## 3.3 Integer Programming

***Integer linear programming problems*** (***ILPP***) are linear programming problems with some or all the variables restricted to integer (discrete) values. When all the variables are constrained to be integers, it is called an ***all*** (***pure***) ***integer programming problem***. In case only some of the variables are restricted to have integer values, the problem is said to be a ***mixed integer programming problem***. The ILPP algorithms are based on exploiting the tremendous computational success of LPP. The strategy of these algorithms involves three steps:

***Step 1:*** Relax the solution space of the ILPP by deleting the integer restriction on all integer variables. The result of the relaxation is a regular LPP.

***Step 2:*** Solve the LPP, and identify its optimum.

***Step 3:*** Starting from the optimum point, add special constraints that iteratively modify the LPP solution space in a manner that will eventually render an optimum extreme point satisfying the integer requirements.

Two general methods have been developed for generating the special constraints in step 3:

1. Cutting - plane method.
2. Branch - and – bound (B & B) method.

### 3.3.1 Gomory's Cutting Plane Method

This systematic procedure for solving pure ILPP was first suggested by R.E. Gomory (1929- ) in 1958. Later, he extends the procedure to cover mixed ILPP. The method consists in first solving the ILPP as ordinary continuous LPP and then introducing additional constraints one after the other to cut (eliminate) certain parts of the solution space until an integral solution is obtained.

**Definition (3.1):**

For all real number $x$, the ***greatest integer function*** (denoted by $[\, x\,]$ ) returns the largest integer less than or equal to *x*. In other words, the greatest integer function rounds down a real number to the nearest integer. The number *x* can be written in the form $x = [x] + e$, where $0 \le e \le 1$.We call $e$ the fractional part of $x$.

## Example (3.8):

1) $[0.41] = 0$ since integers less than 0.41 are: $\dots, -2, -1, 0$ and the greatest one of them is 0. The number 0.41 can be written in the form: $0.41 = 0 + 0.41 = [0.41] + 0.41$.
2) $[-0.41] = -1$ since integers less than $-0.41$ are: $\dots, -3, -2, -1$ and the greatest one of them is $-1$. The number $-0.41$ can be written in the form: $-0.41 = -1 + 0.59 = [-0.41] + 0.59$
3) $[9.73] = 9$
4) $[-7.26] = -8$
5) $[3] = 3$
6) $[-5] = -5$

According to definition (3.1), the structural coefficients and the stipulations can be written as:

$a_{ij} = [a_{ij}] + f_{ij},\ b_i = [b_i] + f_i, where\ 0 \leq f_{ij} \leq 1\ and\ 0 \leq f_i \leq 1\,;$
$i = 1, \dots, m; j = 1, \dots, n$

The steps of Gomory's cutting plane method for pure ILPP are:

***Step 1:* Integerise the constraints**: Transform the constraints so that all the coefficients are whole numbers. For example, the constraint equation:
$\frac{7}{4}x_1 + \frac{1}{5}x_2 + \frac{3}{4}x_3 = \frac{17}{5}$ can be expressed as: $35x_1 + 4x_2 + 15x_3 = 68$.

***Step 2:* Solve the problem**: Ignoring integrality restriction, find the optimal solution to the problem. If the solution is all integers, it is an optimal basic feasible integer solution. If not, proceed to step 3. Ignore non-integer values for slack variables since they represent unused resources only.

***Step 3:* Develop a cutting plane:** From the final table select the constraint with the largest fractional cut. The selected row is called the ***source row***. In case of a tie, choose the constraint having the highest contribution (maximization problem) or the lowest cost (minimization problem). Alternatively select the constraint with $max\ \frac{f_i}{\sum_{j=1}^{n} f_{ij}}$ $\dots..(1)$ . Construct the Gomory's constraint:

$s_i = \sum_{j=1}^{n} f_{ij} y_j - f_i$ .....(2) ( $y_j$ may be decision or slack variable)

And add it to the final table. Add an additional column for $s_i$ also.

***Step 4:* Solve using the dual simplex method:** Solve the augmented ILPP obtained above by the dual simplex method so that the outgoing variable is $s_i$ .
If the optimal solution thus obtained has all integral values, it is an optimal

solution for the given ILPP. If not, repeat step 3 until an optimal feasible integer solution is obtained.

## Remark (3.1):

In mixed ILPP, only constraints corresponding to integer variables are used to construct the cut.

