to\frac{\boldsymbol{\pi}^2}{2}}\frac{-\cos\theta}{-2\sin(2\theta)}$$

$$= \frac{1}{2}\cdot\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{\sin(2\theta)}$$

$$\overset{L'R}{=} \frac{-1}{2}\cdot\lim_{\theta\to\frac{\pi}{2}}\frac{-\sin\theta}{2\cos(2\theta)}$$

$$= \frac{-1}{4}\lim_{\theta\to\frac{\pi}{2}}\frac{\sin\theta}{\cos(2\theta)} = \frac{-1}{4}\cdot\frac{\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)} = \frac{-1}{4}\cdot\frac{1}{-1} = \frac{1}{4}$$

Problems (5.3): Evaluate the following limits, if it exist?

1. $\lim_{w\to 0} \dfrac{3\sin(5w)}{7w}$
2. $\lim_{x\to 0} \dfrac{\sin(3x)}{x\cos(5x)}$
3. $\lim_{t\to 0} \dfrac{5t}{\tan(6t)}$
4. $\lim_{\theta\to 0} \dfrac{\sec\theta - \cos\theta}{\theta^2}$
5. $\lim_{x\to 0} \dfrac{1-\cos^2 x}{2x^2}$
6. $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\theta\cdot\sin\theta}$
7. $\lim_{z\to 0}(\tan(2z)\cdot\csc(4z))$
8. $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\theta^2}$
9. $\lim_{t\to 0} \dfrac{-2\tan t}{5t}$
10. $\lim_{y\to 0} \dfrac{1-\cos y}{\sin y}$

Q2/Proof the following:

(a) $\lim_{\theta\to\frac{\pi}{2}} \dfrac{\sin\theta - 1}{\cos\theta} = 0$ (b) $\lim_{w\to -4} \dfrac{\sin(\pi w)}{w^2-16} = -\dfrac{\pi}{8}$

Differentiation of Trigonometric Functions:

Let u be a function of x, then:

1. $\boldsymbol{\dfrac{d}{dx}(\sin(u)) = \cos(u)\cdot\dfrac{du}{dx}}$
2. $\boldsymbol{\dfrac{d}{dx}(\cos(u)) = -\sin(u)\cdot\dfrac{du}{dx}}$
3. $\boldsymbol{\dfrac{d}{dx}(\tan(u)) = \sec^2(u)\cdot\dfrac{du}{dx}}$
4. $\boldsymbol{\dfrac{d}{dx}(\cot(u)) = -\csc^2(u)\cdot\dfrac{du}{dx}}$
5. $\boldsymbol{\dfrac{d}{dx}(\sec(u)) = \sec(u)\cdot\tan(u)\cdot\dfrac{du}{dx}}$

6. $\frac{d}{dx}(\csc(u)) = -\csc(u)\cdot\cot(u)\cdot\frac{du}{dx}$

Examples: Find the derivatives of the following functions?

(1) $y = \frac{\sin x}{x}$

$$\frac{dy}{dx} = \frac{x\cdot\cos x - \sin x - 1}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

(2) $y = \frac{2}{\cos(3t)}$

$$\frac{dy}{dt} = \frac{\cos(3t)\cdot 0 - 2\cdot(-\sin(3t)\cdot 3)}{\cos^2(3t)} = \frac{6\sin(3t)}{\cos^2(3t)}$$

(3) $y = \cot(z^2)$

$$\frac{dy}{dz} = -\csc^2(z^2)\cdot 2z$$

(4) $y = \sec^2(5x)$

$$\frac{dy}{dx} = 2\cdot\sec(5x)\cdot\sec(5x)\cdot\tan(5x)\cdot 5 = 10\sec^2(5x)\cdot\tan(5x)$$

(5) $y = \sin(\cos w)$

$$\frac{dy}{dw} = \cos(\cos w)\,.\,(-\sin w)\cdot 1 = \cos(\cos w)\,.\,-\sin w$$

Problems (5.4): Find the derivative of the following functions?

