Similar Matrices and Diagonalizable Matrices

Two $n \times n$ matrices A and B are *similar* if and only if there is an invertible matrix P such that $A = PBP^{-1}$ (and then we also have $B = P^{-1}AP = QAQ^{-1}$ where $Q = P^{-1}$).

An $n \times n$ matrix A is *diagonalizable* if and only if it is similar to a diagonal matrix; that is,

An $n \times n$ matrix A is *diagonalizable* if and only if it is similar to a diagon[al]
matrix; that is, there are a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Example

The matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ has *simple* eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Therefore,

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}.$$

Example

The matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ has *simple* eigenvalues $3, 4, 6$ with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Therefore,

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Example

$A_2 = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$ has one *simple* eigenvalue 5 and one *double* eigenvalue 2

with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

$A=PDP^{-1}$, where $P = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ and $D=\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$