



جامعة بغداد

كلية التربية للعلوم الصرفة ابن الهيثم

قسم الرياضيات / المرحلة الاولى

مادة التفاضل والتكامل

الفصل الرابع

الاشتقاق

Differentiation

اعضاء التدريس

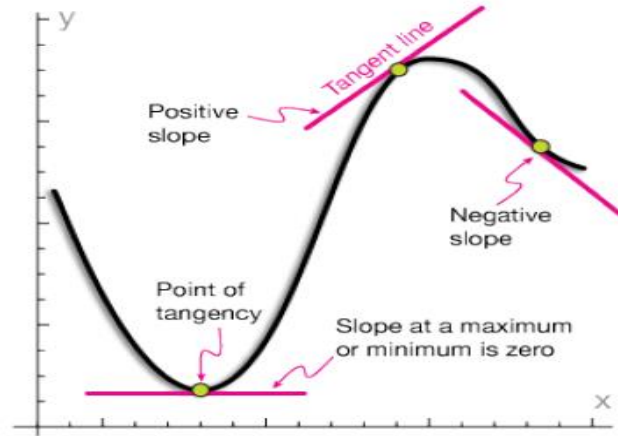
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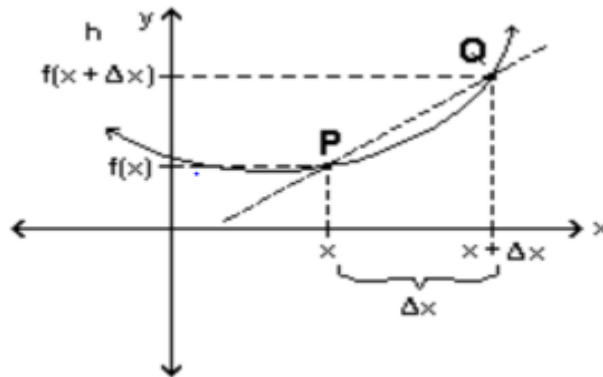
CHAPTER FOUR: Differentiation

For each point on the curve $y = f(x)$, there is a single straight tangent line at the point; the slope of straight tangent of the curve $y = f(x)$ at the point $(x, f(x))$ represents the derivative at that point



Let $P(x, f(x))$ be a fixed point on the curve; and $Q(x + \Delta x, f(x + \Delta x))$ be another point, so $\Delta y = f(x + \Delta x) - f(x)$.

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



Note that: At Δx , decreasing length (close to zero) the straight secant PQ more and more applicability begins on the straight tangent at the point $(x, f(x))$. When $(\Delta x \rightarrow 0)$, knowing that the slope straight tangent at the point $(x, f(x))$ represents a derived function at that point.

$$m_{tan} = \lim_{\Delta x \rightarrow 0} m_{sec} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Remark: When the value of the limit exist, the function is called differentiable function, and f' is called the derivative of f at x .

Remark: The equation of the tangent line at a point (x_1, y_1) is given by the following form:

$$(y - y_1) = m_{tan}(x - x_1)$$

Definition: The normal line of the curve is the line that is perpendicular to the tangent of the curve at a particular

$$m_{\perp} = \frac{-1}{m_{tan}}$$

Remark: The equation of the normal line at a point (x_1, y_1) is given by the following form:

$$(y - y_1) = m_{\perp}(x - x_1)$$

Note: $f'(x) = y' = \frac{dy}{dx} = \frac{df(x)}{dx}$

Example 1: Let $f(x) = 4x - 2$, find $f'(x)$ by using the definition?

Solution:-

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[4(x + \Delta x) - 2] - (4x - 2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{4x + 4\Delta x - 2 - 4x + 2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{4\Delta x}{\Delta x} = 4 \end{aligned}$$

Example 2: Let $f(x) = \sqrt{x}$, find the equation of the tangent line and normal line at the point (4,2) by using the definition?

Solution:-

We need to find: $m_{tan} \Big|_{(4,2)} = f'(x) \Big|_{(4,2)}$

$$\begin{aligned}\Rightarrow f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\&= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\&= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \\ \Rightarrow m_{tan} &= \frac{1}{2\sqrt{x}} \Rightarrow m_{tan} \Big|_{(4,2)} = f'(x) \Big|_{(4,2)} = \frac{1}{2\sqrt{4}} = \frac{1}{4}\end{aligned}$$

Now, we need to find the equation of the tangent line at the point

$$\begin{aligned}(x_1, y_1) &= (4, 2) \\(y - y_1) &= m_{tan}(x - x_1) \\ \Rightarrow y - 2 &= \frac{1}{4}(x - 4) \\ \Rightarrow y &= \frac{1}{4}x + 1\end{aligned}$$

Next, we need to find the equation of the normal line at the point

$$(x_1, y_1) = (4, 2)$$

$$\therefore m_{\perp} = -\frac{1}{m_{tan}} \rightarrow m_{\perp} = -\frac{1}{\frac{1}{4}} = -4$$

$$(y - y_1) = m_{\perp}(x - x_1)$$

$$\Rightarrow (y - 2) = -4(x - 4) \Rightarrow y = -4x + 18$$

Problem 4.1:

1. find $f'(x)$ by using the definition of the following functions:

(a) x^2 (b) $4 - \sqrt{x + 3}$

2. Let $f(x) = x^2$, find the equation of the tangent line and normal line at the point (3,9) by using the definition?

3. Let $f(x) = \sqrt{x + 3}$, find the equation of the tangent line at $x=3$?

Differentiable VS. Continuous:

Theorem: If $f(x)$ is a differentiable function at x_0 , then it is a continuous function at x_0 .