1. $y = \left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right)^3$

2. $y = 4\cos^2(-3w)$

3. $y = \sin^2\left(\frac{3}{z}\right) + \cos^2(z^2)$

4. $y = \sqrt[3]{9x + \cos(2x)}$

5. $y = \frac{\sqrt{2t}}{\cos(3t)}$

6. $y = x^3\cdot\sin(2x^2+3)$

7. $y = \left(\cos^2(1+t) + \sqrt{t+5}\right)^5$

8. $y = \dfrac{7\sqrt{\sec(3\theta)}}{\theta^2}$

9. $y = 2\sin\left(\dfrac{z}{2}\right) - \left(\mathrm{z}\cos\left(\dfrac{2}{z}\right)\right)^3$

10. $y = \sin(3t) \cdot \cos(5t^2)$

Implicit Differentiation of Trigonometric Functions:

Examples: Find y' of the following functions?

(1) $x\sin(2y) = y \cdot \cos(2x)$

$\Longrightarrow x \cdot \cos(2y) \cdot 2y' + \sin(2y) \cdot 1 = y \cdot (-\sin(2x)) \cdot 2 + \cos(2x) \cdot y'$

$\Longrightarrow x \cdot \cos(2y) \cdot 2y' - \cos(2x) \cdot y' = y \cdot (-\sin(2x)) \cdot 2 - \sin(2y)$

$\Longrightarrow (2x \cdot \cos(2y) - \cos(2x)) \cdot y' = -2y \cdot \sin(2x) - \sin(2y)$

$$\Longrightarrow y' = \frac{-2y \cdot \sin(2x) - \sin(2y)}{(2x \cdot \cos(2y) - \cos(2x))}$$

(2) $\cot(xy) + xy = 0$

$\Longrightarrow -\csc^2(xy)(xy' + y \cdot 1) + xy' + y = 0$

$\Longrightarrow -x\csc^2(xy)y' - y\csc^2(xy) + xy' + y = 0$

$\Longrightarrow (-x\csc^2(xy) + x)y' = y\csc^2(xy) - y$

$$\Longrightarrow y' = \frac{y\csc^2(xy) - y}{-x\cos^2(xy) + x}$$

$$\Longrightarrow y' = \frac{y(\cos^2(xy) - 1)}{-x(\csc^2(xy) - 1)}$$

$$\Longrightarrow y' = -\frac{y}{x}$$

Problems (5.5): Find y' of the following functions?

1. $y\sin x + x\sin y = y^2$

2. $\sec^2 y + \csc^2 y = 4\mathrm{x}$

3. $y = \tan(x+y)$

4. $y^2 = \sin^4(2x) + \cos^4(2x)$

5. $\cos(x^2y^2) = x$

6. $x^2y = \dfrac{\cot y}{1+\cos y}$

7. $\sqrt{xy} + \csc(-xy) = y$

8. $y(3+\tan y)^{\frac{1}{3}} = x + 5$

9. $y = \tan y + \sec^2(xy) + \cot(x^2+y^2)$

10. $x^2 = \sin y + \sin(2y)$

The Inverse Trigonometric Functions:

Suppose f be a one-to-one (i.e. 1-1) and onto function.

$$f: X \longrightarrow Y$$

- f is $1-1 \Leftrightarrow x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$$\overset{OR}{\Leftrightarrow} f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

- f is onto $\Leftrightarrow \forall y \in Y\ \exists x \in X \ni y = f(x)$
- $f: X \longrightarrow Y \ni y = f(x) \Leftrightarrow f^{-1}: Y \longrightarrow X \ni x = f^{-1}(y)$

(1) The Inverse of Sine Function:

Let $y = \sin(x)$, $\sin(x): \mathbb{R} \to [-1,1]$

We are going to define a new function which is inverse sine, and we denote it by $\sin^{-1}$ or arcsin.

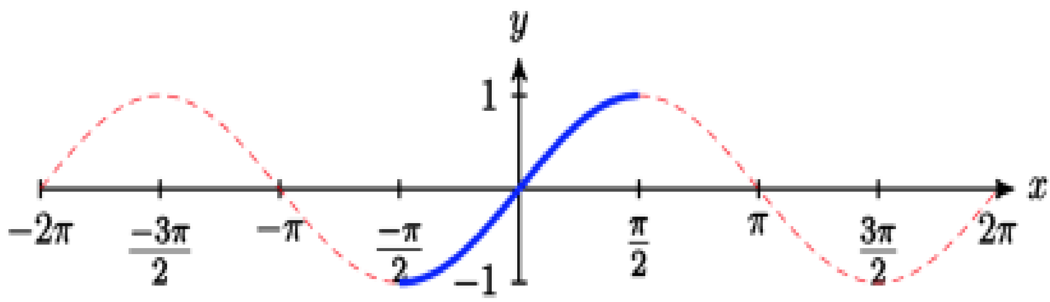
$\therefore \sin^{-1} y = \sin^{-1}(\sin x) \Rightarrow \sin^{-1} y = x$

