Chapter 3: Solving a System of Linear Equations

3.1 BACKGROUND

Systems of linear equations that have to be solved simultaneously arise in problems that include several (possibly many) variables that are dependent on each other. Such problems occur not only in engineering and science but in virtually any discipline (business, statistics, economics, etc.). A system of two (or three) equations with two (or three) unknowns can be solved manually by substitution or other mathematical methods (e.g., Cramer's rule). Solving a system in this way is practically impossible as the number of equations (and unknowns) increases beyond three.

3. 1. 1 Overview of Numerical Methods for Solving a System of Linear Algebraic Equations

The general form of a system of n linear algebraic equations is:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \qquad (3.1)$$

The matrix form of the equations is shown in Fig. 3-1.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$

Figure 3-1: A system of n linear algebraic equations.

Two types of numerical methods, direct and iterative, are used for solving systems of linear algebraic equations. In direct methods, the solution is calculated by performing arithmetic operations with the equations. In iterative methods, an initial approximate solution is assumed and then used in an iterative process for obtaining successively more accurate solutions.

Direct methods

In direct methods, the system of equations that is initially given in the general form, Eqs. (3.1), is manipulated to an equivalent system of equations that can be easily solved. Three systems of equations that can be easily solved are the upper triangular, lower triangular, and diagonal forms. The upper triangular form is shown in Eqs. (3.2),

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{nn}x_n &= b_n \end{aligned}\right\} \qquad (3.2)$$

and is written in a matrix form for a system of four equations in Fig. 3-2.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Figure 3-2: A system of four equations in upper triangular form.

he system in this form has all zero coefficients below the diagonal and is solved by a procedure called back bstitution. It starts with the last equation, which is solved for x_n. The value of x_n is then substituted in the next-to-the-last equation, which is solved for x_{n-1}. The process continues, in the same manner, all the way up to the first equation. In the case of four equations, the solution is given by:

$$x_4 = \frac{b_4}{a_{44}} \quad , \; x_3 = \frac{b_3 - a_{34}x_4}{a_{33}} \quad , \; x_2 = \frac{b_2 - (a_{23}x_3 + a_{24}x_4)}{a_{22}}$$

$$\text{and} \quad x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + a_{14}x_4)}{a_{11}}$$

For a system of n equations in upper triangular form, general formula for the solution using back substitution is:

$$x_n = \frac{b_n}{a_{nn}} \quad , \quad x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}} \quad , i = n-1, n-2, \ldots, 1 \qquad (3.3)$$

In Section 3.2 the upper triangular form and back substitution are used in the Gauss elimination method.

Exc: Write a program for Eq. (3.3).

3.2 GAUSS ELIMINATION METHOD

The Gauss elimination method is a procedure for solving a system of linear equations. In this procedure, a system of equations that are given in a general form is manipulated to be in upper triangular form, which is then solved by using back substitution (see Section 3.1.1). For a set of four equations with four unknowns, the general form is given by:

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 & (3.4a)\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 & (3.4b)\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 & (3.4c)\\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_{41} & (3.4d)
\end{aligned}\right\} \qquad (3.4)$$

The matrix form of the system is shown in Fig. 3-3.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Figure 3-3: Matrix form of a system of four equations.

In the Gauss elimination method, the system of equations is manipulated into an equivalent system of equations that has the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a'_{33} & a'_{34} \\ 0 & 0 & 0 & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Figure 3-4: Matrix form of the equivalent system.

In general, various mathematical manipulations can be used for converting a system of equations from the general form displayed in Eqs. (4.10) to the upper triangular form. One, in particular, the Gauss elimination method, is described next. The procedure can be easily programmed in a computer code.

Gauss elimination procedure (forward elimination)

The Gauss elimination procedure is first illustrated for a system of four equations with four unknowns. The starting point is the set of equations that are given by Eqs. (3.4). Converting the system of equations to the upper triangular form is done in steps.

