Uniform Continuous Functions

Definition: Let (X, d) and (X', d') be two metric spaces, a function $f: X \longrightarrow X'$ is said to be uniformly continuous if:

$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ s.t. if $d(x, y) < \delta$ then $d'(f(x), f(y)) < \varepsilon$

or

$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ s.t. if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Remarks:

1. Every uniform continuous function on X is continuous on X.
2. The converse may not be true.

Example: Let $f: (0,1) \longrightarrow \mathbb{R}$ defined as $f(x) = \frac{1}{x}$, $\forall x \in (0,1)$

f is continuous on $(0,1)$ but not uniform continuous on $(0,1)$. Since

Let $x_n = \frac{1}{n}$, $y_n = \frac{2}{n}$

$$|x_n - y_n| = \left|\frac{1}{n} - \frac{2}{n}\right| = \left|\frac{-1}{n}\right| = \frac{1}{n}$$

By A.P. $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$, let $\varepsilon = \delta$

$$\Rightarrow |x_n - y_n| = \frac{1}{n} < \delta.$$

$$|f(x_n) - f(y_n)| = \left|\frac{1}{x_n} - \frac{1}{y_n}\right| = \left|n - \frac{n}{2}\right| = \frac{n}{2} > \varepsilon$$

$\therefore$ f is not uniform continuous on $(0,1)$.

3. If f is continuous on $\mathbb{R}$, then f is uniform continuous on $[a,b]$, $a,b \in \mathbb{R}$.

Example:

$f(x) = \frac{1}{x}$, $x \in (0,1)$ is continuous on $(0,1)$, and

$f(x) = \frac{1}{x}$, $x \in [a,b] \subset (0,1)$ is uniform continuous on $[a,b]$.

Theorem:

Let $f: X \longrightarrow \mathbb{R}$ be a continuous function on X, if X is compact then f is uniform continuous.

Corollary:

Let $f: [a,b] \longrightarrow \mathbb{R}$ be a continuous function on $[a,b]$ then f is uniform continuous.

proof:

$\because$ $[a,b]$ is closed and bounded

$\therefore$ $[a,b]$ is compact (by Hine Borel)

$\therefore$ f is uniform continuous (by above th.).

Chapter 6

Differentiable Functions

Definition: Let $f: I \to \mathbb{R}$ be a real-valued function, $I=(a,b)$, $x_0 \in I$. We say that f is differentiable at x_0 if

$\exists\ l \in \mathbb{R}$ such that $\frac{f(x)-f(x_0)}{x-x_0}$ converges to l, x goes to x_0

i.e.

$\forall \varepsilon > 0,\ \exists\ \delta(x_0,\varepsilon) > 0$ s.t. $\left|\frac{f(x)-f(x_0)}{x-x_0} - l\right| < \varepsilon$ whenever

$|x-x_0| < \delta(x_0,\varepsilon)$, $(x \in I)$.

l is called the derivative of f at x_0 and denoted by $f'(x_0)$. i.e. $f'(x_0)=l$, $\left(\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = l\right)$.

Theorem:

Let $f: I \to \mathbb{R}$ be a real-valued function, $I=(a,b)$, then f has at most one derivative at $x_0 \in I$.

proof:

Let f has has two derivative l_1, l_2 at $x_0 \in I$ s.t. $l_1 \neq l_2 \Rightarrow |l_1 - l_2| > 0$, let $\varepsilon = |l_1 - l_2| > 0$, $\exists\ \delta_1 > 0$

s.t. $\left|\frac{f(x)-f(x_0)}{x-x_0} - l_1\right| < \frac{\varepsilon}{2}$ whenever $|x-x_0| < \delta_1$

$\exists\ \delta_2 > 0$ s.t. $\left|\frac{f(x)-f(x_0)}{x-x_0} - l_2\right| < \frac{\varepsilon}{2}$ whenever $|x-x_0| < \delta_2$

Choose $\delta = \min\{\delta_1, \delta_2\}$

$$0 < |l_1 - l_2| = \left| \frac{f(x)-f(x_0)}{x-x_0} - l_2 - \left(\frac{f(x)-f(x_0)}{x-x_0} - l_1 \right) \right|$$

$$\leq \left| \frac{f(x)-f(x_0)}{x-x_0} - l_2 \right| + \left| \frac{f(x)-f(x_0)}{x-x_0} - l_1 \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = |l_1 - l_2| \quad C!$$

$\therefore \; l_1 = l_2$.

Theorem:

Let $f: I \to \mathbb{R}$ be a real-valued function, $I = (a,b)$.
If f is differentiable at $x_0 \in I$ then f is continuous at $x_0 \in I$.

Proof:

Since f is diff. at $x_0 \in I \Rightarrow \exists\, l \in \mathbb{R}$ s.t.
$\forall\, \varepsilon > 0$, $\exists\, \delta(x_0, \varepsilon) > 0$ s.t. $\left| \frac{f(x)-f(x_0)}{x-x_0} - l \right| < \varepsilon$
whenever $|x - x_0| < \delta(x_0, \varepsilon)$.

$$|f(x) - f(x_0)| = \left| \frac{f(x)-f(x_0)}{x-x_0} \cdot (x - x_0) \right|$$

$$= \left| \frac{f(x)-f(x_0)}{x-x_0} \right| |x - x_0| = f'(x_0) \cdot 0 = 0 < \varepsilon$$

$\therefore \; f$ is continuous at $x_0 \in I$.

Remark: The converse to above theorem may not be true.

