Chapter Four
Metric Space

Definition (Metric Space):

Let X be any nonempty set, the function $d: X \times X \to \mathbb{R}$ is called metric on X if d satisfies:

M_1: $d(x, y) \geq 0$

M_2: $d(x, y) = 0 \Leftrightarrow x = y$

M_3: $d(x, y) = d(y, x)$

M_4: $d(x, y) \leq d(x, z) + d(z, y)$

$\forall\, x, y, z \in X$

The pair (X, d) is called metric space.

Example (1):

Let $X = \mathbb{R}$, $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as follows $d(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$.

Show that $(\mathbb{R}, d)$ is a metric space.

Answer:

Let $x, y, z \in \mathbb{R}$

M_1: $\because |x - y| \geq 0 \implies \therefore d(x, y) = |x - y| \geq 0$

M_2: $d(x, y) = 0 \Leftrightarrow |x - y| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$

M_3: $d(x, y) = |x - y| = |y - x| = d(y, x)$

M_4: $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$.

$\therefore d$ is metric on $\mathbb{R}$

$(\mathbb{R}, d)$ is metric space called absolute metric (usual metric space).

Some Important Inequality:

1. Cauchy-Schwartz Inequality

   Let $a_1,\ a_2, \dots,\ a_n,\ b_1, b_2, \dots,\ b_n$ are real numbers then

   $$\sum_{i=1}^{n} |a_i + b_i| \le \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}$$

2. Minkowski Inequality

Let $a_1,\ a_2, \dots,\ a_n,\ b_1, b_2, \dots,\ b_n$ are real numbers then

$$\sqrt{\sum_{i=1}^{n} (a_i + b_i)^2} \le \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$$

Example (2):

Let $X = \mathbb{R}^2$, $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined as follows $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$\forall x = (x_1, y_1), y = (x_2, y_2) \in \mathbb{R}^2$. Is $(\mathbb{R}^2, d)$ forms metric space ?

Answer:

Let $x = (x_1, y_1), y = (x_2, y_2),\ Z = (x_3, y_3) \in \mathbb{R}^2$

$M_1: \because \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \ge 0 \implies \therefore d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \ge 0$

M_2: $d(x, y) = 0 \Leftrightarrow \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 0$

$\Leftrightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0$

$\Leftrightarrow x_1 - x_2 = 0$ and $y_1 - y_2 = 0$

$\Leftrightarrow x_1 = x_2$ and $y_1 = y_2 \Leftrightarrow x = y$.

M_3: $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d(y, x)$.

M_4: $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(x_1 - x_3 + x_3 - x_2)^2 + (y_1 - y_3 + y_3 - y_2)^2}$

$\le \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} + \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} = d(x, z) + d(z, y)$. (By using Minkowski Inequality)

$\therefore d$ is metric on $\mathbb{R}^2$, $(\mathbb{R}^2, d)$ is a metric space called (Euclidian metric space).

Example (3): Let X be any nonempty set, $d: X \times X \to \mathbb{R}$ defined as follows $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}, \forall\, x, y \in X$

Show that (X, d) is a metric space.

Answer:

M_1: $d(x,y) \geq 0, \forall\, x, y \in X$

M_2: $d(x,y) = 0 \Leftrightarrow x = y, \forall\, x, y \in X$

M_3: $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases} = \begin{cases} 1, & y \neq x \\ 0, & y = x \end{cases} = d(y,x), \forall\, x, y \in X$

M_4: $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

1. If $x = y$ and $y = z \Longrightarrow x = z$

   $d(x,y) = 0 \leq d(x,z) + d(z,y) = 0$

2. If $x \neq y$ and $y \neq z \Longrightarrow x \neq z$

   $d(x,y) = 1 \leq d(x,z) + d(z,y) = 2$

3. If $x = y$ and $y \neq z \Longrightarrow x \neq z$

   $d(x,y) = 0 \leq d(x,z) + d(z,y) = 2$

4. If $x \neq y$ and $y = z \Longrightarrow x \neq z$

   $d(x,y) = 1 \leq d(x,z) + d(z,y) = 1$

$\therefore d(x,y) \leq d(x,z) + d(z,y), \forall\, x, y, z \in X$

$\therefore$ (X, d) is metric space

Example (4):

Let $X = \mathrm{C}[a,b]$, $d: \mathrm{C}[a,b] \times \mathrm{C}[a,b] \to \mathbb{R}$, defined as follows

$d(f,g) = \max\{|f(x) - g(x)| : x \in [a,b]\}, \forall f,g \in \mathrm{C}[a,b]$

Show that $(\mathrm{C}[a,b], d)$ is a metric space.

Answer: Let $f, g, h \in \mathrm{C}[a,b]$

M_1: $\because |f(x) - g(x)| \geq 0, \forall x \in [a,b] \Longrightarrow \therefore d(f,g) = \max\{|f(x) - g(x)| : x \in [a,b]\} \geq 0$

M_2:

$d(f,g) = 0 \Leftrightarrow \max\{|f(x) - g(x)| : x \in [a,b]\} = 0$

$\Leftrightarrow |f(x) - g(x)| = 0 \Leftrightarrow f(x) - g(x) = 0 \Leftrightarrow f(x) = g(x), \forall x \in [a,b] \Leftrightarrow f = g$

M_3: $d(f,g) = \max\{|f(x) - g(x)| : x \in [a,b]\} = \max\{|g(x) - f(x)| : x \in [a,b]\} = d(g,f)$

M_4:

$d(f,g) = \max\{|f(x) - g(x)| : x \in [a,b]\}$

$= \max\{|f(x) - h(x) + h(x) - g(x)| : x \in [a,b]\}$

$\leq \max\{|f(x) - h(x)| : x \in [a,b]\} + \max\{|h(x) - g(x)| : x \in [a,b]\} = d(f,h) + d(h,g)$

$\therefore$ $(\mathrm{C}[a,b], d)$ is a metric space.