Step 1: In the first step, the first equation is unchanged, and the terms that include the variable x_1 in all the other equations are eliminated. This is done one equation at a time by using the first equation, which is called the ***pivot equation***. The coefficient a_{11} is called the ***pivot coefficient***, or the ***pivot element***. To eliminate the term $a_{i1}x_1$ in Eq. (3.4b), the pivot equation, Eq. (3.4a), is multiplied by $m_{21} = \frac{a_{21}}{a_{11}}$, and then the equation is subtracted from Eq. (3.4b):

$$
\begin{array}{rl}
 & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\
- & m_{21}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) = m_{21}b_1 \\
\hline
\end{array}
$$

$$
0 + \underbrace{(a_{22} - m_{21}a_{12})}_{a'_{22}}x_2 + \underbrace{(a_{23} - m_{21}a_{13})}_{a'_{23}}x_3 + \underbrace{(a_{24} - m_{21}a_{14})}_{a'_{24}}x_4 = \underbrace{b_2 - m_{21}b_1}_{b'_2}
$$

It should be emphasized here that the pivot equation, Eq. (3.4a), itself is not changed. The matrix form of the equations after this operation is shown in Fig. 3-5.

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =
\begin{bmatrix} b_1 \\ b'_2 \\ b_3 \\ b_4 \end{bmatrix}
$$

Figure 3-5: Matrix form of the system after eliminating a_{21}.

Next, the term $a_{31}x_1$ in Eq. (3.4c) is eliminated. The pivot equation, Eq. (3.4a), is multiplied by $m_{31} = \frac{a_{31}}{a_{11}}$ and then is subtracted from Eq. (3.4c):

$$
\begin{array}{rl}
 & a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \\
- & m_{31}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) = m_{31}b_1 \\
\hline
\end{array}
$$

$$
0 + \underbrace{(a_{32} - m_{31}a_{12})}_{a'_{32}}x_2 + \underbrace{(a_{33} - m_{31}a_{13})}_{a'_{33}}x_3 + \underbrace{(a_{34} - m_{31}a_{14})}_{a'_{34}}x_4 = \underbrace{b_3 - m_{31}b_1}_{b'_3}
$$

The matrix form of the equations after this operation is shown in Fig. 3-6.

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =
\begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b_4 \end{bmatrix}
$$

Figure 3-6: Matrix form of the system after eliminating a_{31}.

Next, the term $a_{41}x_1$ in Eq. (3.4d) is eliminated. The pivot equation, Eq. (3.4a), is multiplied by $m_{31} = \frac{a_{31}}{a_{11}}$ and then is subtracted from Eq. (4.3d):

$$
\begin{array}{r}
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \\
- \quad m_{41}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) = m_{41}b_1 \\
\hline
0 + \underbrace{(a_{42} - m_{41}a_{12})}_{a'_{42}}x_2 + \underbrace{(a_{43} - m_{41}a_{13})}_{a'_{43}}x_3 + \underbrace{(a_{44} - m_{41}a_{14})}_{a'_{44}}x_4 = \underbrace{b_4 - m_{41}b_1}_{b'_4}
\end{array}
$$

This is the end of **Step 1.** The system of equations now has the following form:

$$
\left.
\begin{array}{ll}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 & (3.5a) \\
0 + a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 = b'_2 & (3.5b) \\
0 + a'_{32}x_2 + a'_{33}x_3 + a'_{34}x_4 = b'_3 & (3.5c) \\
0 + a'_{42}x_2 + a'_{43}x_3 + a'_{44}x_4 = b'_4 & (3.5d)
\end{array}
\right\} \quad (3.5)
$$

The matrix form of the equations after this operation is shown in Fig. 3-7. Note that the result of the elimination operation is to reduce the first column entries, except a_{11} (the pivot element), to zero.

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}
$$

Figure 3-7: Matrix form of the system after eliminating a_{41}.