## Example (3.9):

Find the optimal solution of the following ILPP

$$max \quad Z = 5x_1 + 6x_2$$
$$S.t \quad 2x_1 + 3x_2 \leq 18$$
$$2x_1 + x_2 \leq 12$$
$$x_1 + x_2 \leq 8$$
$$x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers}$$

## Solution:

The standard form of the LPP (with modification in the objective function and ignoring integrality condition) is:

$$max \quad Z - 5x_1 - 6x_2 = 0$$
$$S.t \quad 2x_1 + 3x_2 + s_1 = 18$$
$$2x_1 + x_2 + s_2 = 12$$
$$x_1 + x_2 + s_3 = 8$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Let $x_1 = x_2 = 0$, then $s_1 = 18, s_2 = 12, s_3 = 8$, and $Z = 0$ and

| B.V. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | b | |
|---|---|---|---|---|---|---|---|
| $s_1$ | 2 | 3 | 1 | 0 | 0 | 18 | |
| $s_2$ | 2 | 1 | 0 | 1 | 0 | 12 | |
| $s_3$ | 1 | 1 | 0 | 0 | 1 | 8 | |
| Z | $-5$ | $\boxed{-6}$ | 0 | 0 | 0 | 0 | |
| $x_2$ | 2/3 | 1 | 1/3 | 0 | 0 | 6 | |
| $s_2$ | 4/3 | 0 | $-1/3$ | 1 | 0 | 6 | |
| $s_3$ | 1/3 | 0 | $-1/3$ | 0 | 1 | 2 | |
| Z | $\boxed{-1}$ | 0 | 2 | 0 | 0 | 36 | |
| $x_2$ | 0 | 1 | 1/2 | $-1/2$ | 0 | 3 | 3 |
| $x_1$ | 1 | 0 | $-1/4$ | 3/4 | 0 | 9/2 | $4\frac{1}{2}$ |
| $s_3$ | 0 | 0 | $-1/4$ | $-1/4$ | 1 | 1/2 | $\frac{1}{2}$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| **Z** | 0 | 0 | 7/4 | 3/4 | 0 | 81/2 |

The non-integer optimal solution is $x_1 = \frac{9}{2}, x_2 = 3,$ and $Z_{max} = 40\frac{1}{2}$ . To construct Gomory's constraint, select $x_1$-row which has the greatest fractional part ½ ($s_3$-row also has the fractional part ½ , but a decision variable is preferred than slack variable).

$$1.x_1 + 0.x_2 - \frac{1}{4}.s_1 + \frac{3}{4}s_2 + 0.s_3 = \frac{9}{2}$$

Or $\quad (1+0)x_1 + \left(-1+\frac{3}{4}\right)s_1 + \left(0+\frac{3}{4}\right)s_2 = 4 + \frac{1}{2}$

By using equation (2), the Gomory's constraint (cut) is:

$$s_4 = \frac{3}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{2} \qquad \Longrightarrow \quad -\frac{3}{4}s_1 - \frac{3}{4}s_2 + s_4 = -\frac{1}{2}$$

The modified table after inserting this equation becomes:

| **B.V.** | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | **b** | |
|---|---|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 1/2 | −1/2 | 0 | 0 | 3 | |
| $x_1$ | 1 | 0 | −1/4 | 3/4 | 0 | 0 | 9/2 | |
| $s_3$ | 0 | 0 | −1/4 | −1/4 | 1 | 0 | 1/2 | |
| $s_4$ | 0 | 0 | −3/4 | $\boxed{-3/4}$ | 0 | 1 | −1/2 | |
| **Z** | 0 | 0 | 7/4 | 3/4 | 0 | 0 | 81/2 | |
| $x_2$ | 0 | 1 | 1 | 0 | 0 | −2/3 | 10/3 | $3\frac{1}{3}$ |
| $x_1$ | 1 | 0 | −1 | 0 | 0 | 1 | 4 | 4 |
| $s_3$ | 0 | 0 | 0 | 0 | 1 | −1/3 | 2/3 | $\frac{2}{3}$ |
| $s_2$ | 0 | 0 | 1 | 1 | 0 | −4/3 | 2/3 | $\frac{2}{3}$ |
| **Z** | 0 | 0 | 1 | 0 | 0 | 1 | 40 | |

