3.2.1 Potential Difficulties When Applying the Gauss Elimination Method

The pivot element is zero

Since the pivot row is divided by the pivot element, a problem will arise during the execution of the Gauss elimination procedure if the value of the pivot element is equal to zero. As shown in the next section, this situation can be corrected by changing the order of the rows. In a procedure called pivoting, the pivot row that has the zero pivot element is exchanged with another row that has a nonzero pivot element.

The pivot element is small relative to the other terms in the pivot row

Significant errors due to rounding can occur when the pivot element is small relative to other elements in the pivot row. This is illustrated by the following example.

Consider the following system of simultaneous equations for the unknowns x_1 and x_2:

$$0.0003x_1 + 12.34x_2 = 12.343$$
$$0.4321\, x_1 + x2 = 5.321 \qquad (3.8)$$

The exact solution of the system is $x_1 = 10$ and $x_2 = 1$. The error due to rounding is illustrated by solving the system using Gaussian elimination on a machine with limited precision so that only four significant figures are retained with rounding. When the first equation of Eqs. (3.8) is entered, the constant on the right-hand side is rounded to 12.34.

The solution starts by using the first equation as the pivot equation and $a_{11}= 0.0003$ as the pivot coefficient. In the first step, the pivot equation is multiplied by $m_{21}= 0.4321/0.0003 = 1440$. With four significant figures and rounding, this operation gives:

$$(1440)(0.0003x_1 + 12.34x_2) = 1440\,(\,12.34)$$

or:

$$0.4320x_1 + 17770x_2 = 17770$$

The result is next subtracted from the second *equation* in Eqs. (3.8):

$$\begin{array}{rl} & 0.4321x_1 + x_2 = 5.321 \\ - & 0.4320x_1 + 17770x_2 = 17770 \\ \hline & 0.0001x_1 - 17770x_2 = -17760 \end{array}$$

After this operation, the system is:

$$0.0003x_1 + 12.34x_2 = 12.34$$
$$0.0001x_1 - 17770x_2 = -17760$$

Note that the a_{21} element is not zero but a very small number. Next, the value of x_2 is calculated from the second equation:

$$x_2 = \frac{-17760}{-17770} = 0.9994$$

Then x_2 is substituted in the first equation, which is solved for x_1:

$$x_1 = \frac{12.34 - 12.34(0.9994)}{0.0003} = \frac{0.01}{0.0003} = 33.33$$

The solution that is obtained for x_1 is obviously incorrect. The incorrect value is obtained because the magnitude of all is small when compared to the magnitude of a_{12}. Consequently, a relatively small error (due to round-off arising from the finite precision of a computing machine) in the value of x_2 can lead to a large error in the value of x_1. The problem can be easily remedied by exchanging the order of the two equations in Eqs. (3.8):

$$0.4321\, x_1 + x_2 = 5.321$$
$$0.0003x_1 + 12.34x_2 = 12.343 \qquad (3.9)$$

Now, as the first equation is used as the pivot equation, the pivot coefficient is $a_{11}= 0.4321$. In the first step, the pivot equation is multiplied by $m_{21} = 0.0003/0.4321 = 0.0006943$. With four significant figures and rounding this operation gives:

$$(0.0006943)(0.4321x_1 + x_2) = 0.0006943\ (5.321)$$

or:

$$0.0003x_1 + 0.0006943x_2 = 0.003694$$

The result is next subtracted from the second equation in Eqs. (3.9):

$$\begin{array}{rrl} & 0.0003x_1 + 12.34x_2 &= 12.34 \\ - & 0.0003x_1 + 0.0006943x_2 &= 0.003694 \\ \hline & 12.34x_2 &= 12.34 \end{array}$$

After this operation, the system is:

$$0.4321x_1 + x_2 = 5.321$$
$$0x1 + 12.34x_2 = 12.34$$

Next, the value of x_2 is calculated from the second equation:

$$x_2 = \frac{12.34}{12.34} = 1$$

Then x_2 is substituted in the first equation that is solved for x_1:

$$x_1 = \frac{5.321 - 1}{0.4321} = 10$$

The solution that is obtained now is the exact solution.