Step 2: In this step, Eqs. (3.5a) and (3.5b) are not changed, and the terms that include the variable x_2 in Eqs. (3.5c) and (3.5d) are eliminated. In this step, Eq. (3.5b) is the pivot equation, and the coefficient a'_{22} is the pivot coefficient. To eliminate the term $a'_{32}x_2$ in Eq. (3.5c), the pivot equation, Eq. (3.5b), is multiplied by $m_{32} = \frac{a'_{32}}{a'_{22}}$ and then is subtracted from Eq. (3.5c):

$$
\begin{array}{r}
a'_{32}x_2 + a'_{33}x_3 + a'_{34}x_4 = b'_3 \\
- \quad m_{32}(a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4) = m_{32}b'_2 \\
\hline
0 + \underbrace{(a'_{33} - m_{32}a'_{23})}_{a''_{33}}x_3 + \underbrace{(a'_{34} - m_{32}a'_{24})}_{a''_{34}}x_4 = \underbrace{b'_3 - m_{32}b'_2}_{b''_3}
\end{array}
$$

The matrix form of the equations after this operation is shown in Fig. 3-8.

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b'_4 \end{bmatrix}
$$

Figure 3-8: Matrix form of the system after eliminating a_{32}.

Next, the term $a'_{42}x_2$ in Eq. (3.5d) is eliminated. The pivot equation, Eq. (3.5b), is multiplied by $m_{42} = \frac{a'_{42}}{a'_{22}}$ and then is subtracted from Eq. (3.5d):

$$
\begin{array}{r}
a'_{42}x_2 + a'_{43}x_3 + a'_{44}x_4 = b'_4 \\
- \quad m_{42}(a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4) = m_{42}b'_2 \\
\hline
0 + \underbrace{(a'_{43} - m_{42}a'_{23})}_{a''_{43}}x_3 + \underbrace{(a'_{44} - m_{42}a'_{24})}_{a''_{44}}x_4 = \underbrace{b'_4 - m_{42}b'_2}_{b''_4}
\end{array}
$$

The matrix form of the equations after this operation is shown in Fig. 3-9.

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & d'_{22} & a'_{23} & d'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & a''_{43} & a''_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b''_4 \end{bmatrix}
$$

Figure 3-9: Matrix form of the system after eliminating a_{42}.

This is the end of **Step 2**. The system of equations now has the following form:

$$
\left.\begin{array}{lr}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 & (3.6a) \\
0 + a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 = b'_2 & (3.6b) \\
0 \; + \; 0 \; + a''_{33}x_3 + a''_{34}x_4 = b''_3 & (3.6c) \\
0 \; + \; 0 \; + a''_{43}x_3 + a''_{44}x_4 = b''_4 & (3.6d)
\end{array}\right\} \quad (3.6)
$$

Step 3: In this step, Eqs. (3.6a), (3.6b), and (3.6c) are not changed, and the term that includes the variable x_3 in Eq. (3.6d) is eliminated. In this step, Eq. (3.6c) is the pivot equation, and the coefficient a''_{33} is the pivot coefficient. To eliminate the term $a''_{43}x_3$ in Eq. (3.6d), the pivot equation is multiplied by $m_{43} = \frac{a''_{43}}{a''_{33}}$ and then is subtracted from Eq. (3.6d):

$$
\begin{array}{r}
a''_{43}x_3 + a''_{44}x_4 = b''_4 \\
- \quad m_{43}(a''_{33}x_3 + a''_{34}x_4) = m_{43}b''_3 \\
\hline
\underbrace{(a''_{44} - m_{43}a''_{34})}_{a'''_{44}}x_4 = \underbrace{b''_4 - m_{43}b''_3}_{b'''_4}
\end{array}
$$

This is the end of **Step 3**. The system of equations is now in an upper triangular form:

$$
\left.\begin{array}{lr}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 & (3.7a) \\
0 + a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 = b'_2 & (3.7b) \\
0 \; + \; 0 \; + a''_{33}x_3 + a''_{34}x_4 = b''_3 & (3.7c) \\
0 \; + \; 0 \; + \; 0 \; + a'''_{44}x_4 = b'''_4 & (3.7d)
\end{array}\right\} \quad (3.7)
$$

The matrix form of the equations is shown in Fig. 3-10. Once transformed to upper triangular form, the equations can be easily solved by using back substitution.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ b'''_4 \end{bmatrix}$$

Figure 3-10: Matrix form of the system after eliminating a_{43} •

The three steps of the Gauss elimination process are illustrated together in Fig. 3-11.