Proof: To prove $f(x)$ is continuous function at x_0 , we need to show:

$$\lim_{x \rightarrow x_0} f(x) = f(x_0) \text{ (i.e., } \lim_{x \rightarrow x_0} [f(x) - f(x_0)] = 0)$$

Suppose that:

$$\Delta x = x - x_0 \Rightarrow x = x_0 + \Delta x \Rightarrow f(x) = f(x_0 + \Delta x)$$

Hence, when $x \rightarrow x_0, \Delta x \rightarrow 0$

$$\begin{aligned}
\lim_{\Delta x \rightarrow 0} [f(x) - f(x_0)] &= \lim_{\Delta x \rightarrow 0} [f(x_0 + \Delta x) - f(x_0)] \\
&= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \cdot \Delta x \right] \\
&= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \cdot \lim_{\Delta x \rightarrow 0} \Delta x \\
&= f'(x_0) \cdot 0 = 0
\end{aligned}$$

Note: The inverse of the above theorem is not true.

(i.e., If $f(x)$ is a continuous at x_0 , then it is not necessary to be differentiable at x_0)

Example: Let $f(x) = |x|$, and $x_0 = 0$.

From the above plot $f(x) = |x|$ is continuous at $x_0 = 0$.

However, $f(x) = |x|$ is not differentiable at $x_0 = 0$.

Proof:

$$\begin{aligned}
\therefore |x| &= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \\
\therefore |\Delta x| &= \begin{cases} \Delta x & \Delta x \geq 0 \\ -\Delta x & \Delta x < 0 \end{cases}
\end{aligned}$$

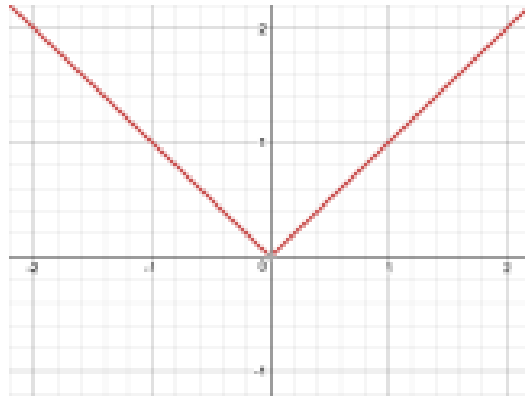
$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{|x + \Delta x| - |x|}{\Delta x} \\
f'(0) &= \lim_{\Delta x \rightarrow 0} \frac{|0 + \Delta x| - |0|}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{|\Delta x|}{\Delta x}
\end{aligned}$$

Hence,

$$L^+ = \lim_{\Delta x \rightarrow 0^+} = 1 \text{ \& } L^- = \lim_{\Delta x \rightarrow 0^-} = -1$$

Since, $L^+ \neq L^- \Rightarrow$ The limit does not exist.

$\therefore f(x)$ is not a differentiable function at $x_0 = 0$



General Theorems of Differentiation

Theorem(1): If $f(x) = c$, c be a constant, then $f'(x) = 0$.

Theorem(2): If f is a differentiable function at x and let c be a constant, then (cf) is differentiable at x and $(cf)'(x) = c f'(x)$.

Theorem(3): If f and g are two differentiable functions at x then $(f + g)$ is differentiable at x and $(f + g)'(x) = f'(x) + g'(x)$.

Remark: In general, If $f_1, f_2, \dots, f_n$ are differentiable functions at x then $(f_1 \pm f_2 \pm \dots \pm f_n)$ is differentiable at x and

$$(f_1 \pm f_2 \pm \dots \pm f_n)'(x) = (f_1)'(x) \pm (f_2)'(x) \pm \dots \pm (f_n)'(x).$$

Theorem(4): If $f(x) = x^n$, where $n > 0$, then $f'(x) = nx^{n-1}$.

Theorem(5): If f and g are two differentiable functions at x then $(f \cdot g)$ is differentiable at x and $(f \cdot g)'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$.

Remark: In general, If f, g and h are differentiable functions at x then $(f \cdot g \cdot h)$ is differentiable at x and $(f \cdot g \cdot h)'(x) = f(x)(gh)'(x) + (f)'(x)(g \cdot h)(x)$

$$\begin{aligned}
&= \mathbf{f(x)(g(x).h'(x) + g'(x).h(x)) + (f)'(x).g(x).h(x)} \\
&= \mathbf{(f)'(x).g(x).h(x) + f(x).g'(x).h(x) + f(x).g(x).h'(x) .}
\end{aligned}$$

Theorem(6): If f and g are two differentiable functions at x and $g(x) \neq 0$ then $\left(\frac{f}{g}\right)$ is differentiable at x and $\left(\frac{f}{g}\right)'(x) = \frac{g(x).f'(x) - f(x).g'(x)}{(g(x))^2}$.

Theorem(7): If g is a differentiable function at x , f is a differentiable function at $g(x)$ and $h = f \circ g$ then $(f \circ g)$ is differentiable at x and

$$h'(x) = (f \circ g)'(x) = f'(g(x)).g'(x).$$

Theorem(8): If f is a differentiable function at x and $y = (f(x))^n$ where $n \in \mathbb{Z}$, then (y) is differentiable at x and

$$y' = \frac{dy}{dx} = ((f(x))^n)' = n(f(x))^{n-1}.(f)'(x).$$

Problem (4.2):

1. Find the derivative of the following functions:

a) $y = \left(\frac{x^2+2}{x+1}\right)^4$

(f) $f(x) = \frac{(1+2x^2)(1+x^3)}{x^2}$

b) $y = (2\sqrt{x} - 1)^3$

(g) $f(x) = \sqrt{x + \sqrt{1 + \sqrt{x}}}$

(c) $y = \sqrt{3 - x^2}$

(h) $f(t) = t^3 - \frac{1}{t^2+1}$

(d) $f(w) = \sqrt{w} + \sqrt[3]{w} + \sqrt[4]{w}$

(i) $f(t) = \frac{\sqrt{t^2+1}}{(t+2)^4}$

(e) $f(x) = (x^3 + 2)^2(1 - x^2)^3$

(j) $f(z) = z^2(z^2 + 1)^{-\frac{1}{3}}$

2. Let $f(x) = x$ and $g(x) = x^2$, what is the value of x that makes the tangent line of two curves are parallel.

3. Let $f(x) = \frac{1}{\sqrt{2}}$, what is the value of x that make the tangent of the curve when it is parallel to the line $x + 8y = 10$

Chain Rule:

1. If $y = f(x)$ and $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

2. If $y = f(x)$ and $t = g(x)$, then $\frac{dy}{dt} = \frac{\frac{dy}{dx}}{\frac{dt}{dx}}$

Example1: Let $y = 3x - 1$ and $x = 2t$, find $\frac{dy}{dt}$?

