The Inverse Laplace Transform by Partial Fraction Expansion

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Distinct Real Roots

Example: Distinct Real Roots.Find the inverse Laplace Transform of:

$$F(s) = \frac{s+1}{s(s+2)} = \frac{A_1}{s} + \frac{A_2}{s+2}$$

Solution:

We can find the two unknown coefficients using the "cover-up" method.

$$A_1 = \left.\frac{s+1}{s(s+2)}\right|_{s=0} = \frac{1}{2}$$

$$A_2 = \left.\frac{s+1}{s(s+2)}\right|_{s=-2} = \frac{-1}{-2} = \frac{1}{2}$$

So

$$F(s) = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s+2}$$

and

$$f(t) = \frac{1}{2}U(t) + \frac{1}{2}e^{-2t}U(t)$$

(where U(t) is the unit step function) or expressed another way

$$f(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}, \quad t > 0$$

The last two expressions are somewhat cumbersome. Unless there is confusion about the result, we will assume that all of our results are implicitly 0 for t<0, and we will write the result as

$$f(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}$$

Repeated Real Roots

Consider next an example with repeated real roots (in this case at the origin, s=0).

Example: Repeated Real Roots

Find the inverse Laplace Transform of the function F(s).

$$F(s) = \frac{s^2+1}{s^2(s+2)} = \frac{A_1}{s+2} + \frac{A_2}{s} + \frac{A_3}{s^2}$$

Solution:

We can find two of the unknown coefficients using the "cover-up" method.

$$A_1 = \left.\frac{s^2+1}{s^2(s+2)}\right|_{s=-2} = \frac{5}{4}$$

$$A_3 = \left.\frac{s^2+1}{s^2(s+2)}\right|_{s=0} = \frac{1}{2}$$

We find the other term using cross-multiplication:

$$s^2+1 = s^2(s+2)\left(\frac{A_1}{s+2} + \frac{A_2}{s} + \frac{A_3}{s^2}\right)$$

$$= s^2A_1 + s(s+2)A_2 + (s+2)A_3$$

Equating like powers of "s" gives us:

power of "s"	Equation
s^2	$1 = A_1 + A_2$
s^1	$0 = 2A_2 + A_3$
s^0	$1 = 2A_3$

We could have used these relationships to determine A_1, A_2, and A_3. But A_1 and A_3 were easily found using the "cover-up" method. The top relationship tells us that A_2=-0.25, so

$$F(s)=\frac{5}{4}\frac{1}{s+2}-\frac{1}{4}\frac{1}{s}+\frac{1}{2}\frac{1}{s^2}$$

and

$$f(t)=\frac{5}{4}e^{-2t}-\frac{1}{4}+\frac{1}{2}t$$

(where, again, it is implicit that f(t)=0 when t<0).

Many texts use a method based upon differentiation of the fraction when there are repeated roots. The technique involves differentiation of ratios of polynomials which is prone to errors. Details are here if you are interested.

Complex Roots

Another case that often comes up is that of complex conjugate roots. Consider the fraction:

$$F(s)=\frac{s+3}{(s+5)(s^2+4s+5)}$$

The second term in the denominator cannot be factored into real terms. This leaves us with two possibilities - either accept the complex roots, or find a way to include the second order term.

Example: Complex Conjugate Roots (Method 1)

Using the complex (first order) roots

Simplify the function F(s) so that it can be looked up in the Laplace Transform table.

$$F(s)=\frac{s+3}{(s+5)(s^2+4s+5)}$$

Solution:

If we use complex roots, we can expand the fraction as we did before. This is not typically the way you want to proceed if you are working by hand, but may be easier for computer solutions (where complex numbers are handled as easily as real numbers). To perform the expansion, continue as before.