In general, a more accurate solution is obtained when the equations are arranged (and rearranged every time a new pivot equation is used) such that the pivot equation has the largest possible pivot element. This is explained in more detail in the next section.

Round-off errors can also be significant when solving large systems of equations even when all the coefficients in the pivot row are of the same order of magnitude. This can be caused by a large number of operations (multiplication, division, addition, and subtraction) associated with large systems.

3.3 GAUSS ELIMINATION WITH PIVOTING

In the Gauss elimination procedure, the pivot equation is divided by the pivot coefficient. This, however, cannot be done if the pivot coefficient is zero. For example, for the following system of three equations:

$$0x_1 + 2x_2 + 3x_3 = 46$$
$$4x_1 - 3x_2 + 2x_3 = 16$$
$$2x_1 + 4x_2 - 3x_3 = 12$$

the procedure starts by taking the first equation as the pivot equation and the coefficient of x_1, which is 0, as the pivot coefficient. To eliminate the term $4x_1$ in the second equation, the pivot equation is supposed to be multiplied by 4/0 and then subtracted from the second equation. Obviously, this is not possible when the pivot element is equal to zero. The division by zero can be avoided if the order in which the equations are written is changed such that in the first equation the first coefficient is not zero. For example, in the system above, this can be done by exchanging the first two equations.

In the general Gauss elimination procedure, an equation (or a row) can be used as the pivot equation (pivot row) only if the pivot coefficient (pivot element) is not zero. If the pivot element is zero, the equation (i.e., the row) is exchanged with one of the equations (rows) that are below, which has a nonzero pivot coefficient. This exchange of rows, illustrated in Fig. 3-12, is called pivoting.

After the first step, the second equation has a pivot element that is equal to zero.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Using pivoting, the second equation is exchanged with the third equation.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & 0 & a'_{23} & a'_{24} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_3 \\ b'_2 \\ b'_4 \end{bmatrix}$$

Figure 3-12: Illustration of pivoting.

Additional comments about pivoting

• If during the Gauss elimination procedure a pivot equation has a pivot element that is equal to zero, then if the system of equations that are being solved has a solution, an equation with a nonzero element in the pivot position can always be found.

• The numerical calculations are less prone to error and will have fewer round-off errors if the pivot element has a larger numerical absolute value compared to the other elements in the same row. Consequently, among all the equations that can be exchanged to be the pivot equation, it is better to select the equation whose pivot element has the largest absolute numerical value. Moreover, it is good to employ pivoting for the purpose of having a pivot equation with the pivot element that has the largest absolute numerical value at all times (even when pivoting is not necessary).

3.4 LU DECOMPOSITION METHOD

Background

The Gauss elimination method consists of two parts. The first part is the elimination procedure in which a system of linear equations that is given in a general form, [a][x] = [b], is transformed into an equivalent system of equations [a'][x] = [b'] in which the matrix of coefficients [a'] is upper triangular. In the second part, the equivalent system is solved by using back substitution. The elimination procedure requires many mathematical operations and significantly more computing time than the back substitution calculations. During the elimination procedure, the matrix of coefficients [a] and the vector [b] are both changed. This means that if there is a need to solve systems of equations that have the same left-hand-side terms (same coefficient matrix [a]) but different right-hand-side constants (different vectors [b]), the elimination procedure has to be carried out for each [b] again. Ideally, it would be better if the operations on the matrix of coefficients [a] were dissociated from those on the vector of constants [b] . In this way, the elimination procedure with [a] is done only once and then is used for solving systems of equations with different vectors [b] .

One option for solving various systems of equations [a][x] = [b] that have the same coefficient matrices [a] but different constant vectors [b] is to first calculate the inverse of the matrix [a] . Once the inverse matrix $[a]^{-1}$ is known, the solution can be calculated by: $[x] = [a]^{-1}[b]$.

Calculating the inverse of a matrix, however, requires many mathematical operations, and is computationally inefficient. A more efficient method of solution for this case is the LU decomposition method. In the LU decomposition method, the operations with the matrix [a] are done without using or changing, the vector [b], which is used only in the substitution part of the solution. The LU decomposition method can be used for solving a single system of linear equations, but it is especially advantageous for solving systems that have the same coefficient matrices [a] but different constant vectors [b].