Solution:

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} \\ &= (3) \cdot (2) = 6\end{aligned}$$

OR: $y = 3x - 1 = 3(2t) - 1 = 6t - 1$

Example2: Let $y = t^2 - 1$ and $x = 2t + 3$, find $\frac{dy}{dx}$?

Solution:-

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t = \frac{x-3}{2}$$

Problems (4.3): Find $\frac{dy}{dx}$ for the following functions:

1. $y = u^3 + 1$, $u = x^2 + 3$

5. $y = z^{\frac{2}{3}}$, $z = x^2 + 1$

2. $y = 3t^2 - 1$, $x = 6t - 1$

6. $y = w^2 - w^{-1}$, $w = 3x$

3. $y = \frac{t^2}{1+t}$, $x = \frac{t}{2+t}$

7. $y = 2v^3 + \frac{2}{v^3}$, $v = (2x + 2)^{\frac{2}{3}}$

4. $y = t^2$, $x = \frac{t}{1-t}$

8. $y = \frac{u^2}{u^2+1}$, $u = \sqrt{2x+1}$

Implicit Differentiation

Example1: Let $x^2 + xy + y^5 = 0$, find $\frac{dy}{dx}$ and $\frac{dx}{dy}$?

Solution:-

To find $\frac{dy}{dx}$, we derive implicitly for x by considering y is an implicit function of x .

$$\because x^2 + xy + y^5 = 0$$

$$\rightarrow 2x \frac{dx}{dx} + \left(x \frac{dy}{dx} + y \frac{dx}{dx} \right) + 5y^4 \frac{dy}{dx} = 0$$

$$\rightarrow 2x + xy' + y + 5y^4 y' = 0$$

$$\rightarrow xy' + 5y^4 y' = -2x - y$$

$$\rightarrow (x + 5y^4)y' = -2x - y \rightarrow y' = \frac{dy}{dx} = \frac{-2x - y}{x + 5y^4}$$

To find $\frac{dx}{dy}$, we derive implicitly for y by considering x is an implicit function of y .

$$\because x^2 + xy + y^5 = 0$$

$$\rightarrow 2x \frac{dx}{dy} + \left(x \frac{dy}{dy} + y \frac{dx}{dy} \right) + 5y^4 \frac{dy}{dy} = 0$$

$$\rightarrow 2x \frac{dx}{dy} + x + y \frac{dx}{dy} + 5y^4 = 0$$

$$\rightarrow 2x \frac{dx}{dy} + y \frac{dx}{dy} = -x - 5y^4$$

$$\rightarrow (2x + y) \frac{dx}{dy} = -x - 5y^4 \rightarrow \frac{dx}{dy} = \frac{-x - 5y^4}{2x + y}$$

Note that: $\frac{dx}{dy} = x' = \frac{1}{y'} = \frac{1}{\frac{dy}{dx}}$

Example 2: Find the equation of the tangent line and normal line of the curve $x^2 + y^2 = 2$ at $(1,1)$.

Solution:- $\because x^2 + y^2 = 2$

$$2x \frac{dx}{dx} + 2y \frac{dy}{dx} = 0 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = \frac{-x}{y}$$

$$\text{Hence, } y'|_{(1,1)} = m_{\tan} \Big|_{(1,1)} = \frac{-1}{1} = -1$$

The equation of the tangent line: $(y - y_1) = m_{\tan} (x - x_1)$

$$\Rightarrow (y - 1) = -1(x - 1)$$

$$\Rightarrow (y - 1) = -x + 1$$

$$\Rightarrow y = -x + 2$$

$$\text{Since, } m_{\perp}|_{(1,1)} = \frac{-1}{m_{\tan}} \Rightarrow m_{\perp} \Big|_{(1,1)} = \frac{-1}{-1} = 1$$

The equation of the normal line: $(y - y_1) = m_{\perp} (x - x_1)$

$$\Rightarrow (y - 1) = 1(x - 1) \Rightarrow (y - 1) = x - 1 \Rightarrow y = x$$

Problems (4.4):

- 1 Find the slop of the tangent line of the curve $x^2 + xy + y^2 = 7$ at the point (1,2).
- 2 Find the slop of the tangent line of the circle equation $8x^2 + 8y^2 = 232$ at the point (-5,2).
- 3 Find the equation of the tangent line and the normal line of the curve $xy^2 + yx^2 + y^2 = 0$ at the point (1,1).
- 4 Find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ for the following functions:

a) $x^3y^2 + 2xy - x + 3y = 6$ b) $x^2 + x^3 = y + y^4$

c) $\frac{1}{x} + \frac{1}{y} = x + y$

d) $x^2 - \sqrt{xy} + y^2 = 6$

e) $x^3 + y^3 - 9xy = 0$

f) $xy^2 + yx^2 = 3y^3$

g) $2 - y^3 + yx^2 = 5$

h) $(1 + x^2y)^3 + x\sqrt{y} = 9$

High-Order Derivative

Let $y = f(x)$, then:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = y' = y^{(1)} \text{ [First Derivative]}$$

$$f''(x) = \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x} = \frac{d^2y}{dx^2} = y'' = y^{(2)} \text{ [Second Derivative]}$$

$$f'''(x) = \lim_{\Delta x \rightarrow 0} \frac{f''(x + \Delta x) - f''(x)}{\Delta x} = \frac{d^3y}{dx^3} = y''' = y^{(3)} \text{ [Third Derivative]}$$