$$F(s) = \frac{s+3}{(s+5)(s^2+4s+5)} = \frac{s+3}{(s+5)(s+2-j)(s+2+j)}$$

$$= \frac{A_1}{(s+5)} + \frac{A_2}{(s+2-j)} + \frac{A_3}{(s+2+j)}$$

where

$$A_1 = (s+5)F(s)\Big|_{s=-5}$$

$$A_2 = (s+2-j)F(s)\Big|_{s=-2+j}$$

$$A_3 = (s+2+j)F(s)\Big|_{s=-2-j} = A_2^*$$

Note that A2 and A3 must be complex conjugates of each other since they are equivalent except for the sign on the imaginary part. Performing the required calculations:

$$A_1 = (s+5)F(s)\Big|_{s=-5} = \frac{s+3}{(s+5)(s^2+4s+5)}\Bigg|_{s=-5} = -0.2$$

$$A_2 = (s+2-j)F(s)\Big|_{s=-2+j} = \frac{s+3}{(s+5)(s+2+j)}\Bigg|_{s=-2+j}$$

$$= \frac{-2+j+3}{(-2+j+5)(-2+j+2+j)} = 0.1-0.2j$$

$$A_3 = (s+2+j)F(s)\Big|_{s=-2-j} = A_2^* = 0.1+0.2j$$

so

$$F(s) = \frac{-0.2}{s+5} + \frac{0.1-0.2j}{s+2-j} + \frac{0.1+0.2j}{s+2+j}$$

The inverse Laplace Transform is given below (Method 1).

Example: Complex Conjugate Roots (Method 2)

Method 2 - Using the second order polynomial

Simplify the function F(s) so that it can be looked up in the Laplace Transform table.

$$F(s) = \frac{s+3}{(s+5)(s^2+4s+5)}$$

Solution:

Another way to expand the fraction without resorting to complex numbers is to perform the expansion as follows.

$$F(s) = \frac{s+3}{(s+5)(s^2+4s+5)} = \frac{A}{s+5} + \frac{Bs+C}{s^2+4s+5}$$

Note that the numerator of the second term is no longer a constant, but is instead a first order polynomial. From above (or using the cover-up method) we know that A=-0.2. We can find the quantities B and C from cross-multiplication.

$$\begin{aligned} s+3 &= (s^2+4s+5)(s+5)\left(\frac{A}{s+5} + \frac{Bs+C}{s^2+4s+5}\right) \\ &= (s^2+4s+5)A + (s+5)(Bs+C) \\ &= (A+B)s^2 + (4A+5B+C)s + (5A+5C) \end{aligned}$$

If we equate like powers of "s" we get

order of coefficient	left side coefficient	right side coefficient
2nd (s^2)	0	A+B
1st (s^1)	1	4A+5B+C
0th (s^0)	3	5A+5C

Since we already know that A=-0.2, the first expression (0=A+B) tells us that B=0.2, and the last expression (3=5A+5C) tells us that C=0.8. We can use the middle expression (1=4A+5B+C) to check our calculations. Finally, we get

$$F(s) = \frac{-0.2}{s+5} + \frac{0.2s+0.8}{s^2+4s+5}$$

The inverse Laplace Transform is given below (Method 2).

Some Comments on the two methods for handling complex roots

The two previous examples have demonstrated two techniques for performing a partial fraction expansion of a term with complex roots. The first technique was a simple extension of the rule for dealing with distinct real roots. It is conceptually simple, but can be difficult when working by hand because of the need for using complex numbers; it is easily done by computer. The second technique is easy to do by hand, but is conceptually a bit more difficult. It is easy to show that the two resulting partial fraction representations are equivalent to each other. Let's first examine the result from Method 1 (using two techniques).

We start with Method 1 with no particular simplifications.Method 1 - force technique

$$F(s) = \frac{-0.2}{s+5} + \frac{0.1-0.2j}{s+2-j} + \frac{0.1+0.2j}{s+2+j}$$

$$\begin{aligned} f(t) &= -0.2e^{-5t} + (0.1-0.2j)e^{(-2+j)t} + (0.1+0.2j)e^{(-2-j)t} \\ &= -0.2e^{-5t} + e^{-2t}\left[0.1\left(e^{jt}+e^{-jt}\right) - 0.2j\left(e^{jt}-e^{-jt}\right)\right] \\ &= -0.2e^{-5t} + e^{-2t}\left[0.2\frac{\left(e^{jt}+e^{-jt}\right)}{2} + 0.4\frac{\left(e^{jt}-e^{-jt}\right)}{2j}\right] \\ &= -0.2e^{-5t} + 0.2\cos(t)e^{-2t} + 0.4\sin(t)e^{-2t} \end{aligned}$$