The LU decomposition method

The *LU* decomposition method is a method for solving a system of linear equations *[a] [x] = [b]* . In this method the matrix of coefficients *[a]* is decomposed (factored) into a product of two matrices *[L]* and *[U]*:

$$[a] = [L][U] \qquad (3.10)$$

where the matrix *[L]* is a lower triangular matrix and *[U]* is an upper triangular matrix. With this decomposition, the system of equations to be solved has the form:

$$[L][U][x] = [b] \qquad (3.11)$$

To solve this equation, the product *[U][x]* is defined as:

$$[U][x] = [y] \qquad (3.12)$$

and is substituted in Eq. (3.11) to give:

$$[L][y] = [b] \qquad (3.13)$$

Now, the solution *[x]* is obtained in two steps. First, Eq. (3.13) is solved for *[y]*. Then, the solution *[y]* is substituted in Eq. (3.12), and that equation is solved for *[x]*. Since the matrix *[L]* is a lower triangular matrix, the solution *[y]* in Eq. (3.13) is obtained by using the **forward substitution** method. Once *[y]* is known and is substituted in Eq. (3.12), this equation is solved by using **back substitution**, since *[U]* is an upper triangular matrix. For a given matrix *[a]* several methods can be used to determine the corresponding *[L]* and *[U]*. One of them is related to the Gauss elimination method are described next.

3.4.1 LU Decomposition Using the Gauss Elimination Procedure

When the Gauss elimination procedure is applied to a matrix *[a]*, the elements of the matrices *[L]* and *[U]* are actually calculated. The upper triangular matrix *[U]* is the matrix of coefficients *[a]* that is obtained at the end of the procedure, as shown in Figs. 3-4 and 3- 11. The lower triangular matrix *[L]* is not written explicitly during the procedure, but the elements that make up the matrix are actually calculated along the way. The elements of *[L]* on the diagonal are all **1**, and the elements below the diagonal are the **multipliers** m_{ij} that multiply the pivot equation when it is used to eliminate the elements below the pivot coefficient. For the case of a system of four equations, the matrix of coefficients *[a]* is *(4 x 4)*, and the decomposition has the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix}$$

A numerical example illustrating LU decomposition is given next. It uses the information in the solution of Example 3- 1, where a system of four equations is solved by using the Gauss elimination method. The matrix *[a]* can be written from the given set of equations in the problem statement, and the matrix *[U]* can be written from the set of equations at the end of step 3 (page 35). The matrix *[L]* can be written by using the multipliers that are calculated in the solution. The decomposition has the form:

$$\begin{bmatrix} 4 & -2 & -3 & 6 \\ -6 & 7 & 6.5 & -6 \\ 1 & 7.5 & 6.25 & 5.5 \\ -12 & 22 & 15.5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.5 & 1 & 0 & 0 \\ 0.25 & 2 & 1 & 0 \\ -3 & 4 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & -3 & 6 \\ 0 & 4 & 2 & 3 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

3.5 ITERATIVE METHODS

A system of linear equations can also be solved by using an iterative approach. The process, in principle, is the same as in the fixed-point iteration method used for solving a single nonlinear equation. In an iterative process for solving a system of equations, the equations are written in an explicit form in which each unknown is written in terms of the other unknown. The explicit form for a system of four equations is illustrated in Fig. 3-13.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4 \end{aligned}$$

(*a*)

Writing the equations in an explicit form.

$$\begin{aligned} x_1 &= [b_1 - (a_{12}x_2 + a_{13}x_3 + a_{14}x_4)]/a_{11} \\ x_2 &= [b_2 - (a_{21}x_1 + a_{23}x_3 + a_{24}x_4)]/a_{22} \\ x_3 &= [b_3 - (a_{31}x_1 + a_{32}x_2 + a_{34}x_4)]/a_{33} \\ x_4 &= [b_4 - (a_{21}x_1 + a_{42}x_2 + a_{43}x_3)]/a_{44} \end{aligned}$$

(*b*)

Figure 3-13: Standard (a) and explicit (b) forms of a system of four equations.