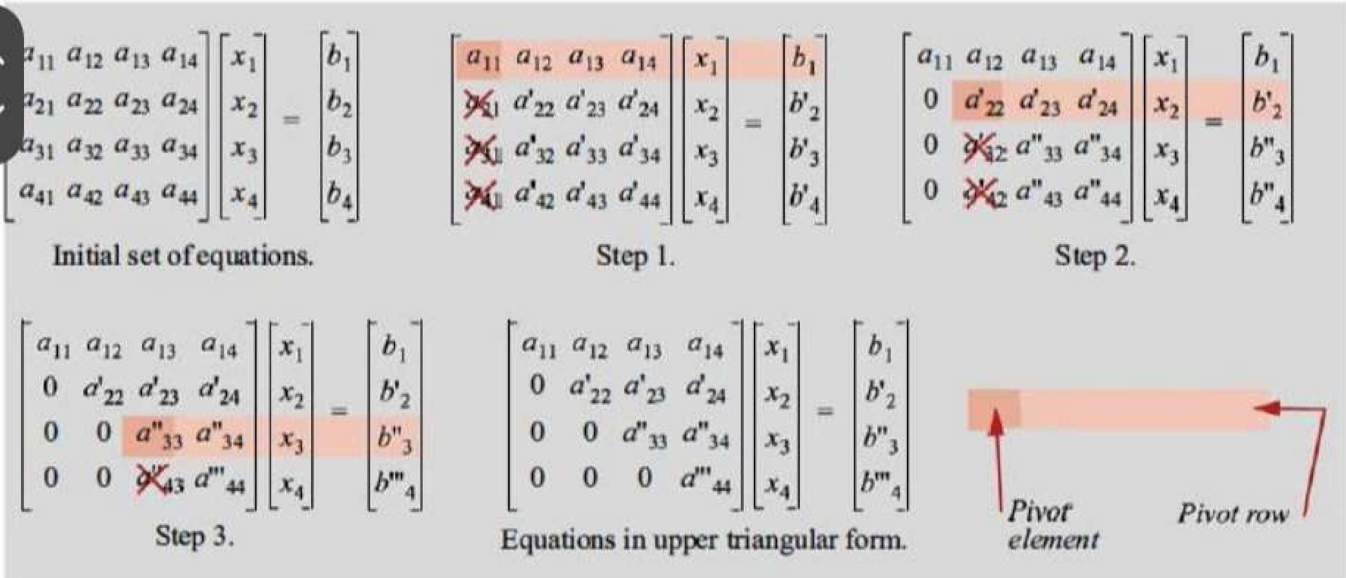

Figure 3-11: Gauss elimination procedure.

Example 3-1: Solve the following system of four equations using the Gauss elimination method.

$$4x_1 - 2x_2 - 3x_3 + 6x_4 = 12$$
$$-6x_1 + 7x_2 + 6.5x_3 - 6x_4 = -6.5$$
$$x_1 + 7.5x_2 + 6.25x_3 + 5.5x_4 = 16$$
$$-12x_1 + 22x_2 + 15.5x_3 - x_4 = 17$$

SOLUTION: The solution follows the steps presented in the previous pages.

Step 1: The first equation is the pivot equation, and 4 is the pivot coefficient.