$\therefore y = \sin x \Leftrightarrow x = \sin^{-1} y$

$\sin(x): \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow [-1,1]$

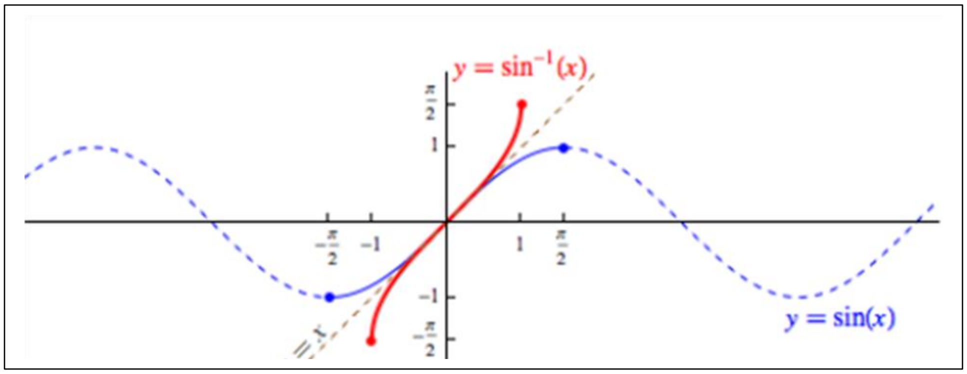
$\because \sin\ is\ 1-1\ and\ onto \Rightarrow \exists\ \sin^{-1} \ni \sin^{-1}: [-1,1] \Rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$D_{\sin^{-1}} = [-1,1] = R_{\sin x}$



$R_{\sin^{-1}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = D_{\sin x}$

Note: $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$



Remark: $\sin^{-1}$ *is* an odd function. **(i.e., $\sin^{-1}(-x) = -\sin^{-1}(x)$)**

(2) The Inverse of Cosine Function:

Let $y = \cos(x)$, $\cos(x): \mathbb{R} \to [-1,1]$

We are going to define a new function which is inverse cosine, and we denote it by $\cos^{-1}$ or arccos.