The solution process starts by assuming initial values for the unknowns (first estimated solution). In the first iteration, the first assumed solution is substituted on the right-hand side of the equations, and the new values that are calculated for the unknowns are the second estimated solution. In the second iteration, the second solution is substituted back in the equations to give new values for the unknowns, which are the third estimated solution. The iterations continue in the same manner, and when the method does work, the solutions that are obtained as successive iterations converge toward the actual solution. For a system with n equations, the explicit equations for the $[x_j]$ unknowns are:

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{j=n} a_{ij}x_j\right), i = 1,2, \dots, n \qquad (3.14)$$

Condition for convergence

For a system of n equations $[a][x] = [b]$, a sufficient condition for convergence is that in each row of the matrix of coefficients $[a]$ the absolute value of the diagonal element is greater than the sum of the absolute values of the off-diagonal elements.

$$|a_{ii}| > \sum_{j=1, j\neq i}^{j=n} |a_{ij}| \qquad (3.15)$$

This condition is sufficient but not necessary for convergence when the iteration method is used. When the condition (3.15) is satisfied, the matrix $[a]$ is classified as diagonally dominant, and the iteration process converges toward the solution. The solution, however, might converge even when Eq. (3.15) is not satisfied. Two specific iterative methods for executing the iterations, the Jacobi and Gauss-Seidel methods, are presented next. The difference between the two methods is in the way that the new calculated values of the unknowns are used.

3. 5. 1 Jacobi Iterative Method

In the Jacobi method, an initial (first) value is assumed for each of the unknowns $x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}$. If no information is available regarding the approximate values of the unknown, the initial value of all the unknowns can be assumed to be zero. The second estimate of the solution $x_1^{(2)}, x_2^{(2)}, \dots, x_n^{(2)}$ is calculated by substituting the first estimate in the right-hand side ofEqs. (3.14):

$$x_i^{(2)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{j=n} a_{ij}x_j^{(1)}\right), i = 1,2, \dots, n$$

In general, the (k + 1) th estimate of the solution is calculated from the (k) th estimate by:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{j=n} a_{ij}x_j^{(k)}\right), i = 1,2, \dots, n$$

The iterations continue until the differences between the values that are obtained in successive iterations are small. The iterations can be stopped when the absolute value of the estimated relative error of all the unknowns is smaller than some predetermined value:

$$\left|\frac{x_i^{(k+1)} - x_i^{(k)}}{x_i^{(k)}}\right| < \epsilon, i = 1,2, \dots, n$$

Example 3.3 Solve the following equations by Jacobi's method.

$$15x + 3y - 2z = 85$$
$$2x + 10y + z = 51$$
$$x - 2y + 8z = 5$$

Solution In the above equations:

$$|15| > |3| + |-2|$$
$$|10| > |2| + |1|$$
$$|8| > |1| + |-2|$$

then Jacobi's method is applicable. We rewrite the given equations as follows:

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) = \frac{1}{15}(85 - 3y + 2z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) = \frac{1}{10}(51 - 2x - z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) = \frac{1}{8}(5 - x + 2y)$$

Let the initial approximations be:

$x^0 = y^0 = z^0 = 0$

Iteration 1:

$$\boxed{x_1} = \frac{d_1}{a_1} = \frac{85}{15} = \frac{17}{3}$$

$$\boxed{y_1} = \frac{d_2}{b_2} = \frac{51}{10}$$

$$\boxed{z_1} = \frac{d_3}{c_3} = \frac{5}{8}$$

Iteration 2:

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1) = \frac{1}{15}\left(85 - 3\times\frac{51}{10} - (-2)\times\frac{5}{8}\right)$$

$$\boxed{x_2} = 4.73$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1) = \frac{1}{10}\left(51 - 2\times\frac{17}{3} - 1\times\frac{5}{8}\right)$$

$$\boxed{y_2} = 3.904$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1) = \frac{1}{8}\left(5 - 1\times\frac{17}{3} - (-2)\times\frac{51}{10}\right)$$

$$\boxed{z_2} = 1.192$$

Iteration 3:

$$\boxed{x_3} = \frac{1}{15}(85 - 3\times 3.904 + 2\times 1.192) = 5.045$$

$$\boxed{y_3} = \frac{1}{10}(51 - 2\times 4.73 - 1\times 1.192) = 4.035$$

$$\boxed{z_3} = \frac{1}{8}(5 - 1\times 4.173 + 2\times 3.904) = 1.010$$

Iteration 4:

$$\boxed{x_4} = \frac{1}{15}(85 - 3\times 4.035 + 2\times 1.010) = 4.994$$

$$\boxed{y_4} = \frac{1}{10}(51 - 2\times 5.045 - 1\times 1.010) = 3.99$$

$$\boxed{z_4} = \frac{1}{8}(5 - 1\times 5.045 + 2\times 4.035) = 1.003$$

Iteration 5:

$$\boxed{x_5} = \frac{1}{15}(85 - 3\times 3.99 + 2\times 1.003) = 5.002$$

$$\boxed{y_5} = \frac{1}{10}(51 - 2\times 4.994 - 1\times 1.003) = 4.001$$

$$\boxed{z_5} = \frac{1}{8}(5 - 1\times 4.994 + 2\times 3.99) = 0.998$$

Iteration 6:

$$\boxed{x_6} = \frac{1}{15}(85 - 3\times 4.001 + 2\times 0.998) = 5.0$$
$$\boxed{y_6} = \frac{1}{10}(51 - 2\times 5.002 - 1\times 0.998) = 4.0$$
$$\boxed{z_6} = \frac{1}{8}(5 - 1\times 5.002 + 2\times 4.001) = 1.0$$

Iteration 7:

$$\boxed{x_7} = \frac{1}{15}(85 - 3\times 4 + 2\times 1) = 5.0$$
$$\boxed{y_7} = \frac{1}{10}(51 - 2\times 5 - 1\times 1) = 4.0$$
$$\boxed{z_7} = \frac{1}{8}(5 - 1\times 5 + 2\times 4) = 1.0$$

Example 3.4: Use the Jacobi iterative scheme to obtain the solutions of the system of equations correct to three decimal places.

$$x + 2y + z = 0$$
$$3x + y - z = 0$$
$$x - y + 4z = 3$$

Solution

Rearrange the equations in such a way that all the diagonal terms are dominant.

$$3x + y - z = 0$$
$$x + 2y + z = 0$$
$$x - y + 4z = 3$$

Computing for x, y and z we get:

$$x = (z - y)/3$$
$$y = (-x - z)/2$$
$$z = (3 + y - x)/4$$

The iterative equation can be written as:

$$x^{(r+1)} = (z^{(r)} - y^{(r)})/3$$
$$y^{(r+1)} = (-x^{(r)} - z^{(r)})/2$$
$$z^{(r+1)} = (3 - x^{(r)} + y^{(r)})/4$$

The initial vector is not specified in the problem. Hence we choose

$$x^{(0)} = y^{(0)} = z^{(0)} = 1$$

Then, the first iteration gives:

$$x^{(1)} = (z^{(0)} - y^{(0)})/3 = (1 - 1)/3 = 0$$
$$y^{(1)} = (-x^{(0)} - z^{(0)})/2 = (-1 - 1)/2 = -1.0$$
$$z^{(1)} = (3 - x^{(0)} + y^{(0)})/4 = (3 - 1 + 1)/4 = 0.750$$

similarly, second iteration yields:

$$x^{(2)} = (z^{(1)} - y^{(1)})/3 = (0.75 + 1.0)/3 = 0.5833$$
$$y^{(2)} = (-x^{(1)} - z^{(1)})/2 = (-0 - 0.75)/2 = -0.3750$$
$$z^{(2)} = (3 - x^{(1)} + y^{(1)})/4 = (3 - 0 - 0)/4 = 0.500$$

Subsequent iterations result in the following:

$x^{(3)} = 0.29167$	$y^{(3)} = -0.34165$	$z^{(3)} = 0.51042$
$x^{(4)} = 0.32986$	$y^{(4)} = -0.40104$	$z^{(4)} = 0.57862$
$x^{(5)} = 0.32595$	$y^{(5)} = -0.45334$	$z^{(5)} = 0.56728$
$x^{(6)} = 0.34021$	$y^{(6)} = -0.44662$	$z^{(6)} = 0.55329$
$x^{(7)} = 0.3333$	$y^{(7)} = -0.44675$	$z^{(7)} = 0.55498$
$x^{(8)} = 0.33391$	$y^{(8)} = -0.44414$	$z^{(8)} = 0.55498$
$x^{(9)} = 0.33304$	$y^{(9)} = -0.44445$	$z^{(9)} = 0.5555$

so to three decimal places the approximate solution:

$$x = 0.333 \quad y = -0.444 \quad z = 0.555$$

3. 5. 2 Gauss-Seidel Iterative Method

In the Gauss-Seidel method, initial (first) values are assumed for the unknowns x_2, x_3, ..., x_n (all of the unknowns except x_1). If no information is available regarding the approximate value of the unknowns, the initial value of all the unknowns can be assumed to be zero. The first assumed values of the unknowns are substituted in Eq. (3.14) with $i = 1$ to calculate the value of x_1. Next, Eq. (3.14) with $i = 2$ is used for calculating a new value for x_2. This is followed by using Eq. (3.14) with $i = 3$ for calculating a new value for x_3. The process continues until $i = n$, which is the end of the first iteration. Then, the second iteration starts with $i = 1$ where a new value for x_1 is calculated, and so on. In the Gauss-Seidel method, the current values of the unknowns are used for calculating the new value of the next unknown. In other words, as a new value of an unknown is calculated, it is immediately used for the next application of Eq. (3.14). (In the Jacobi method, the values of the unknowns obtained in one iteration are used as a complete set for calculating the new values of the unknowns in the next iteration. The values of the unknowns are not updated in the middle of the iteration.) Applying Eq. (3.14) to the Gauss-Seidel method gives the iteration formula:

$$\left.\begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{11}}\left(b_1 - \sum_{j=1,j\neq i}^{j=n} a_{1j}x_j^{(k)}\right) \\ x_i^{(k+1)} &= \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{j=i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{j=n} a_{ij}x_j^{(k)}\right), i = 2, \dots, n-1 \\ x_n^{(k+1)} &= \frac{1}{a_{nn}}\left(b_n - \sum_{j=1}^{j=n-1} a_{nj}x_j^{(k+1)}\right) \end{aligned}\right\} \quad (3.16)$$

Example 3.5: Solve the following equations by Gauss-Seidal method.

$$8x + 2y - 2z = 8$$
$$x - 8y + 3z = -4$$
$$2x + y + 9z = 12$$

Solution

In the above equations:

$$|8| > |2| + |-2|$$
$$|-8| > |1| + |3|$$
$$|9| > |2| + |1|$$

So, the conditions of convergence are satisfied and we can apply Gauss-Seidal method. Then we rewrite the given equations as follows:

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y^0 - c_1 z^0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z^0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

Let the initial approximations be:

$$x0 = y0 = z0 = 0$$

Iteration 1:

$$\boxed{x_1} = \frac{d_1}{a_1} = \frac{8}{8} = 1.0$$

$$\boxed{y_1} = \frac{1}{b_2}(d_2 - a_2 x_1) = \frac{1}{-8}(-4 - 1 \times 1.0) = 0.625$$

$$\boxed{z_1} = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1) = \frac{1}{9}(12 - 2) = 2 \times 1.0 - 1 \times 0.625) = 1.042$$

Iteration 2:

$$\boxed{x_2} = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1) = \frac{1}{8}(8 - 2 \times 0.625 - (-2) \times 1.042) = 1.104$$

$$\boxed{y_2} = \frac{1}{b_2}(d_2 - a_2 x_2 - c_2 z_1) = \frac{1}{-8}(-4 - 1 \times 1.104 - 3 \times 1.042) = 1.029$$

$$\boxed{z_2} = \frac{1}{c_3}(d_3 - a_3 x_2 - b_3 y_2) = \frac{1}{9}(12 - 2 \times 1.104 - 1 \times 1.029) = 0.974$$