$\vdots$

$$f^{(n)}(x) = \lim_{\Delta x \rightarrow 0} \frac{f^{(n-1)}(x + \Delta x) - f^{(n-1)}(x)}{\Delta x} = \frac{d^{(n)}y}{dx^{(n)}} = y^{(n)}, n \in N [n^{\text{th}} \text{Derivative}]$$

Notes: $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right), \frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right), \dots, \frac{d^ny}{dx^n} = \frac{d}{dx} \left(\frac{d^{(n-1)}y}{dx^{(n-1)}} \right)$

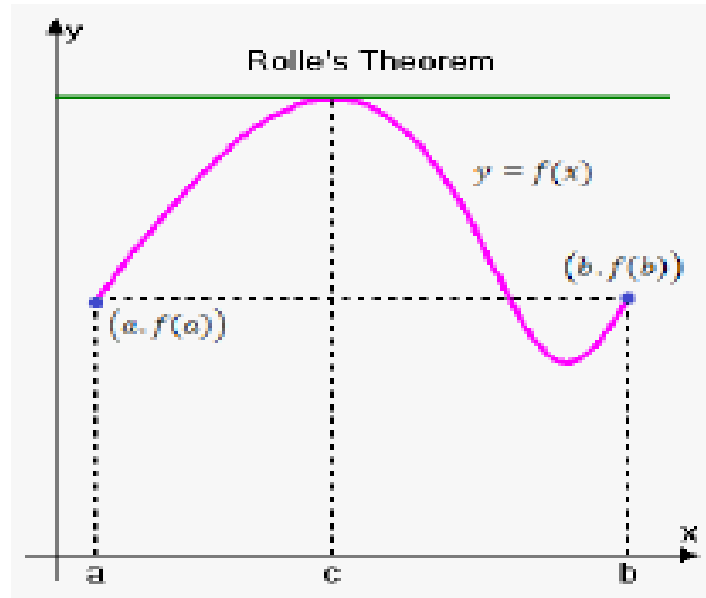
Example: Let $y = 2x^3 + x^2 - 1$, Find $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$ and $y^{(5)}$?
Solution:-

$$\begin{aligned} \because y &= 2x^3 + x^2 - 1 \\ \Rightarrow y^{(1)} &= 6x^2 + 2x \\ \Rightarrow y^{(2)} &= 12x + 2 \\ \Rightarrow y^{(3)} &= 12 \\ \Rightarrow y^{(4)} &= 0 \Rightarrow y^{(5)} = 0 \end{aligned}$$

Problems (4.5): Find y', y'' and y''' for the following:

1. $y = x^7 - x^2 + 4x + 33$
2. $y = -4 + 2x^2 - 7x^3 + x^4$
3. $y = \frac{1}{2}x^2 - 100$
4. $y = x^3 - 9x - 5$
5. $y = -x^3 - 9x^2 - 23$
6. $y = -3x^2 - 4x^3 + x^4$

Rolle's Theorem: Let $f(x)$ be a continuous function on $[a,b]$, and f is differentiable on (a, b) . If $f(a)=f(b)$, then there exist $c \in (a, b)$ such that $f'(c) = 0$.



Example 1: Let $f(x) = x^2 - 3x + 2$. Show that $f(x)$ satisfy Rolle's theorem on $[1,2]$.

Solution:-

$f(x)$ is continuous on $[1,2]$. (because $f(x)$ is a polynomial function)

$f(x)$ is differentiable on $(1,2)$. (because $f(x)$ is a polynomial function)

$$a = 1 \text{ and } b = 2$$

$$f(a) = f(1) = 1^2 - 3(1) + 2 = 0$$

$$f(b) = f(2) = 2^2 - 3(2) + 2 = 0$$

$$\Rightarrow f(a) = f(b)$$

From above Rolle's theorem is satisfied, and hence $\exists c \in (1,2)$ s.t.

$$f'(c) = 0$$

$$\because f'(x) = 2x - 3$$

$$\Rightarrow f'(c) = 2c - 3 = 0$$

$$\Rightarrow 2c - 3 = 0 \Rightarrow c = \frac{3}{2} \in (1,2)$$

Example 2: Let $f(x) = 1 - |x|$. Show that $f(x)$ does not satisfy Rolle's theorem on $[-1,1]$.

Solution:-

$f(x)$ is continuous on $[-1,1]$. (because $f(x)$ is a polynomial function) But, $f(x)$ is not differentiable at $= 0$?

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{1 - |x + \Delta x| - 1 + |x|}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-|x + \Delta x| + |x|}{\Delta x}$$

$$f'(0) = \lim_{\Delta x \rightarrow 0} \frac{-|0 + \Delta x| + |0|}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-|\Delta x|}{\Delta x}$$

$$L^+ = \lim_{\Delta x \rightarrow 0^+} \frac{-\Delta x}{\Delta x} = -1, L^- = \lim_{\Delta x \rightarrow 0^-} \frac{-(-\Delta x)}{\Delta x} = 1$$

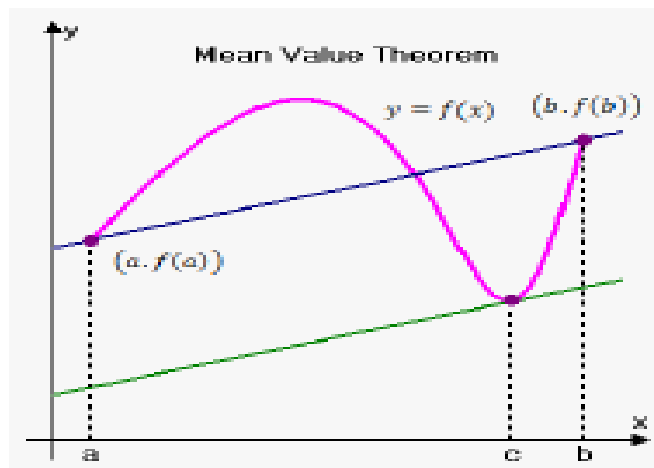
$\therefore L^+ \neq L^- \rightarrow$ the limit does not exist at 0.

Hence, $f'(0)$ does not exist.