Multiply the pivot equation by m_{21} = (-6)/ 4 = -1.5 and subtract it from the second equation:

$$\begin{array}{rl} & -6x_1 + 7x_2 + 6.5x_3 - 6x_4 = -6.5 \\ - & (-1.5)(4x_1 - 2x_2 - 3x_3 + 6x_4) = (-6/4) \cdot 12 \\ \hline & 0x_1 + 4x_2 + 2x_3 + 3x_4 = 11.5 \end{array}$$

Multiply the pivot equation by m_{31}= (1I4) =0 .25 and subtract it from the third equation:

$$\begin{array}{rl} & x_1 + 7.5x_2 + 6.25x_3 + 5.5x_4 = 16 \\ - & (0.25)(4x_1 - 2x_2 - 3x_3 + 6x_4) = (1/4) \cdot 12 \\ \hline & 0x_1 + 8x_2 + 7x_3 + 4x_4 = 13 \end{array}$$

Multiply the pivot equation by m_{41}=(-12)/4 = -3 and subtract it from the fourth equation:

$$\begin{array}{rl} & -12x_1 + 22x_2 + 15.5x_3 - x_4 = 17 \\ - & (-3)(4x_1 - 2x_2 - 3x_3 + 6x_4) = -3 \cdot 12 \\ \hline & 0x_1 + 16x_2 + 6.5x_3 + 17x_4 = 53 \end{array}$$

At the end of Step 1, the four equations have the form:

$$4x_1 - 2x_2 - 3x_3 + 6x_4 = 12$$
$$4x_2 + 2x_3 + 3x_4 = 11.5$$
$$8x_2 + 7x_3 + 4x_4 = 13$$
$$16x_2 + 6.5x_3 + 17x_4 = 53$$

Step 2: The second equation is the pivot equation, and 4 is the pivot coefficient. Multiply the pivot equation by $m_{32} = 8/4 = 2$ and subtract it from the third equation:

$$\begin{array}{rrl} & 8x_2 + 7x_3 + 4x_4 & = 13 \\ - & 2(4x_2 + 2x_3 + 3x_4) & = 2 \cdot 11.5 \\ \hline & 0x_2 + 3x_3 - 2x_4 & = -10 \end{array}$$

Multiply the pivot equation by $m_{42} = 16/4 = 4$ and subtract it from the fourth equation:

$$\begin{array}{rrl} & 16x_2 + 6.5x_3 + 17x_4 & = 53 \\ - & 4(4x_2 + 2x_3 + 3x_4) & = 4 \cdot 11.5 \\ \hline & 0x_2 - 1.5x_3 + 5x_4 & = 7 \end{array}$$

At the end of Step 2, the four equations have the form:

$$4x_1 - 2x_2 - 3x_3 + 6x_4 = 12$$
$$4x_2 + 2x_3 + 3x_4 = 11.5$$
$$3x_3 - 2x_4 = -10$$
$$-1.5x_3 + 5x_4 = 7$$

Step 3: The third equation is the pivot equation, and 3 is the pivot coefficient. Multiply the pivot equation by $m_{43} = (-1.5)/3 = -0.5$ and subtract it from the fourth equation:

$$\begin{array}{rrl} & -1.5x_3 + 5x_4 & = 7 \\ - & -0.5(3x_3 - 2x_4) & = -0.5 \cdot -10 \\ \hline & 0x_3 + 4x_4 & = 2 \end{array}$$

At the end of Step 3, the four equations have the form:

$$4x_1 - 2x_2 - 3x_3 + 6x_4 = 12$$
$$4x_2 + 2x_3 + 3x_4 = 11.5$$
$$3x_3 - 2x_4 = -10$$
$$4x_4 = 2$$

Once the equations are in this form, the solution can be determined by back substitution. The value of x_4 is determined by solving the fourth equation:

$$x_4 = 2/4 = 0.5$$

Next, x_4 is substituted in the third equation, which is solved for x_3 :

$$x_3 = \frac{-10 + 2x_4}{3} = \frac{-10 + 2(0.5)}{3} = -3$$

Next, x_4 and x_3 are substituted in the second equation, which is solved for x_2:

$$x_2 = \frac{11.5 - 2x_3 - 3x_4}{4} = \frac{11.5 - 2(-3) - 3(0.5)}{4} = 4$$

Lastly, x_4, x_3 and x_2 are substituted in the first equation, which is solved for x_1 :

$$x_1 = \frac{12 + 2x_2 + 3x_3 - 6x_4}{4} = \frac{12 + 2(4) + 3(-3) - 6(0.5)}{4} = 2$$