( $\left|\frac{7/2}{-3/4}\right| = 4.7, \left|\frac{3/4}{-3/4}\right| = 1$ ) . $x_2$ has a non-integer value ( 10/3 ). Since the fractional part of $s_2$ and $s_3$ are equal (=2/3), then from equation (1):

$$\frac{f_i}{\sum_{j=1}^{n} f_{ij}} \text{ for } s_2 - \text{equation} = \frac{2/3}{2/3} = 1$$

$$\frac{f_i}{\sum_{j=1}^{n} f_{ij}} \text{ for } s_3 - \text{equation} = \frac{2/3}{2/3} = 1$$

Since both ratios are equal, we choose $s_2$ arbitrarily to construct second Gomory's cut as follows:

$$1.s_1 + 1.s_2 - \frac{4}{3}s_4 = \frac{2}{3}$$

Or $(1+0)s_1 + (1+0)s_2 + \left(-2+\frac{2}{3}\right)s_4 = 0 + \frac{2}{3}$

Then from equation (2):

$s_5 = \frac{2}{3}s_4 - \frac{2}{3} \quad \Rightarrow \quad -\frac{2}{3}s_4 + s_5 = -\frac{2}{3}$

The modified table after inserting this equation becomes

| B.V. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | b |
|---|---|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 1 | 0 | 0 | $-2/3$ | 0 | 10/3 |
| $x_1$ | 1 | 0 | $-1$ | 0 | 0 | 1 | 0 | 4 |
| $s_3$ | 0 | 0 | 0 | 0 | 1 | $-1/3$ | 0 | 2/3 |
| $s_2$ | 0 | 0 | 1 | 1 | 0 | $-4/3$ | 0 | 2/3 |
| $s_5$ | 0 | 0 | 0 | 0 | 0 | $\boxed{-2/3}$ | 1 | $-2/3$ |
| Z | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 40 |
| $x_2$ | 0 | 1 | 1 | 0 | 0 | 0 | $-1$ | 4 |
| $x_1$ | 1 | 0 | $-1$ | 0 | 0 | 0 | 1/2 | 3 |
| $s_3$ | 0 | 0 | 0 | 0 | 1 | 0 | $-1/2$ | 1 |
| $s_2$ | 0 | 0 | 1 | 1 | 0 | 0 | $-2$ | 2 |
| $s_4$ | 0 | 0 | 0 | 0 | 0 | 1 | $-3/2$ | 1 |
| Z | 0 | 0 | 1 | 0 | 0 | 0 | 3/2 | 39 |

The optimal solution is: $x_1 = 3, x_2 = 4$, and $Z_{max} = 39$.

## Example (3.10):

Discuss, graphically, the effect of the cuts in example (3.9) on the feasible solutions space.

**Solution:**

For $2x_1 + 3x_2 = 18 \Rightarrow$ if $x_1 = 0$ then (0, 6) is the intersection point with the $x_2-$axis.

And if $x_2 = 0$ then (9, 0) is the intersection point with the $x_1-$axis.

For $2x_1 + x_2 = 12 \quad \Rightarrow$ if $x_1 = 0$ then (0, 12) is the intersection point with the $x_2-$axis.

And if $x_2 = 0$ then (6, 0) is the intersection point with the $x_1-$axis.

For $x_1 + x_2 = 8 \quad \Rightarrow$ if $x_1 = 0$ then (0, 8) is the intersection point with the $x_2-$axis.

And if $x_2 = 0$ then (8, 0) is the intersection point with the $x_1-$axis.

The point B is resulting from the intersection of the lines representing the first and the second constraints, so we use these constraints to find the coordinates of B.

$$\cancel{2x_1} + 3x_2 = 18$$
$$\underline{\cancel{\mp 2x_1} \mp x_2 = \mp 12}$$

$$\Rightarrow 2x_2 = 6 \Rightarrow x_2 = 3 \xRightarrow{(constr.2)} x_1 = \frac{12-3}{2} = 4.5$$

The feasible solution region space is the shaded area OABC in figure (3.4) whose corners are the points $O(0,0)$, $A(0,6)$, $B(4.5,3)$, and $C(6,0)$.

| **Corner** | **Value of Z** |
|---|---|
| O(0,0) | Z=0 |
| $A(0,6)$ | $Z = 5 \times 0 + 6 \times 6 = 36$ |
| $B(4.5,3)$ | $Z = 5 \times 4.5 + 6 \times 3 = 40.5$ * |
| $C(6,0)$ | $Z = 5 \times 6 + 6 \times 0 = 30$ |

From the table, we see that the greatest value of Z occurs at corner B, then $x_1 = 4.5$ , $x_2 = 3$, and $Z_{max} = 40.5$.