Therefore, $f(x)$ does not satisfy Rolle's theorem on $[-1,1]$.

The Mean Value Theorem: Let $f(x)$ be a continuous function on $[a, b]$ and f is differentiable on (a, b) then there exist at least one point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Note: Rolle's theorem is a special case from the Mean Value Theorem.

Example1: Find the value of c that satisfies the Mean Value Theorem, where $f(x) = x^2$, $x \in [0, 1]$.

Solution:-

$f(x)$ is continuous on $[0, 2]$. (because $f(x)$ is a polynomial function)

$f(x)$ is differentiable on $(0, 2)$.

From above Mean Value theorem is satisfied, and hence $\exists c \in (0, 2)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\because a = 0 \Rightarrow f(a) = f(0) = 0^2 = 0$$

$$\because b = 2 \Rightarrow f(b) = f(2) = 2^2 = 4$$

$$\Rightarrow f'(x) = 2x \Rightarrow f'(c) = 2c$$

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{4 - 0}{2 - 0} = 2$$

$$\Rightarrow 2c = 4 \Rightarrow c = 2 \in (0, 2).$$

Example 2: Let $f(x) = x^3 - 3x$ and $f: [a, 0] \rightarrow R$ where f satisfies the Mean Value Theorem at $c = -1$, find the value of a

Solution:- $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$\Rightarrow f'(x) = 3x^2 - 3 \Rightarrow f'(c) = f'(-1) = 3(-1)^2 - 3 = 0$$

$$\because a = ? \text{ and } b = 0 \Rightarrow$$

Hence, $0 = \frac{f(0) - f(a)}{0 - a} = \frac{3(0)^2 - 3 - 3(a)^2 + 3}{0 - a} = \frac{a^2 - 3a}{a}$

$$\Rightarrow a^2 - 3a = 0 \Rightarrow a^2 = 3 \Rightarrow a = \pm\sqrt{3} \Rightarrow a = -\sqrt{3} = -1.7.$$

Problems (4.6):

1. Check whether the following functions satisfy the Rolle's theorem or not?

a) $f(x) = (2 - x)^2$ on $[0, 4]$ b) $f(x) = 9x + 3x^2 - x^3$ on $[-1, 1]$.

2. Find the value of c that satisfies the Mean Value Theorem, where $f(x) = x^2 - 6x + 4$, $x \in [-1, 7]$.

3. Let $f(x) = x^2 - 4x$, and $f: [0, b] \rightarrow R$ where f satisfies the Mean value theorem at $c=2$, find the value of b ?

L'Hopitals Rule:

Let f and g be differentiable functions at x_0 , $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$, where $\lim_{x \rightarrow x_0} g'(x) \neq 0$. (or $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$)

Then $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$.

Example1: Find $\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 1}$.

Solution:-

$$\begin{aligned} \because \lim_{x \rightarrow 1} x^2 - 3x + 2 &= 0 \text{ and } \lim_{x \rightarrow 1} x^2 - 1 = 0 \\ \therefore \lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 1} &= \lim_{x \rightarrow 1} \frac{2x - 3}{2x} = \frac{1}{2} \end{aligned}$$

Example2: Find $\lim_{x \rightarrow 0} \frac{2 - \sqrt{x+4}}{x}$.

Solution:-

$$\begin{aligned} \because \lim_{x \rightarrow 0} 2 - \sqrt{x+4} &= 0 \text{ and } \lim_{x \rightarrow 0} x = 0 \\ \lim_{x \rightarrow 0} \frac{2 - \sqrt{x+4}}{x} &= \lim_{x \rightarrow 0} \frac{0 - \frac{1}{2}(x+4)^{\frac{1}{2}}}{1} \\ &= -\frac{1}{2} \cdot \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+4}} = -\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4} \end{aligned}$$

Another Method: by multiplying by the conjugate:

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{2 - \sqrt{x+4}}{x} \\
&= \lim_{x \rightarrow 0} \frac{2 - \sqrt{x+4}}{x} \cdot \frac{2 + \sqrt{x+4}}{2 + \sqrt{x+4}} \\
&= \lim_{x \rightarrow 0} \frac{4 - (x-4)}{x(2 + \sqrt{x+4})} \\
&= \lim_{x \rightarrow 0} \frac{-x}{(2 + \sqrt{x+4})} = \frac{-1}{2 + \sqrt{0+4}} = \frac{-1}{2+2} = \frac{-1}{4}
\end{aligned}$$

Problems (4.7): Find the following limits if it exists:

$$1) \lim_{x \rightarrow 2} \frac{x^2 + 2x - 8}{x^2 - 9x + 14}$$

$$2) \lim_{x \rightarrow 0} \frac{\sqrt{x+9} - 3}{x}$$

$$3) \lim_{x \rightarrow 1} \frac{x^2 + 5x + 4}{x^2 - 4x - 5}$$

$$4) \lim_{x \rightarrow 0} \frac{4x^3 + 3x^2 - 8x + 1}{x^3 - 2x^2 + 3x - 6}$$

$$5) \lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 1}$$

$$6) \lim_{x \rightarrow 0} \frac{x^3 + 4x^2 - 5x}{x^3 - 2x}$$

Increasing and Decreasing Functions:

Definition: A function f is defined on an interval $[a, b]$ is said to be **increasing** on $[a, b]$ if $\forall x_1, x_2 \ni a \leq x_1 < x_2 \leq b \Rightarrow f(x_1) < f(x_2)$.