$\therefore \cos^{-1} y = \cos^{-1}(\cos x) \Longrightarrow \cos^{-1} y = x$

$\therefore y = \cos x \Leftrightarrow x = \cos^{-1} y$

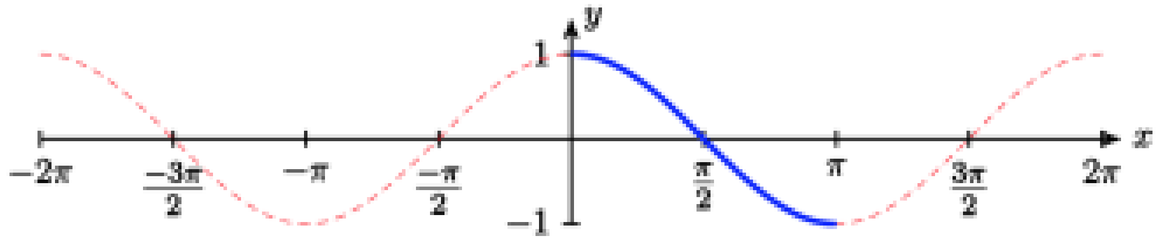
$\cos: [0, \pi] \Longrightarrow [-1,1]$

$\because$ *cos is* $1-1$ *and onto* $\Longrightarrow \exists \cos^{-1} \ni$

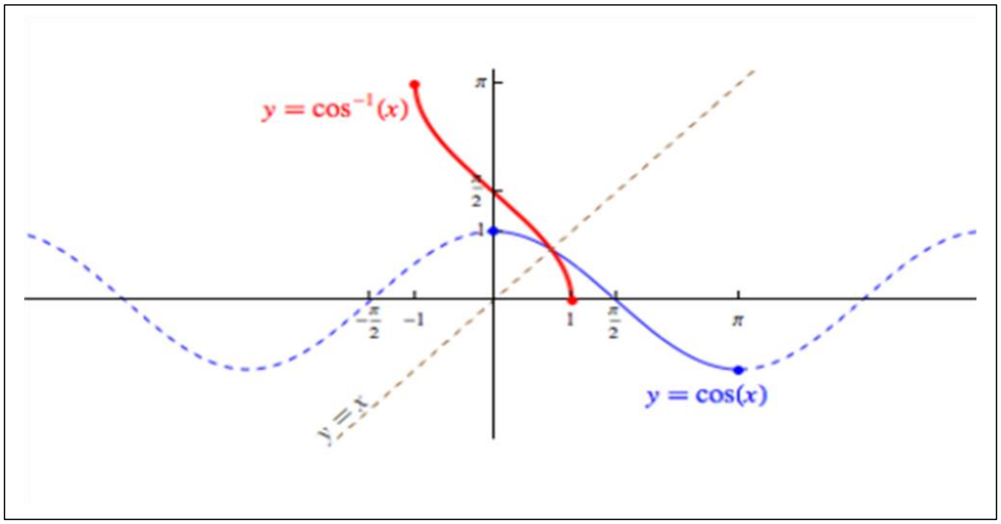
$\cos^{-1}: [-1,1] \Longrightarrow [0,\pi]$

$D_{\cos^{-1}} = [-1,1] = R_{\cos x}$

$R_{\cos^{-1}} = [0,\pi] = D_{\cos x}$



Note: $\cos^{-1}(x) \neq \frac{1}{\cos(x)}$



Remark: $\cos^{-1}$ is neither even nor odd function.

Note: $\boldsymbol{\cos^{-1}(-x) = \pi - \cos^{-1}(x)}$

(3) The Inverse of Tangent Function:

Let $y = \tan(x)$, $\tan(x): \mathbb{R} \setminus \left\{x: x = \frac{\pi}{2} + n\pi; n \in \mathbb{I}\right\} \longrightarrow \mathbb{R}$

We are going to define a new function which is inverse tangent, and we denote it by $\tan^{-1}$ or arctan.

$\therefore \tan^{-1} y = \tan^{-1}(\tan x) \Longrightarrow \tan^{-1} y = x$

$\therefore y = \tan x \Leftrightarrow x = \tan^{-1} y$

$\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$

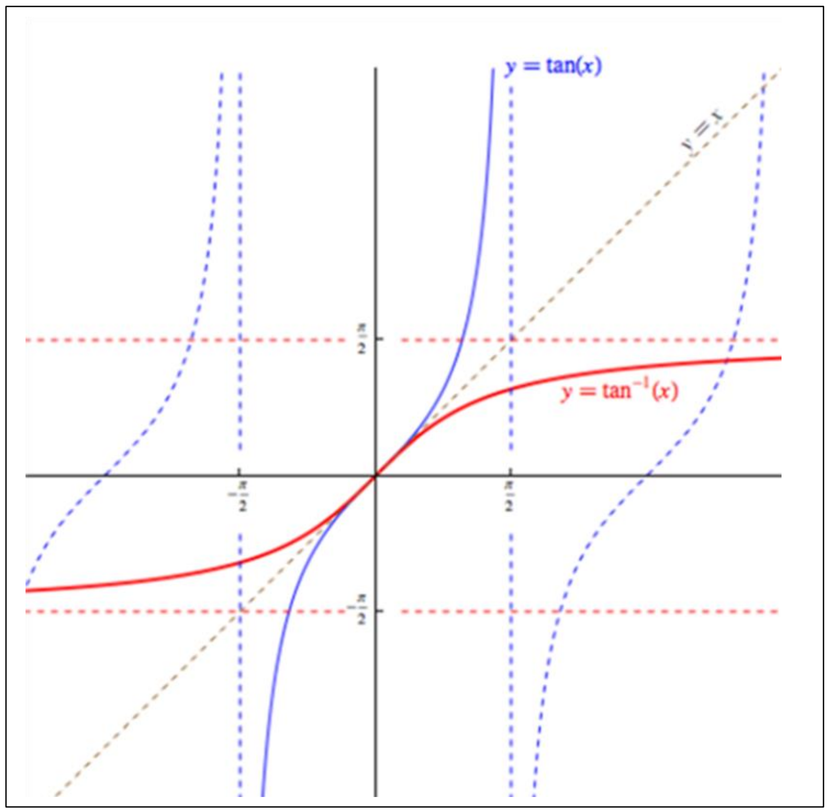
$\because \tan\ is\ 1-1\ and\ onto, \Longrightarrow \exists \tan^{-1} \ni$