The first cut is: $0 \leq s_4 = \frac{3}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{2}$ ...(*), that is:

$-\frac{3}{4}s_1 - \frac{3}{4}s_2 \leq -\frac{1}{2}$ .... (**)

From the first and second constraints in the standard form:

$s_1 = 18 - 2x_1 - 3x_2$

$s_2 = 12 - 2x_1 - x_2$

Substitute in (**), then:

$$-\frac{3}{4}(18 - 2x_1 - 3x_2) - \frac{3}{4}(12 - 2x_1 - x_2) \leq -\frac{1}{2}$$

$$\frac{-54-36}{4} + \frac{3}{2}x_1 + \frac{9}{4}x_2 + \frac{3}{2}x_1 + \frac{3}{4}x_2 \leq -\frac{1}{2}$$

$$\Longrightarrow 3x_1 + 3x_2 \leq 22$$

$3x_1 + 3x_2 = 22 \Longrightarrow$ if $x_1 = 0$ then (0, 7.3) is the intersection point with the $x_2$ – axis.

And if $x_2 = 0$ then (7.3, 0) is the intersection point with the $x_1$ – axis.

The first cut intersects the first constraint in the point (4,10/3), this solution is not optimal. The second constraint is: $0 \leq s_5 = \frac{2}{3}s_4 - \frac{2}{3}$, that is

$-\frac{2}{3}s_4 \leq -\frac{2}{3}$ ...(***)

From equation (*):

$s_4 = \frac{3}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{2} = \frac{3}{4}(18 - 2x_1 - 3x_2\ ) + \frac{3}{4}(12 - 2x_1 - x_2\ ) - \frac{1}{2}$

$s_4 = 22 - 3x_1 - 3x_2$

Substitute $s_4$ in (***) , the result is the second cut in terms of $x_1$ and $x_2$:

$2x_1 + 2x_2 \leq 14$

$2x_1 + 2x_2 = 14 \Rightarrow$ if $x_1 = 0$ then (0, 7) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then (7, 0) is the intersection point with the $x_1 -$ axis.