(The last line used Euler's identity for cosine and sine)

We now repeat this calculation, but in the process we develop a general technique (that proves to be useful when using MATLAB to help with the partial fraction expansion. We know that F(s) can be represented as a partial fraction expansion as shown below:

Method 1 - a more general technique

$$F(s) = \frac{s+3}{(s+5)(s^2+4s+5)} = \frac{s+3}{(s+5)(s+2-j)(s+2+j)}$$

$$= \frac{A_1}{(s+5)} + \frac{A_2}{(s+2-j)} + \frac{A_3}{(s+2+j)}$$

We know that A_2 and A_3 are complex conjugates of each other:

$$A_3 = A_2^*$$

Let

$$A_2 = Ke^{j\phi}$$
$$A_3 = Ke^{-j\phi}$$

where $K = |A_2|$, $\phi = \angle A_2 = \tan^{-1}\left(\frac{\text{Im}(A_2)}{\text{Re}(A_2)}\right)$ (note)

We can now find the inverse transform of the complex conjugate terms by treating them as simple first order terms (with complex roots).

$$\frac{A_2}{(s+2-j)} + \frac{A_3}{(s+2+j)} = \frac{Ke^{j\phi}}{(s+2-j)} + \frac{Ke^{-j\phi}}{(s+2+j)}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{Ke^{j\phi}}{(s+2-j)} + \frac{Ke^{-j\phi}}{(s+2+j)}\right\} &= Ke^{j\phi}e^{-(2-j)t} + Ke^{-j\phi}e^{-(2+j)t} \\
&= Ke^{-2t}\left(e^{j(t+\phi)} + e^{-j(t+\phi)}\right) \\
&= 2Ke^{-2t}\frac{\left(e^{j(t+\phi)} + e^{-j(t+\phi)}\right)}{2} \\
&= Me^{-2t}\cos(t+\phi) \\
&= Me^{\sigma t}\cos(\omega t+\phi)
\end{aligned}$$

In this expression M=2K. The frequency (ω) and decay coefficient (σ) are determined from the root of the denominator of A_2 (in this case the root of the term is at s=-2+j; this is where the term is equal to zero). The frequency is the imaginary part of the root (in this case, ω=1), and the decay coefficient is the real part of the root (in this case, σ=-2).

Using the cover-up method (or, more likely, a computer program) we get

$$A_1 = (s+5)F(s)\big|_{s=-5} = -0.2$$
$$A_2 = (s+2-j)F(s)\big|_{s=-2+j} = 0.1 - 0.2j$$
$$A_3 = (s+2+j)F(s)\big|_{s=-2-j} = A_2^* = 0.1 + 0.2j$$

and

$$K = |A_2| = |A_2| = \sqrt{0.1^2 + 0.2^2} = 0.224$$
$$\phi = \angle K = \tan^{-1}(0.2 / 0.1) = 63.4°$$
$$M = 0.448$$

This yields

$$f(t) = -0.2e^{-5t} + 0.448e^{-2t}\cos(t + 63.4°)$$

It is easy to show that the final result is equivalent to that previously found, i.e.,

$$f(t) = -0.2e^{-5t} + 0.448e^{-2t}\cos(t + 63.4°)$$
$$= -0.2e^{-5t} + 0.2\cos(t)e^{-2t} + 0.4\sin(t)e^{-2t}$$

While this method is somewhat difficult to do by hand, it is very convenient to do by computer. This is the approach used on the page that shows MATLAB techniques.

Finally we present Method 2, a technique that is easier to work with when solving problems for hand (for homework or on exams) but is less useful when using MATLAB.