$\tan^{-1}: \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$D_{\tan^{-1}} = \mathbb{R} = R_{\tan x}$

$$R_{\tan^{-1}} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = D_{\tan x}$$

Note: $\tan^{-1}(x) \neq \frac{1}{\tan(x)}$



Remark: $\tan^{-1}$ *is* an odd function. $\boldsymbol{i.e}$, $(\mathbf{tan}^{-1}(-x) = -\,tan^{-1}(x))$

The Derivative of Inverse Trigonometric Functions:

Let u be a function of x, then:

1. $$\frac{d}{dx}\left(\sin^{-1}(u)\right) = \frac{1}{\sqrt{1 - u^2}} \cdot \frac{du}{dx}$$

2. $$\frac{d}{dx}\left(\cos^{-1}(u)\right) = \frac{-1}{\sqrt{1 - u^2}} \cdot \frac{du}{dx}$$

3. $$\frac{d}{dx}\left(\tan^{-1}(u)\right) = \frac{1}{1 + u^2} \cdot \frac{du}{dx}$$

4. $$\frac{d}{dx}\left(\cot^{-1}(u)\right) = \frac{-1}{1 + u^2} \cdot \frac{du}{dx}$$

5. $$\frac{d}{dx}\left(\sec^{-1}(u)\right) = \frac{1}{|u|\sqrt{u^2 - 1}} \cdot \frac{du}{dx}$$

6. $$\frac{d}{dx}\left(\csc^{-1}(u)\right) = \frac{-1}{|u|\sqrt{u^2 - 1}} \cdot \frac{du}{dx}$$

Examples: Find the derivatives of the following functions:

(1) $f(x) = \sin^{-1}(x^2)$

$$\Longrightarrow f'(x) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot 2x$$

(2) $g(t) = \cos^{-1}(\sqrt{t})$

$$\Longrightarrow g'(t) = \frac{-1}{\sqrt{1 - (\sqrt{t})^2}} \cdot \frac{1}{2} t^{\frac{-1}{2}}$$

(3) $y = \sin^{-1}\sqrt{1 - \sqrt{\theta}}$

$$\Longrightarrow y' = \frac{1}{\sqrt{1 - (\sqrt{1 - \sqrt{\theta}})^2}} \cdot \frac{1}{2}(1 - \sqrt{\theta})^{\frac{-1}{2}} \cdot \frac{1}{2}\theta^{\frac{-1}{2}}$$

(4) $f(x) = \cot^{-1}\left(\frac{1-x}{1+x}\right)$

$$\Longrightarrow f'(x) = \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \cdot \frac{(1+x)\cdot(-1)-(1-x)\cdot 1}{(1+x)^2}$$

(5) $g(x) = \sec^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

$$\Longrightarrow g'(x) = \frac{1}{\left|\frac{\sqrt{1+x^2}}{x}\right|\sqrt{\left(\frac{1+x^2}{x}\right)^2 - 1}} \cdot \frac{x\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x - \sqrt{1+x^2}\cdot 1}{x^2}$$

(6) $y = x\cdot\csc^{-1}\left(\frac{1}{x}\right) + \sqrt{1-x^2}$

$$\Longrightarrow y' = x\cdot\frac{-1}{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2}-1}}\cdot\frac{x\cdot 0 - 1.1}{x^2} + \csc^{-1}\left(\frac{1}{x}\right)\cdot 1 + \frac{1}{2}(1-x^2)^{\frac{-1}{2}}\cdot -2x$$

Problems (5.6):

1 Find y' of the following functions?

(a) $y = \sin^{-1}\left(\frac{x-1}{x+1}\right)$

(b) $y = \theta\cdot(\sin^{-1}(\theta))^2 - 2\theta + 2\sqrt{1-\theta}\cdot\sin^{-1}(\theta)$

(c) $y = t\cdot\cos^{-1}(2t) - \frac{1}{2}\sqrt{1-4t^2}$

(d) $y = \frac{\cos^{-1}(2x)}{\sqrt{1+4x^2}}$

(e) $y = \cos^{-1}\left(\frac{3}{t}\right) + \frac{t}{1-t^2}$

(f) $y = \sec^{-1}\left(\sqrt{w^2+4}\right)$

(g) $y = \sin(\tan^{-1} x)$

(h) $y = \tan^{-1}(3\tan 2z)$

(i) $y = \sec^{-1}(5x^2)$

$(j)\ \ y = \dfrac{\cot^{-1}(3\theta)}{1+\theta^2}$

2 Find y' of the following functions?

(a) $x\sin y + x^3 = \tan^{-1} y$

(b) $\sin^{-1}(xy) = \cos^{-1}(x+y)$

$(c)\ \ \cos 3y - \cos^{-1}(y) - \dfrac{\sin x}{2x} = 0$

<u>Algebra of Inverse Trigonometric Functions:</u>

Some properties for the inverse trigonometric functions:

$*\ \ \boldsymbol{\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)}$

$*\ \ \boldsymbol{\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)}$

$*\ \ \boldsymbol{\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)}$

Examples: Evaluate the following:

1 $\mathbf{\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = ?}$

$$\therefore \cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \cos\cos^{-1}\left(\frac{1}{2}\right) = \mathrm{I}\left(\frac{1}{2}\right) = \frac{1}{2}$$

2 $\mathbf{\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = ?}$

Let $\alpha = \cos^{-1}\frac{\sqrt{2}}{2}$

$$\Longrightarrow \alpha = \cos^{-1}\frac{1}{\sqrt{2}} \Longrightarrow \cos\alpha = \frac{1}{\sqrt{2}} \Longrightarrow \alpha = \frac{\pi}{4} = 45°$$

$$\therefore \sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

3 $\mathbf{\csc(\sec^{-1}(2)) = ?}$

Let $\alpha = \sec^{-1}(2)$

$$\Longrightarrow \alpha = \sec^{-1}(2) = \cos^{-1}\left(\frac{1}{2}\right) \Longrightarrow \cos\alpha = \frac{1}{2} \Longrightarrow \alpha = \frac{\pi}{3} = 60°$$

$$\therefore \csc(\sec^{-1}(2)) = \csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

4 $\mathbf{\cot\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = ?}$

Let $\alpha = \sin^{-1}\frac{1}{2} \quad \Longrightarrow \sin\alpha = \frac{1}{2} \Longrightarrow \alpha = \frac{\pi}{6}$

$$\therefore \cot\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \cot\left(\frac{\pi}{6}\right) = \cot\frac{\pi}{6}$$

$$= \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

5 $\mathbf{cos\left(sin^{-1}\left(\frac{8}{10}\right)\right) = ?}$

Let $\alpha = \sin^{-1}\left(\frac{8}{10}\right) \Longrightarrow \sin\alpha = \frac{8}{10} \Longrightarrow \mathbf{sin^2}\alpha = \frac{64}{100}$

$$\because \mathbf{cos^2}\alpha = 1 - \mathbf{sin^2}\alpha \Longrightarrow \mathbf{cos^2}\alpha = 1 - \frac{64}{100} = \frac{36}{100}$$

$\Longrightarrow \cos(\alpha) = \frac{6}{10}$

6 $\mathbf{cos\left(sin^{-1}\left(\frac{1}{3}\right) - tan^{-1}\left(\frac{1}{2}\right)\right) = ?}$

Let $\alpha = \sin^{-1}\left(\frac{1}{3}\right) \Longrightarrow \sin\alpha = \frac{1}{3}$

Let $\beta = \tan^{-1}\left(\frac{1}{2}\right) \Longrightarrow \tan\beta = \frac{1}{2}$

$\because \cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$$\therefore \cos\left(\sin^{-1}\left(\frac{1}{3}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right) = \cos(\alpha - \beta)$$

$$= cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$$

$$= \frac{2\sqrt{2}}{3} . \frac{2}{\sqrt{5}} + \frac{1}{3} . \frac{1}{\sqrt{5}} = \frac{4\sqrt{2} + 1}{3\sqrt{5}}$$

<u>Problems (5.7):</u> Evaluate the following?

1. $\sec\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$
2. $\cos(cot^{-1}(1))$
3. $\tan\left(sin^{-1}\left(-\frac{1}{2}\right)\right)$
4. $\csc\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$
5. $\cos^{-1}\left(-\sin\left(\frac{\pi}{6}\right)\right)$
6. $\cos\left(\cos^{-1}\left(\frac{3}{4}\right) - \cot^{-1}\left(\frac{1}{4}\right)\right)$

Hyperbolic Functions:

1. $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$
2. $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$
3. $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
4. $\coth(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
5. $\operatorname{sech}(x) = \dfrac{2}{e^x + e^{-x}}$
6. $\operatorname{csch}(x) = \dfrac{2}{e^x - e^{-x}}$