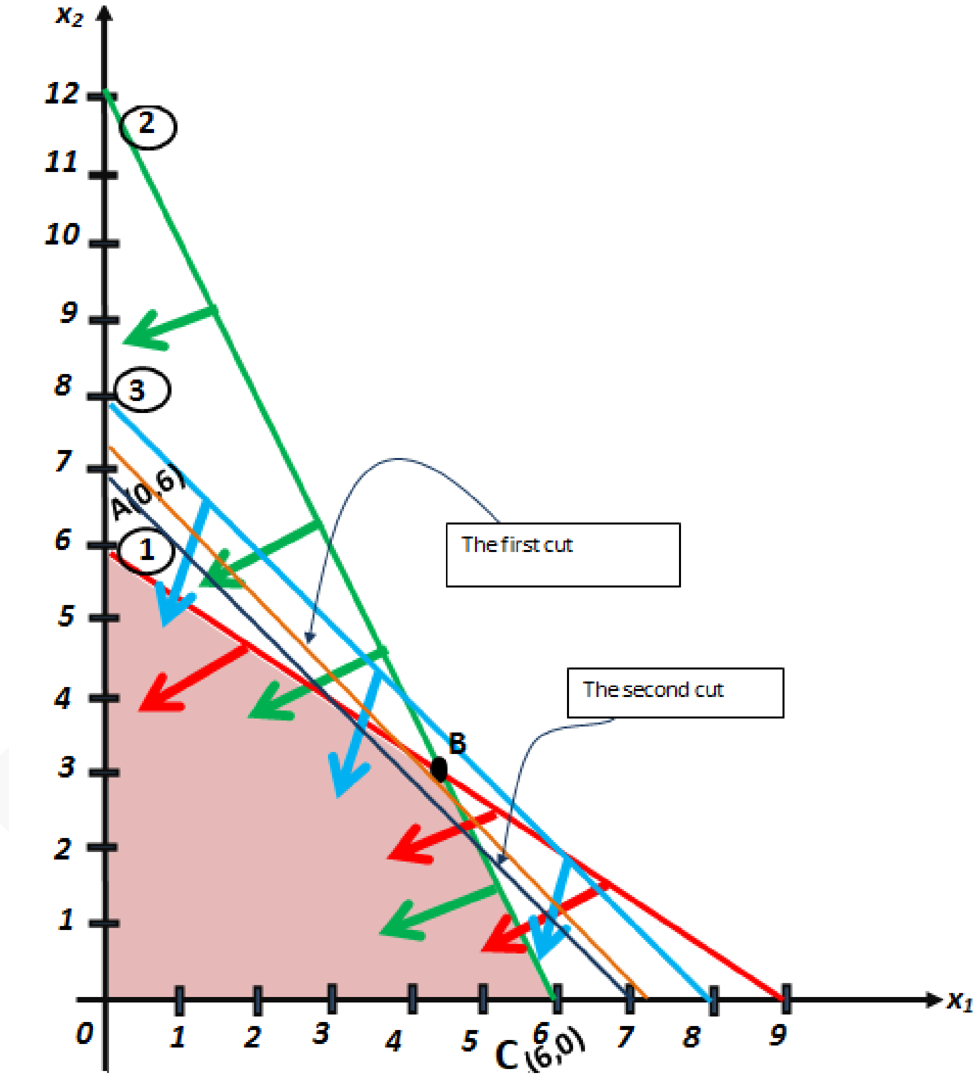


**Figure (3.4)**

The second cut passes through the point C(4,3) which is the optimal solution. Each cut neglecting a part of the feasible solutions set as we see in figures (3.4) and (3.5). Figure (3.5) shows the parts of the feasible solution set in which the extreme point exists.

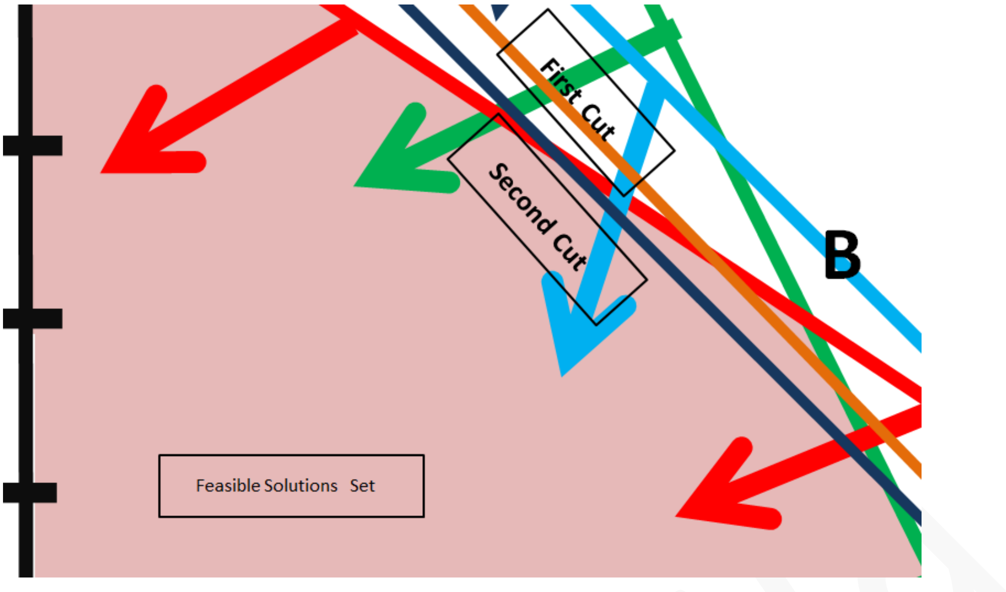


**Figure (3.5)**

## Example (3.11):

Find the optimal solution of the following ILPP

$max \quad Z = -4x_1 + 5x_2$

$S.t \quad -\frac{3}{5}x_1 + \frac{3}{5}x_2 \le \frac{6}{5}$

$2x_1 + 4x_2 \le 12$

$x_1, x_2 \ge 0, x_1 \text{ and } x_2 \text{ are integers}$

**Solution:**

First of all, multiplying the first constraint by (5), so it will be:

$-3x_1 + 3x_2 \le 6$

The standard form of the LPP (with modification in the objective function and ignoring integrality condition) is:

$max \quad Z + 4x_1 - 5x_2 = 0$

$S.t \quad -3x_1 + 3x_2 + s_1 = 6$

$2x_1 + 4x_2 + s_2 = 12$

$x_1, x_2, s_1, s_2 \ge 0$

Let $x_1 = x_2 = 0$, then $s_1 = 6, s_2 = 12$, and $Z = 0$ and

| B.V. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | b |
|---|---|---|---|---|---|
| $s_1$ | $-3$ | 3 | 1 | 0 | 6 |
| $s_2$ | 2 | 4 | 0 | 1 | 12 |
| Z | 4 | $\boxed{-5}$ | 0 | 0 | 0 |
| $x_2$ | $-1$ | 1 | 1/3 | 0 | 2 |

| | | | | | |
|---|---|---|---|---|---|
| $s_2$ | 6 | 0 | $-4/3$ | 1 | 4 |
| **Z** | $\boxed{-1}$ | 0 | 5/3 | 0 | 10 |
| $x_2$ | 0 | 1 | 1/9 | 1/6 | 8/3 |
| $x_1$ | 1 | 0 | $-2/9$ | 1/6 | 2/3 |
| **Z** | 0 | 0 | 13/9 | 1/6 | 32/3 |

$2\frac{2}{3}$

$\frac{2}{3}$

The non-integer optimal solution is $x_1 = \frac{2}{3}, x_2 = \frac{8}{3}$, and $Z_{max} = \frac{32}{3}$ . Since the fractional part of $x_1$ and $x_2$ are equal (=2/3), then from equation (1):

$\frac{f_i}{\sum_{j=1}^{n} f_{ij}}$ for $x_1 -$ equation $= \frac{2/3}{(7/9)+(1/6)} = 12/17$

$\frac{f_i}{\sum_{j=1}^{n} f_{ij}}$ for $x_2 -$ equation $= \frac{2/3}{(1/9)+(1/6)} = 12/5$

$x_2$ –equation is selected as the source row and Gomory's cut is:

$1. x_2 + \frac{1}{9}s_1 + \frac{1}{6}s_2 = \frac{8}{3}$

Or $(1+0)x_2 + (0+\frac{1}{9})s_1 + (0+\frac{1}{6})s_2 = 2 + \frac{2}{3}$

Then from equation (2):

$s_3 = \frac{1}{9}s_1 + \frac{1}{6}s_2 - \frac{2}{3} \quad \Longrightarrow \quad -\frac{1}{9}s_1 - \frac{1}{6}s_2 + s_3 = -\frac{2}{3}$

The modified table after inserting this equation becomes

| B.V. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | b |
|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 1/9 | 1/6 | 0 | 8/3 |
| $x_1$ | 1 | 0 | $-2/9$ | 1/6 | 0 | 2/3 |
| $s_3$ | 0 | 0 | $-1/9$ | $\boxed{-1/6}$ | 1 | $-2/3$ |
| **Z** | 0 | 0 | 13/9 | 1/6 | 0 | 32/3 |
| $x_2$ | 0 | 1 | 0 | 0 | 1 | 2 |
| $x_1$ | 1 | 0 | $-3/9$ | 0 | 1 | 0 |
| $s_2$ | 0 | 0 | 6/9 | 1 | $-6$ | 4 |
| **Z** | 0 | 0 | 4/3 | 0 | 1 | 10 |

The optimal solution is $x_1 = 0, x_2 = 2$, and $Z_{max} = 10$.

## Exercises 3.3 (In addition to the text book exercises)

Find the optimal solution of the following ILPP:

**1.** $max \quad Z = 14x_1 + 20x_2$

$S.t \quad -2x_1 + 6x_2 \leq 12$

$14x_1 + 2x_2 \leq 70$

$$x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers}$$

**2.** $\quad max \quad Z = 2x_1 + 10x_2 + x_3$

$\quad S.t \quad 5x_1 + 2x_2 + x_3 \leq 15$

$\quad\quad 2x_1 + x_2 + 7x_3 \leq 20$

$\quad\quad x_1 + 3x_2 + 2x_3 \leq 25$

$\quad\quad x_1, x_2, x_3 \geq 0 \text{ and are integers}$

### 3.3.2 Branch-and-Bound ( B&B) Method

The first B&B algorithm was developed in 1960 by A.H.Land and A.G.Doig for the general mixed and pure ILPP. In this method also, the problem is first solved as a continuous LPP ignoring the integrality condition. Assume a maximization (minimization) problem, set an initial lower (upper) bound $Z = -\infty\ (\infty)$ on the optimum objective value of ILPP. Set $i = 0$.