Example: Let $f: I \longrightarrow \mathbb{R}$ defined as $f(x) = |x|$, $\forall x \in I$, $I = (-a, a)$
f is continuous $\forall x_0 \in I$ but it is not differentable at $x_0 = 0 \in I$. Since

Let $\langle \frac{1}{n} \rangle$ sequence in I s.t. $\frac{1}{n} \longrightarrow 0$ (A.P.), $\frac{1}{n} \neq 0, \forall n$
$\frac{-1}{n} \longrightarrow 0$ (A.P.), $\frac{-1}{n} \neq 0, \forall n$

$$\frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n} - 0} = \frac{\frac{1}{n} - 0}{\frac{1}{n} - 0} = 1$$

$$\frac{f(\frac{-1}{n}) - f(0)}{\frac{-1}{n} - 0} = \frac{\frac{1}{n} - 0}{\frac{-1}{n} - 0} = -1$$

But $-1 \neq 1 \Rightarrow f(x) = |x|$ is not diff. at $x_0 = 0$.

Theorem:

Let $f, g: I \longrightarrow \mathbb{R}$ be two differentiable real-valued functions at $x_0 \in I$, $I = (a,b)$, then:

i) $f \mp g$ is differentiable at $x_0 \in I$ and $(f \mp g)'_{(x_0)} = f'(x_0) \mp g'(x_0)$

ii) cf is differentiable at $x_0 \in I$ and $(cf)'_{(x_0)} = c f'(x_0)$, where c is constant.

iii) $f \cdot g$ is differentiable at $x_0 \in I$ and $(f \cdot g)'_{(x_0)} = f(x_0) g'(x_0) + g(x_0) f'(x_0)$.

iv) $\frac{f}{g}$ is differentiable at $x_0 \in I$ and

$$\left(\frac{f}{g}\right)'_{(x_0)} = \frac{g(x_0) f'(x_0) - f(x_0) \cdot g'(x_0)}{\left(g(x_0)\right)^2}, \text{ where } g(x_0) \neq 0, x_0 \in I.$$

Proof: For $(f+g)$,

$\because f$ is diff. at $x_0 \in I \Rightarrow \forall \varepsilon > 0, \exists \delta_1(x_0, \varepsilon) > 0$ s.t.

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon/2 \text{ whenever } |x - x_0| < \delta_1(x_0, \varepsilon)$$

$\because g$ is diff. at $x_0 \in I \Rightarrow \forall \varepsilon > 0, \exists \delta_2(x_0, \varepsilon) > 0$ s.t.

$$\left| \frac{g(x) - g(x_0)}{x - x_0} - g'(x_0) \right| < \varepsilon/2 \text{ whenever } |x - x_0| < \delta_2(x_0, \varepsilon)$$

Choose $\delta = \min\{\delta_1, \delta_2\}$

$$\left| \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} - (f+g)'_{(x_0)} \right|$$

$$= \left| \frac{f(x) + g(x) - f(x_0) - g(x_0)}{x - x_0} - f'(x_0) - g'(x_0) \right|$$

$$\leq \left| \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0) \right| + \left| \frac{g(x)-g(x_0)}{x-x_0} - g'(x_0) \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore$ $f+g$ is diff. at $x_0 \in I$ and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

Proof (iv):

$$\lim_{x \to x_0} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x-x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)\,g(x_0) - f(x_0)\,g(x)}{(x-x_0)\,g(x)\,g(x_0)}$$

$$= \lim_{x \to x_0} \frac{f(x)\,g(x_0) + f(x_0)\,g(x_0) - f(x_0)\,g(x_0) - g(x)\,f(x_0)}{(x-x_0)\;g(x)\,g(x_0)}$$

$$= \lim_{x \to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0} \cdot g(x_0) - f(x_0)\,\frac{g(x)-g(x_0)}{x-x_0} \right] \cdot \frac{1}{g(x)\,g(x_0)}$$

$$= \left[f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0) \right] \cdot \frac{1}{g(x_0)\,g(x_0)}$$

$$= \frac{g(x_0)\,f'(x_0) - f(x_0)\,g'(x_0)}{g^2(x_0)}$$

$\therefore$ $\left(\frac{f}{g}\right)$ is diff. at x_0 and $\left(\frac{f}{g}\right)'(x_0) = \dfrac{g(x_0)\,f'(x_0) - f(x_0)\,g'(x_0)}{(g(x_0))^2}$, $g(x_0) \neq 0$.

Note: If $g(x)$ is continuous at x_0 and $g(x_0) \neq 0$, then $\frac{1}{g(x)}$ is continuous at x_0 i.e. $\lim_{x \to x_0} \frac{1}{g(x)} = \frac{1}{g(x_0)}$.

Theorem (Chain Rule):

Let $f: I \longrightarrow \mathbb{R}$, $g: J \longrightarrow \mathbb{R}$ such that $f(I) \subseteq J$ and f differentiable at $x_0 \in I$, g differentiable at $f(x_0) \in J$. Then $g \circ f$ is differentiable at $x_0 \in I$ and $(g \circ f)'(x_0) = f'(x_0) \cdot g'(f(x_0))$.

Theorem (Roll Theorem):

Let f be a continuous function on $[a,b]$ and differentiable on (a,b), if $f(a) = f(b)$ then $\exists\, c \in (a,b)$ s.t. $f'(c) = 0$.

Theorem (Mean Value Theorem):

Let f be a continuous function on $[a,b]$ and differentiable on (a,b), then $\exists\, c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem (Intermediate Value Theorem):

Let f be a continuous function on $[a,b]$ and $a < b$, $\forall\, z \in \mathbb{R}$ and $f(a) \leq z \leq f(b)$, then $\exists\, c \in [a,b]$ s.t. $f'(c) = z$.