Example: Let $f(x) = x^3$ on $[-2, 2]$.

$$-2 < -1 \Rightarrow f(-2) = -8 < -1 = f(-1)$$

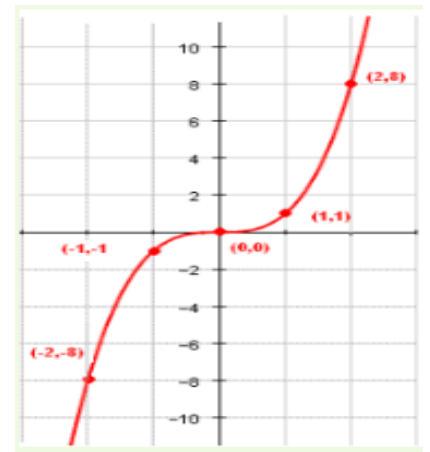
$$-1 < 1 \Rightarrow f(-1) = -1 < 1 = f(1)$$

$$1 < 2 \Rightarrow f(1) = 1 < 8 = f(2)$$

$\vdots$

$$\therefore \forall a, b \in [-2, 2] \Rightarrow f(a) < f(b)$$

$$\therefore f(x) \text{ is an increasing function on } [-2, 2]$$



Definition: A function f is defined on an interval $[a, b]$ is said to be **decreasing** on $[a, b]$ if $\forall x_1, x_2 \ni a \leq x_1 < x_2 \leq b \Rightarrow f(x_1) > f(x_2)$.

Example: Let $f(x) = -x^3$ on $[-2, 2]$.

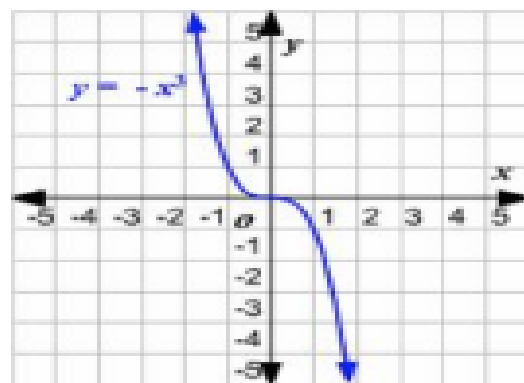
$$-2 < -1 \Rightarrow f(-2) = 8 > 1 = f(-1)$$

$$-1 < 1 \Rightarrow f(-1) = 1 > -1 = f(1)$$

$$1 < 2 \Rightarrow f(1) = -1 > -8 = f(2)$$

$$\therefore \forall a, b \in [-2, 2] \Rightarrow f(a) > f(b)$$

$\therefore f(x)$ is a decreasing function on $[-2, 2]$.



Definition: Let f be defined and continuous function on $[a, b]$, and let $x_0 \in [a, b]$, then $(x_0, f(x_0))$ is said to be a Critical Point of $f \Leftrightarrow f'(x_0) = 0$ or $f'(x)$ is not defined.

Example 1: Let $f(x) = x^2$ be defined and continuous on $[-1, 1]$. Find the critical points (if exists)?

Solution: $f'(x) = 2x$ When $f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

Hence, $(x_0, f(x_0)) = (0, 0)$ is a critical point.

Example 2: Let $f(x) = \frac{x^4}{3} - \frac{x^2}{2}$ be defined and continuous on all the real numbers. Find the critical points (if exists)?

Solution:

$$\text{When } f'(x) = 0 \Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0 \text{ or } x = 1$$

Hence, $(0, f(0)) = (0, 0)$ and $(1, f(1)) = \left(1, -\frac{1}{6}\right)$ are the critical points.

Example 3: Let $f(x) = |x|$ be defined and continuous on $[-1, 1]$. Find the critical points (if exists)?

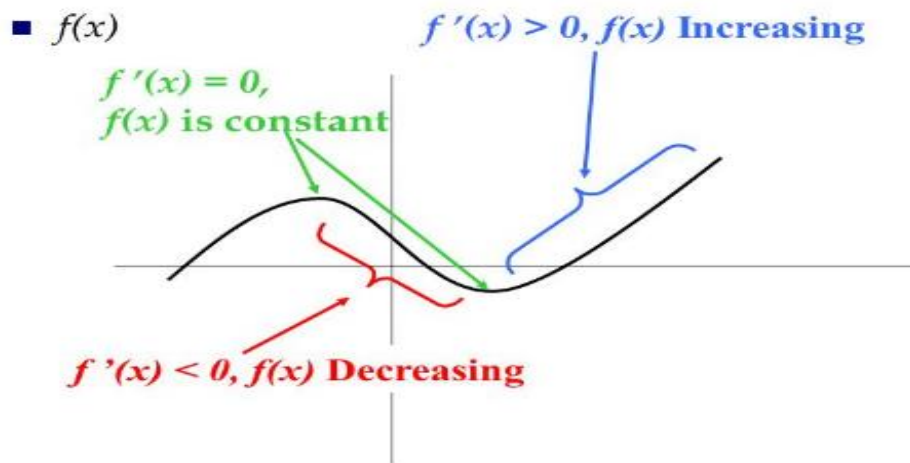
Solution:

$$0 \in [-1, 1], \text{ but } f'(0) \text{ does not exist}$$

Hence, $(0, f(0)) = (0, 0)$ is critical points.

Theorem: Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b) , then:

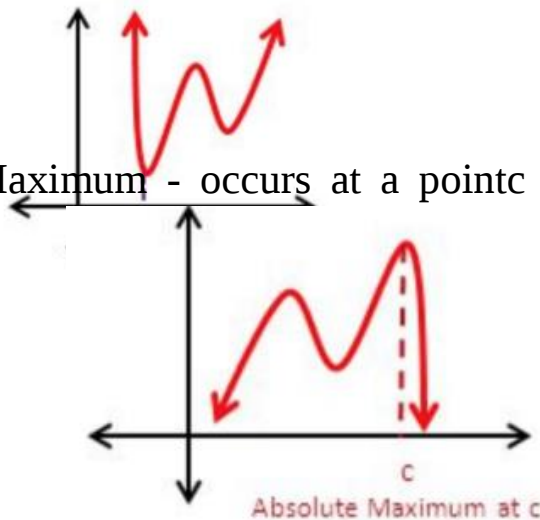
1. If $f'(x) > 0 \forall x \in (a, b)$, then f is increasing on $[a, b]$.
2. If $f'(x) < 0 \forall x \in (a, b)$, then f is decreasing on $[a, b]$.
3. If $f'(x) = 0 \forall x \in (a, b)$, then f is constant on $[a, b]$.