Iteration 3:

$$\boxed{x_3} = \frac{1}{a_1}(d_1 - b_1 y_2 - c_1 z_2) = \frac{1}{8}(8 - 2 \times 1.029 - (-2) \times 0.974) = 0.986$$

$$\boxed{y_3} = \frac{1}{b_2}(d_2 - a_2 x_3 - c_2 z_2) = \frac{1}{-8}(-4 - 1 \times 0.986 - 3 \times 0.974) = 0.989$$

$$\boxed{z_3} = \frac{1}{c_3}(d_3 - a_3 x_3 - b_3 y_3) = \frac{1}{9}(12 - 2 \times 0.986 - 1 \times 0.989) = 1.004$$

Iteration 4:

$$\boxed{x_4} = \frac{1}{8}(8 - 2 \times 0.989 - (-2) \times 1.004) = 1.004$$

$$\boxed{y_4} = \frac{1}{-8}(-4 - 1 \times 1.004 - 3 \times 1.004) = 1.002$$

$$\boxed{z_4} = \frac{1}{9}(12 - 2 \times 1.004 - 1 \times 1.002) = 0.999$$

Iteration 5:

$$\boxed{x_5} = \frac{1}{8}(8 - 2\times 1.002 - (-2)\times 0.999) = 0.999$$

$$\boxed{y_5} = \frac{1}{-8}(-4 - 1\times 0.999 - 3\times 0.999) = 1.0$$

$$\boxed{z_5} = \frac{1}{9}(12 - 2\times 0.999 - 1\times 1.0) = 1.0$$

Iteration 6:

$$\boxed{x_6} = \frac{1}{8}(8 - 2\times 1 + 2\times 1) = \boxed{1.0}$$

$$\boxed{y_6} = \frac{1}{-8}(-4 - 1\times 1.0 - 3\times 1.0) = \boxed{1.0}$$

$$\boxed{z_6} = \frac{1}{9}(12 - 2\times 1.0 - 1\times 1.0) = \boxed{1.0}$$

Example 3.6: Using the Gauss-Seidal method solve the system of equations correct to three decimal places.

$$x + 2y + z = 0$$
$$3x + y - z = 0$$
$$x - y + 4z = 3$$

Solution

Rearranging the given equations to give dominant diagonal elements, we obtain

$3x + y - z = 0$

$x + 2y + z = 0$

$x - y + 4z = 3$ (E.1)

Equation (E.1) can be rewritten as

$x = (z - y)/3$

$y = -(x + z)/2$

$z = (3 + x + y)/4$ (E.2)

Writing Eq.(E.2) in the form of Gauss-Seidal iterative scheme, we get:

$$x^{(r+1)} = (z^{(r)} - y^{(r)})/3$$
$$y^{(r+1)} = -(x^{(r+1)} - z^{(r)})/2$$
$$z^{(r+1)} = (3 - x^{(r+1)} + y^{(r+1)})/4$$

We start with the initial value

$$x(0) = y(0) = z(0) = 1$$

The iteration scheme gives:

$$x^{(1)} = (z^{(0)} - y^{(0)})/3 = (1 - 1)/3 = 0$$
$$y^{(1)} = (-x^{(1)} - z^{(0)})/2 = (0 - 1)/2 = -0.5$$
$$z^{(1)} = (3 - x^{(1)} + y^{(1)})/4 = (3 - 0 - 0.5)/4 = 0.625$$

The second iteration gives:

$$x^{(2)} = (z^{(1)} - y^{(1)})/3 = (0.625 + 0.5)/3 = 0.375$$
$$y^{(2)} = (-x^{(2)} - z^{(1)})/2 = (-0.375 - 0.625)/2 = -0.50$$
$$z^{(2)} = (3 - x^{(2)} + y^{(2)})/4 = (3 - 0.375 - 0.5)/4 = 0.53125$$

Subsequent iterations result in:

$$x^{(3)} = 0.34375 \qquad y^{(3)} = -0.4375 \qquad z^{(3)} = 0.55469$$
$$x^{(4)} = 0.33075 \qquad y^{(4)} = -0.44271 \qquad z^{(4)} = 0.55664$$
$$x^{(5)} = 0.33312 \qquad y^{(5)} = -0.44488 \qquad z^{(5)} = 0.5555$$
$$x^{(6)} = 0.33346 \qquad y^{(6)} = -0.44448 \qquad z^{(6)} = 0.55552$$

Hence, the approximate solution is as follows:

$$x = 0.333,\ y = -0.444,\ z = 0.555$$

3.6 USE OF MATLAB Built IN FUNCTIONS FOR SOLVING A SYSTEM OF LINEAR EQUATIONS

MATLAB has mathematical operations and built-in functions that can be used for solving a system of linear equations and for carrying out other matrix operations that are described in this chapter.