Method 2 - Completing the square

Review of procedure for completing the square.

$$F(s) = \frac{-0.2}{s+5} + \frac{0.2s + 0.8}{s^2 + 4s + 5}$$
$$= \frac{-0.2}{s+5} + \frac{0.2s + 0.8}{(s+2)^2 + 1}$$
$$f(t) = -0.2e^{-5t} + 0.2\cos(t)e^{-2t} + 0.4\sin(t)e^{-2t}$$

Thus it has been shown that the two methods yield the same result. Use Method 1 with MATLAB and use Method 2 when solving problems with pencil and paper.

Example - Combining multiple expansion methods

Find the inverse Laplace Transform of

$$F(s) = \frac{5s^2 + 8s - 5}{s^2(s^2 + 2s + 5)}$$

Solution:

The fraction shown has a second order term in the denominator that cannot be reduced to first order real terms. As discussed in the page describing partial fraction expansion, we'll use two techniques. The first technique involves expanding the fraction while retaining the second order term with complex roots in the denominator. The second technique entails "Completing the Square."

$$F(s) = \frac{5s^2 + 8s - 5}{s^2\left(s^2 + 2s + 5\right)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{Bs + C}{s^2 + 2s + 5}$$

Since we have a repeated root, let's cross-multiply to get

$$\begin{aligned} 5s^2 + 8s - 5 &= s^2\left(s^2 + 2s + 5\right)\left(\frac{A_1}{s} + \frac{A_2}{s^2} + \frac{Bs + C}{s^2 + 2s + 5}\right) \\ &= A_1\left(s^3 + 2s^2 + 5s\right) + A_2\left(s^2 + 2s + 5\right) + Bs^3 + Cs^2 \end{aligned}$$

Then equating like powers of s

Power of s	Equation
s^3	$0=A_1+B$
s^2	$5=2A_1+A_2+C$
s^1	$8=5A_1+2A_2$
s^0	$-5=5A_2$

Starting at the last equation

$$\begin{aligned} A_2 &= -1 \\ A_1 &= \frac{8+2}{5} = 2 \\ C &= 5 - 4 + 1 = 2 \\ B &= -A_1 = -2 \end{aligned}$$

So

$$F(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{-2s + 2}{s^2 + 2s + 5}$$

The last term is not quite in the form that we want it, but by completing the square we get

$$F(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{-2s + 2}{(s+1)^2 + 4}$$

$$f(t) = 2 - t + e^{-t}\left(-2\cos(2t) + 2\sin(2t)\right)$$

Example - Repeat Previous Example, Using Brute Force (قوة الاسس)

Find the inverse Laplace Transform of

$$F(s) = \frac{5s^2 + 8s - 5}{s^2\left(s^2 + 2s + 5\right)} = \frac{5s^2 + 8s - 5}{s^2(s + 1 - 2j)(s + 1 + 2j)}$$

Solution:

We can express this as four terms, including two complex terms (with $A_3=A_4^*$)

$$F(s)=\frac{5s^2+8s-5}{s^2\left(s^2+2s+5\right)}=\frac{A_1}{s}+\frac{A_2}{s^2}+\frac{A_3}{s+1-2j}+\frac{A_4}{s+1+2j}$$

Cross-multiplying we get (using the fact that (s+1-2j)(s+1+2j)=(s^2+2s+5))

$$\begin{aligned}5s^2+8s-5&=\left(\frac{A_1}{s}+\frac{A_2}{s^2}+\frac{A_3}{s+1-2j}+\frac{A_4}{s+1+2j}\right)\cdot\left(s^2\left(s+1-2j\right)\left(s+1+2j\right)\right)\\&=A_1s\left(s+1-2j\right)\left(s+1+2j\right)+A_2\left(s+1-2j\right)\left(s+1+2j\right)+A_3s^2\left(s+1+2j\right)+A_4s^2\left(s+1-2j\right)\\&=A_1s\left(s^2+2s+5\right)+A_2\left(s^2+2s+5\right)+A_3s^2\left(s+1+2j\right)+A_4s^2\left(s+1-2j\right)\end{aligned}$$