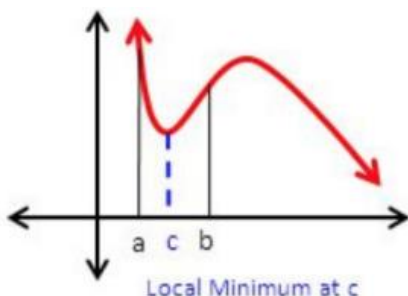
Definitions:

Absolute Minimum-occurs at a point c if $f(c) \leq f(x)$ for x all values in the domain.

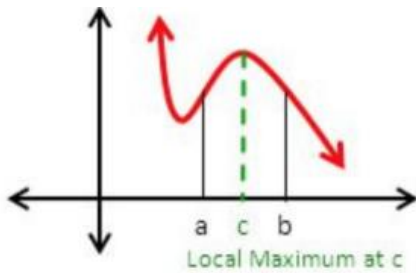
Absolute Maximum - occurs at a point c if $f(c) \geq f(x)$ for all x values in the domain.



Local Minimum-occurs at a point c in an open interval, (a, b) , in the domain if $f(c) \leq f(x)$ for all x values in the open interval.



Local Maximum-occurs at a point c in an open interval, (a, b) , in the domain if $f(c) \geq f(x)$ for all x values in the open interval.



Example: Let $f(x)$ be define on $[-4,5]$ as given in the following plot. Find the absolute maximum, absolute minimum, local maximum and local minimum points.

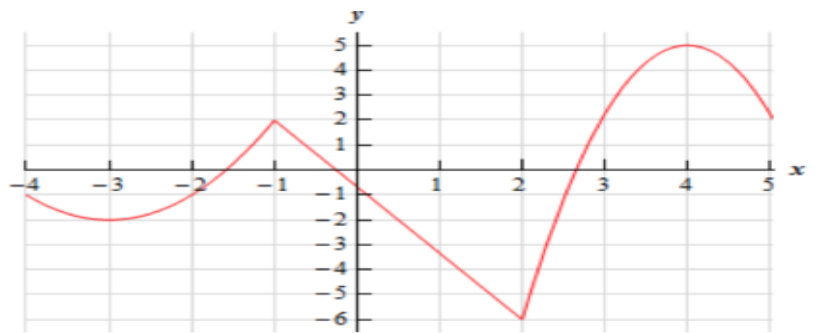
Solution:

Absolute Maximum: $(4, 5)$

Absolute minimum: $(2, -6)$

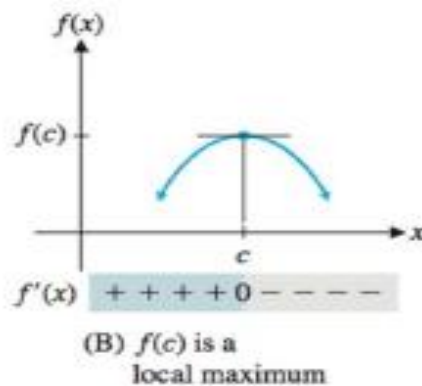
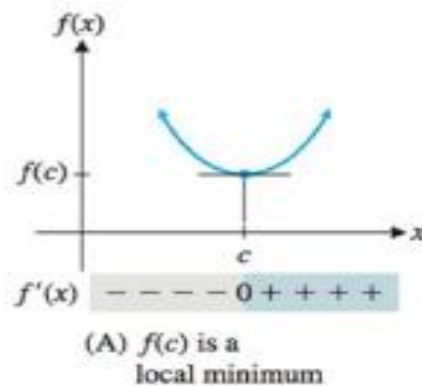
Local maximum: $(-1, 2)$ and $(4, 5)$

Local maximum: $(-3, -2)$ and $(2, -6)$



First Derivative Test

- 1) If the sign changes from “+” to “-“ at c , then c is called a **local maximum point**.
- 2) If the sign changes from “-“ to “+“ at c , then c is called a **local minimum point**.



Example 1: Let $f(x) = x^3 - 6x^2 + 1$. Using the First Derivative Test, find the local maximum and minimum points.

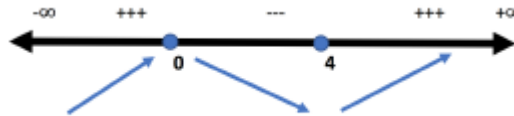
Solution:

First, we need to find the critical points ($f'(x) = 0$):

$$\begin{aligned}\because f(x) &= x^3 - 6x^2 + 1 \Rightarrow f'(x) = 3x^2 - 12x \\ f'(x) &= 0 \Rightarrow 3x^2 - 12x = 0 \Rightarrow 3x(x - 4) = 0\end{aligned}$$

Hence, $f(x)$ has critical points at $x = 0, 4$.

Increasing Intervals: $(-\infty, 0)$ and $(4, \infty)$
Decreasing Interval: $(0, 4)$



$f(x)$ has local maximum at $x = 0$, and $(0, 1)$ is a local maximum point.
 $f(x)$ has local minimum at $x = 4$, and $(4, -31)$ is a local minimum point.

Example 2: Let $f(x) = x^3 - 6x^2 + 9x - 8$ on $(0, 5)$. Using the First Derivative Test, find the local maximum and minimum points.

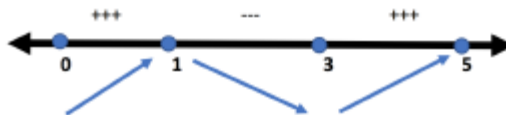
Solution:

First, we need to find the critical points ($f'(x) = 0$):

$$\begin{aligned}\because f(x) &= x^3 - 6x^2 + 9x - 8 \Rightarrow f'(x) = 3x^2 - 12x + 9 \\ f'(x) &= 0 \Rightarrow 3x^2 - 12x + 9 = 0 \\ \Rightarrow 3(x^2 - 4x + 3) &= 0 \Rightarrow (x - 1)(x - 3) = 0\end{aligned}$$

Hence, $f(x)$ has critical points at $x = 1, 3$.