Remarks:

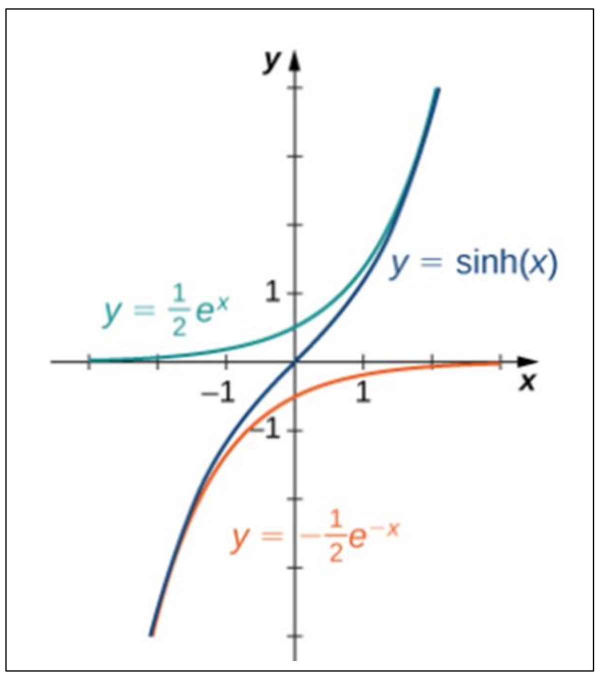
* $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$
* $\coth(x) = \dfrac{1}{\tanh(x)} = \dfrac{\cosh(x)}{\sinh(x)}$
* $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$
* $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)}$

The Graph of Hyperbolic Functions:

1. $y = \sinh(x)$

   Domain $:= \mathbb{R}$

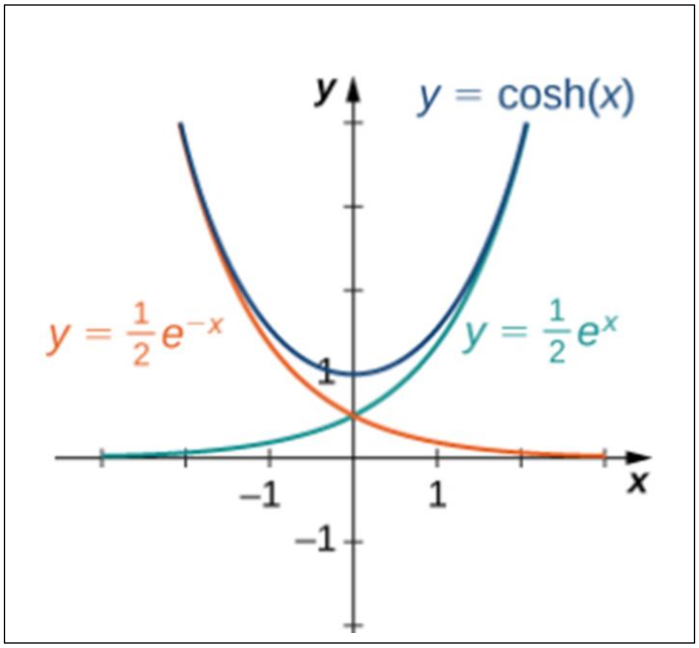
   Range $:= \mathbb{R}$



2. $y = \cosh(x)$

   Domain: $=\mathbb{R}$

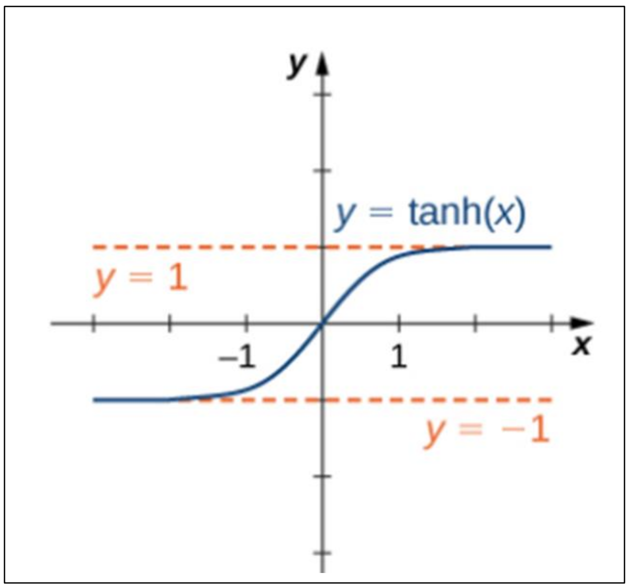
   Range $:= [1, \infty)$



3. $y = \tanh(x)$

   Domain: $=\mathbb{R}$

   Range $:= (-1,1)$



Some Facts about Hyperbolic Functions:

1. $\cosh^2(x) - \sinh^2(x) = 1$
2. $1 - \tanh^2(x) = \operatorname{sech}^2(x)$
3. $\coth^2(x) - 1 = \operatorname{csch}^2(x)$

4. $\cosh(-x) = \cosh(x)$ "Even Function",

   $\sinh(-x) = -\sinh(x)$ "Odd Function",

   $\tanh(-x) = -\tanh(x)$ "Odd Function"

5. $\cosh(x) + \sinh(x) = e^x$,

   $\cosh(x) - \sinh(x) = e^{-x}$

6. $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$,

   $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$,

   $$\tanh(x + y) = \frac{\tanh(x) + \tanh(y)}{1 - \tanh(x)\tanh(y)}$$

7. $\cosh^2(x) = \frac{1}{2}(\cosh(2x) + 1)$,

   $$\sinh^2(x) = \frac{1}{2}(\cosh(2x) - 1)$$

8. $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$,

   $\sinh(2x) = 2\sinh(x)\cosh(x)$

<u>The Derivative of Hyperbolic Functions:</u>

Let u be a function of x, then:

1 $\frac{d}{dx}(\sinh(u)) = \cosh(u) \cdot \frac{du}{dx}$

2 $\frac{d}{dx}(\cosh(u)) = \sinh(u) \cdot \frac{du}{dx}$

3 $\frac{d}{dx}(\tanh(u)) = \operatorname{sech}^2(u) \cdot \frac{du}{dx}$

4 $\frac{d}{dx}(\coth(u)) = -\operatorname{csch}^2(u) \cdot \frac{du}{dx}$

5 $\frac{d}{dx}(\mathbf{sech}(\boldsymbol{u})) = -\mathbf{sech}(\boldsymbol{u}) \cdot \mathbf{tanh}(\boldsymbol{u}) \cdot \frac{d\boldsymbol{u}}{dx}$

6 $\frac{d}{dx}(\mathbf{csch}(\boldsymbol{u})) = -\mathbf{csch}(\boldsymbol{u}) \cdot \mathbf{coth}(\boldsymbol{u}) \cdot \frac{d\boldsymbol{u}}{dx}$

Examples: Find the derivatives of the following functions:

- $\sinh(3x)$

  $\Longrightarrow y' = 3\cosh(3x)$

- $y = \cosh^2(5x)$

  $\Longrightarrow y' = 2\cosh(5x) \cdot \sinh(5x).5$

- $\tanh(2x)$

  $\Longrightarrow y' = \operatorname{sech}^2(2x) \cdot 2$

- $y = \coth(\tan(x))$

  $\Longrightarrow y' = -\operatorname{csch}^2(\tan x) \cdot \sec^2 x$

- $y = \operatorname{sech}^3 x$

  $\Longrightarrow y' = 3\operatorname{sech}^2(x) \cdot (-\operatorname{sech}(x)\tanh(x) \cdot 1)$

- $y = 4\operatorname{csch}\left(\frac{x}{4}\right)$

  $\Longrightarrow y' = 4 \cdot \left(-\operatorname{csch}\left(\frac{x}{4}\right)\right) \cdot \coth\left(\frac{x}{4}\right) \cdot \frac{1}{4}$

Problems (5.8): Find y' of the following:

1 $y = \frac{\cosh(x)}{x}$

2 $y = \sinh^2(3w)$

3 $y = e^w \cdot \cosh(w)$

4 $\sin^{-1}(x) = \text{sech}(y)$

5 $y = \tanh\left(\frac{4t+1}{5}\right)$

6 $\tan(x) = \tanh^2(y)$

7 $x = cosh(cos(y))$

8. $\sinh(y) = \sec(x)$

8. $y = \coth\left(\frac{1}{\theta}\right)$

9. $y^2 + x\cosh y + \sinh^2 x = 50$

10. $y = \cosh^2(5x) - \sinh^2(5x)$

11. $y = \text{csch}^3(\sqrt{2x})$

12. $\sinh(y) = \tanh(x)$