3.6.1 Solving a System of Equations Using MATLAB's Left and Right Division

Left division \ : Left division can be used to solve a system of n equations written in matrix form *[a][x]=[b]*, where *[a]* is the *(n* x *n)* matrix of coefficients, *[x]* is an *(n* x *1)* column vector of the unknowns, and *[b]* is an *(n* x *1)* column vector of constants.

x=a\b

For example, the solution of the system of equations in Example 3-1 is calculated by (Command Window):

```
>> a=[4 -2 -3 6; -6 7 6.5 -6; 1 7.5 6.25 5.5; -12 22 15.5 -1];
>> b=[12; -6.5; 16; 17];
>> x=a\b
x =
    2.0000
    4.0000
   -3.0000
    0.5000
```

Right division / : Right division is used to solve a system of n equations written in matrix form *[x][a] = [b]*, where *[a]* is the *(n* x *n)* matrix of coefficients, *[x]* is a *(1* x *n)* row vector of the unknowns, and *[b]* is a *(1* x *n)* row vector of constants.

x=b/a

For example, the solution of the system of equations in Example 3-1 is calculated by (Command Window):

```
>> a=[4 -6 1 -12; -2 7 7.5 22; -3 6.5 6.25 15.5; 6 -6 5.5 -1];
>> b=[12 -6.5 16 17];
>> x=b/a
x =
    2.0000    4.0000    -3.0000    0.5000
```

Notice that the matrix *[a]* used in the right division calculation is the transpose of the matrix used in the left division calculation.

3.6.2 Solving a System of Equations Using MATLAB Inverse Operation

In MATLAB, the inverse of a matrix *[a]* can be calculated either by raising the matrix to the power of -1 or by using the **inv(** ***a*** **)** function. Once the inverse is calculated, the solution is obtained by multiplying the vector *[b]* by the inverse. This is demonstrated for the solution of the system in Example 4-1.

```
>> a=[4 -2 -3 6; -6 7 6.5 -6; 1 7.5 6.25 5.5; -12 22 15.5 -1];
>> b=[12; -6.5; 16; 17];
>> x=a^-1*b
x =
    2.0000
    4.0000
   -3.0000
    0.5000
```

The same result is obtained by typing >>x = inv(a)*b.

3.7 Problems

1. Solve the following system of equations using the Gauss elimination method:

$$2x_1 + x_2 - x_3 = 1$$
$$x_1 + 2x_2 + x_3 = 8$$
$$-x_1 + x_2 - x_3 = -5$$

2. Consider the following system of two linear equations:

$$0.0003x_1 + 1.566x_2 = 1.569$$
$$0.3454x_1 - 2.436x_2 = 1.018$$

(a) Solve the system with the Gauss elimination method using rounding with four significant figures.
(b) Switch the order of the equations, and solve the system with the Gauss elimination method using rounding with four significant figures.
Check the answers by substituting the solution back in the equations.

3. Solve the following set of simultaneous linear equations using the Jacobi's method.

   a. $2x - y + 5z = 15$
   $2x + y + z = 7$
   $x + 3y + z = 10$

   b. $20x + y - 2z = 17$
   $3x + 20y - z = -18$
   $2x - 3y + 20z = 25$

   c. $5x + 2y + z = 12$
   $x + 4y + 2z = 15$
   $x + 2y + 5z = 20$

4. Solve the following system of simultaneous linear equations using the Gauss-Seidal method.

   a. $4x - 3y + 5z = 34$
   $2x - y - z = 6$
   $z + y + 4z = 15$

   b. $2x - y + 5z = 15$
   $2x + y + z = 7$
   $x + 3y + z = 10$