Increasing Intervals: $(0, 1)$ and $(3, 5)$
Decreasing Interval: $(1, 3)$



$f(x)$ has local maximum at $x = 1$, and $(1, -4)$ is a local maximum point.
 $f(x)$ has local minimum at $x = 3$, and $(3, -8)$ is a local minimum point.

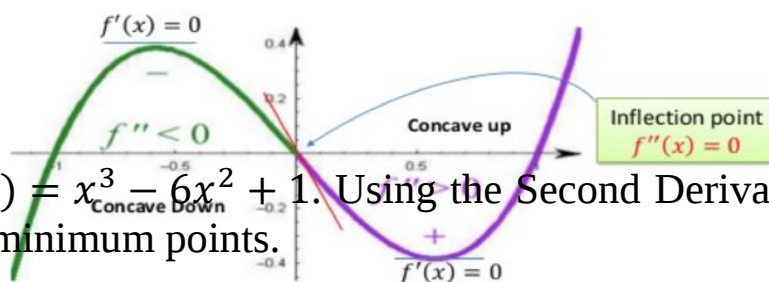
Definition: The graph of a differentiable function $y = f(x)$ is concave up on an interval where y' is increasing and is concave down on an interval where y' is decreasing .

Definition: An **inflection point** is a point on a curve where the curve change from being concave down (going up, then down) to concave up (going down, then up), or the other way around.

Second Derivative Test for Local Maxima and Minima

If $f'(x) = 0$ and $f''(x) > 0$ then f has a local maximum at $x=c$. (i.e. f is concave up)

If $f'(x) = 0$ and $f''(x) < 0$ then f has a local minimum at $x=c$. (i.e. f is concave down)



Example 1: Let $f(x) = x^3 - 6x^2 + 1$. Using the Second Derivative Test, find the local maximum and minimum points.

Solution:

First, we need to find the critical points ($f'(x) = 0$):

$$\begin{aligned} \because f(x) &= x^3 - 6x^2 + 1 \Rightarrow f'(x) = 3x^2 - 12x \\ f'(x) &= 0 \Rightarrow 3x^2 - 12x = 0 \Rightarrow 3x(x - 4) = 0 \end{aligned}$$

Hence, $f(x)$ has critical points at $x = 0, 4$.

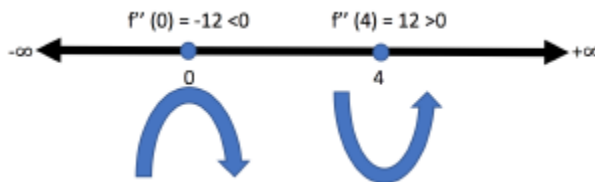
$$\begin{aligned} \because f'(x) &= 3x^2 - 12x \Rightarrow f''(x) = 6x - 12 \\ f''(x) &= 0 \Rightarrow 6x - 12 = 0 \Rightarrow x = 2 \end{aligned}$$

$$\because f''(0) = -12 \Rightarrow f(x) \text{ "Concave Down" on } (-\infty, 2),$$

and has local Maximum at " $x = 0$ ".

$$\because f''(4) = 12 \Rightarrow f(x) \text{ "Concave Up" on } (2, \infty),$$

and has local Minimum at " $x = 4$ ".
 $f(x)$ has an inflection point at $x = 2$ because the function concave down then concave up.



Example 2: Let $f(x) = x^3 - 6x^2 + 9x - 8$ on $(0,5)$. Using the Second Derivative Test, find the local maximum and minimum points.

Solution:

First, we need to find the critical prints ($f'(x) = 0$):

$$\because f(x) = x^3 - 6x^2 + 9x - 8 \Rightarrow f'(x) = 3x^2 - 12x + 9$$

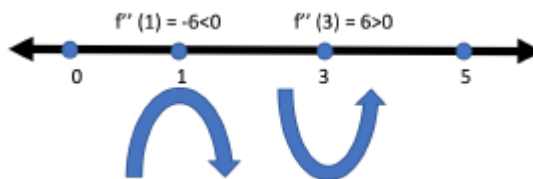
$$f'(x) = 0 \Rightarrow 3x^2 - 12x + 9 = 0$$

$$\Rightarrow 3(x^2 - 4x + 3) = 0 \Rightarrow (x - 1)(x - 3) = 0$$

Hence, $f(x)$ has critical points at $x = 1, 3$.

$$\because f'(x) = 3x^2 - 12x + 9 \Rightarrow f''(x) = 6x - 12$$

$$f''(x) = 0 \Rightarrow 6x - 12 = 0 \Rightarrow x = 2$$



$\because f''(1) = -6 \Rightarrow f(x)$ "Concave Down" on $(0, 2)$, and has local Maximum at " $x = 1$ ".

$\because f''(3) = 6 \Rightarrow f(x)$ "Concave Up" on $(2, 5)$, and has local Minimum at " $x = 3$ ".

$f(x)$ has inflection point at $x=2$ because the function concave down then concave up.

Problems (4.8):

- 1 By using the First Derivative Test, check whether the critical points are local maximum or minimum points, and specify the increasing and decreasing intervals.

(a) $f(x) = \frac{1}{2}x^2 - x, x \in [0,2]$

(b) $f(x) = \frac{x^2}{3} + \frac{5}{2}x^2 + 6x, x \in \mathbb{R}$

(c) $f(x) = x^3 - x, x \in [-2,2]$

- 2 By using the Second Derivative Test, check whether the critical points are local maximum or minimum points, and specify the concave up and concave down intervals.

(a) $f(x) = \frac{x^3}{6} - \frac{x^2}{2} - 4, x \in [-2,5]$

(b) $f(x) = \frac{x^3}{3} + \frac{5}{2}x^2 + 6x, x \in \mathbb{R}$

(c) $f(x) = \frac{x^4}{12} - \frac{x^2}{6} - x^2, x \in [-3,3]$