Then equating like powers of s

Power of s	Equation
s^3	$0=A_1+A_3+A_4$
s^2	$5=2A_1+A_2+(1+2j)A_3+(1-2j)A_4$
s^1	$8=5A_1+2A_2$
s^0	$-5=5A_2$

We could solve by hand, or use MATLAB:

So,

$$\begin{aligned}A_1&=2\\A_2&=-1\\A_3&=-1-1j=\sqrt{2}\angle 225°\\A_3&=-1+1j=\sqrt{2}\angle 135°\end{aligned}$$

and

$$F(s)=\frac{2}{s}-\frac{1}{s^2}+\frac{-1-j}{s+1-2j}+\frac{-1+j}{s+1+2j}$$

We will use the notation derived above (<u>Method 1 - a more general technique</u>). The root of the denominator of the A_3 term in the partial fraction expansion is at s=-1+2j (i.e., the denominator goes to 0 when s=-1+2j), the magnitude of A_3 is √2, and the angle of A3 is 225°. So, M=2√2, φ=225°, ω=2, and σ=-1. Solving for f(t) we get

$$f(t) = 2 - t + Me^{\sigma t}\cos(\omega t + \varphi)$$
$$= 2 - t + 2\sqrt{2}e^{-t}\cos(2t + 225^\circ)$$

This expression is equivalent to the one obtained in the previous example.

ORDER OF NUMERATOR POLYNOMIAL EQUALS ORDER OF DENOMINATOR

When the Laplace Domain Function is not strictly proper (i.e., the order of the numerator is different than that of the denominator) we can not immediatley apply the techniques described above.

Example: Order of Numerator Equals Order of Denominator

Find the inverse Laplace Transform of the function F(s).

$$F(s) = \frac{3s^2 + 2s + 3}{s^2 + 3s + 2}$$

Solution:

For the fraction shown below, the order of the numerator polynomial is not less than that of the denominator polynomial, therefore we first perform long division

$$\begin{array}{r|l} & 3 \\ s^2 + 3s + 2 & 3s^2 + 2s + 3 \\ & \underline{3s^2 + 9s + 6} \\ & -7s - 3 \end{array}$$

Now we can express the fraction as a constant plus a proper ratio of polynomials.

$$F(s) = 3 + \frac{-7s - 3}{s^2 + 3s + 2} = 3 + \frac{-7s - 3}{(s+1)(s+2)}$$
$$= 3 + \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

Using the cover up method to get A_1 and A_2 we get

$$F(s) = 3 + \frac{4}{s+1} - \frac{11}{s+2}$$

$$f(t) = 3\delta(t) + 4e^{-t} - 11e^{-2t}$$

EXPONENTIALS IN THE NUMERATOR

The last case we will consider is that of exponentials in the numerator of the function.

Example: Exponentials in the numerator .Find the inverse Laplace Transform of the function F(s).

$$F(s) = \frac{s\left(1 + e^{-1.5s} + e^{-2.2s}\right) + e^{-1.5s}}{s(s+2)}$$

Solution:

The exponential terms indicate a time delay (see the time delay property). The first thing we need to do is collect terms that have the same time delay.

$$F(s) = \frac{s}{s(s+2)} + e^{-1.5s}\frac{s+1}{s(s+2)} + e^{-2.2s}\frac{s}{s(s+2)}$$

$$= \frac{1}{(s+2)} + e^{-1.5s}\frac{s+1}{s(s+2)} + e^{-2.2s}\frac{1}{(s+2)}$$

We now perform a partial fraction expansion for each time delay term (in this case we only need to perform the expansion for the term with the 1.5 second delay), but in general you must do a complete expansion for each term.

$$F(s) = \frac{1}{(s+2)} + e^{-1.5s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{(s+2)}\right) + e^{-2.2s}\frac{1}{(s+2)}$$

Now we can do the inverse Laplace Transform of each term (with the appropriate time delays)

$$f(t) = e^{-2t}\gamma(t) + \left(\frac{1}{2} - \frac{1}{2}e^{-2(t-1.5)}\right)\gamma(t-1.5) + e^{-2(t-2.2)}\gamma(t-2.2)$$