***Step 1:* (Fathoming / bounding)**. Select $Z_i$ , the next subproblem to be examined. Solve $Z_i$ , and attempt to fathom it using one of three conditions:

a) The optimal Z-value of $Z_i$ cannot yield a better objective value than the current lower bound.
b) $Z_i$ yields a better feasible integer solution than the current lower bound.
c) $Z_i$ has no feasible solution.

Two cases will arise:

a) If $Z_i$ is fathomed and a better solution is found, update the lower bound. If all subproblems have been fathomed, stop; the optimum ILPP is associated with the current finite lower bound. If no finite lower bound exists, the problem has no feasible solution. Else, set $i = i + 1$, and repeat step 1.
b) If $Z_i$ is not fathomed, go to step 2 for branching.

***Step 2:* (branching)**. Select one of the integer variables $x_j$, whose optimum value $x_j^*$ is not integer. Eliminate the region:

$$[x_j^*] < x_j < [x_j^*] + 1$$

By creating two LP subproblems that correspond to:

$$x_j \leq [x_j^*] \text{ and } x_j \geq [x_j^*] + 1$$

$x_j$ is called the ***branching variable***. These two conditions are mutually execlusive and when applied separately to the continuous LPP, form two different subproblems.Thus the original problem is branched (or partitioned) into two subproblems (also called ***nodes***). Geometrically, it means that the branching process eliminates that portion of the feasible region that contains no feasible integer solution. Set $i = i + 1$, and go to step 1.

## Example (3.12):

Find the optimal solution of the following ILPP:

$$\max \quad Z = x_1 + x_2$$
$$S.t \quad 2x_1 + 5x_2 \leq 16$$
$$6x_1 + 5x_2 \leq 30$$
$$x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers}$$

**Solution:**

For $2x_1 + 5x_2 = 16 \Rightarrow$ if $x_1 = 0$ then (0, 3.2) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then (8, 0) is the intersection point with the $x_1 -$ axis.

For $6x_1 + 5x_2 = 30 \Rightarrow$ if $x_1 = 0$ then (0, 6) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then (5, 0) is the intersection point with the $x_1 -$ axis.

The graphical representation is:

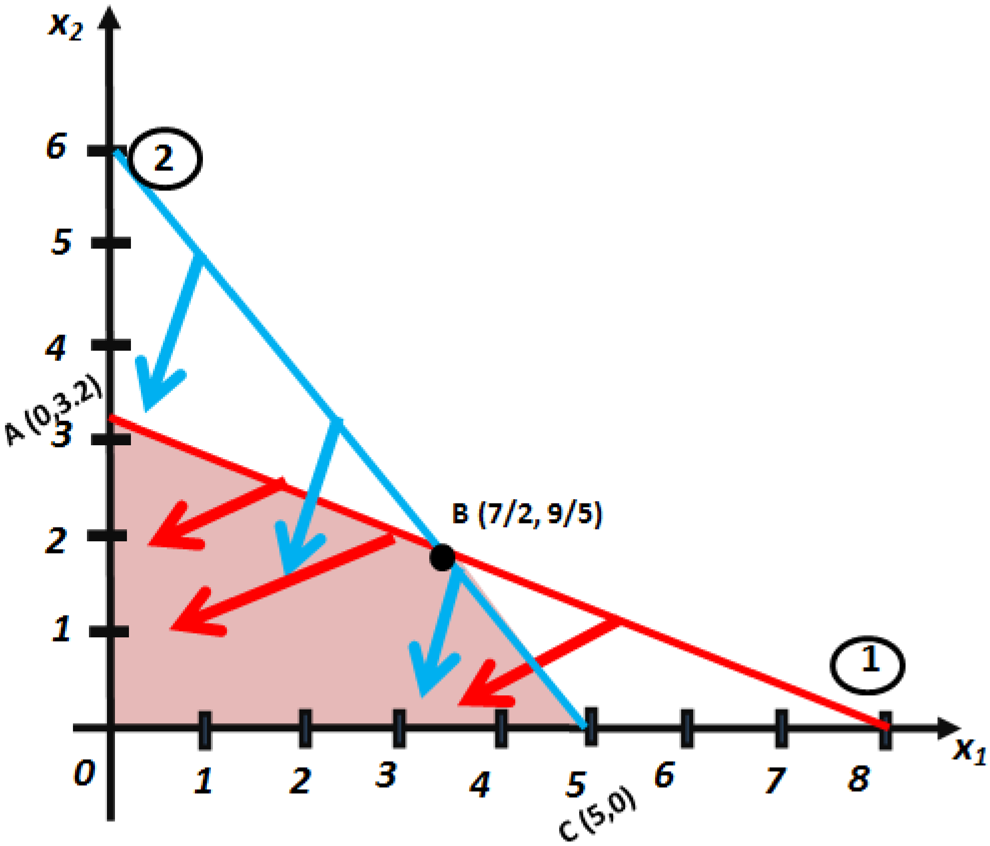


**Figure (3.6)**

The point B is resulting from the intersection of the lines representing the first and the second constraints, so we use these constraints to find the coordinates of B.

$$2x_1 + \cancel{5x_2} = 16$$
$$\mp 6x_1 \mp \cancel{5x_2} = \mp 30$$

---

$$\Rightarrow -4x_1 = -14 \Rightarrow x_1 = 7/2 \xRightarrow{(constr.2)} x_2 = \frac{16-7}{5} = 9/5$$

The feasible solution region space is the shaded area OABC in figure (3.6) whose corners are the points $O(0,0)$, $A(0,3.2)$, $B(7/2,9/5)$, and $C(5,0)$.

| Corner | Value of Z |
|---|---|
| O(0,0) | Z=0 |
| $A(0,3.2)$ | $Z = 0 + 3.2 = 3.2$ |
| $B(7/2,9/5)$ | $Z = \frac{7}{2} + \frac{9}{5} = \frac{53}{10} = 5.3$ * |
| $C(5,0)$ | $Z = 5 + 0 = 5$ |

From the table, we see that the greatest value of Z occurs at corner B, then $x_1 = 7/2$, $x_2 = 9/5$, and $Z_{max} = 5.3$. The solution is not optimal, then choose $x_1 = 3.5$ as a branching variable, $3 \leq x_1 \leq 4$. Construct two new problems by adding the constructs: $x_1 \leq 3, x_1 \geq 4$.

**Subproblem $Z_1$:**

$$\begin{aligned} max \quad & Z = x_1 + x_2 \\ S.t \quad & 2x_1 + 5x_2 \leq 16 \\ & 6x_1 + 5x_2 \leq 30 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$x_1$ and $x_2$ are integers

The solution is: $x_1 = 3$, $x_2 = 2, Z_{max} = 5$

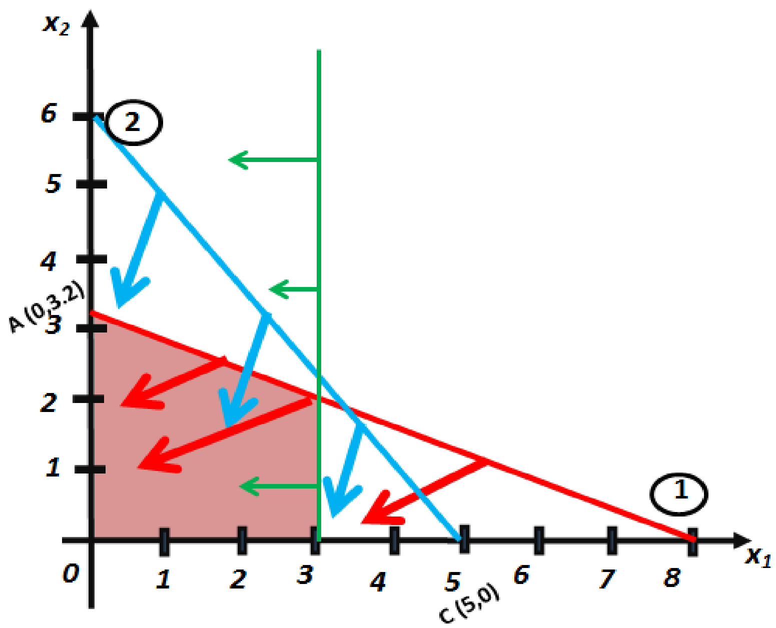


Figure (3.7)

**Subproblem $Z_2$:**

$max \quad Z = x_1 + x_2$

$S.t \quad 2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1 \geq 4$

$x_1, x_2 \geq 0$ , $x_1$ and $x_2$ are integers

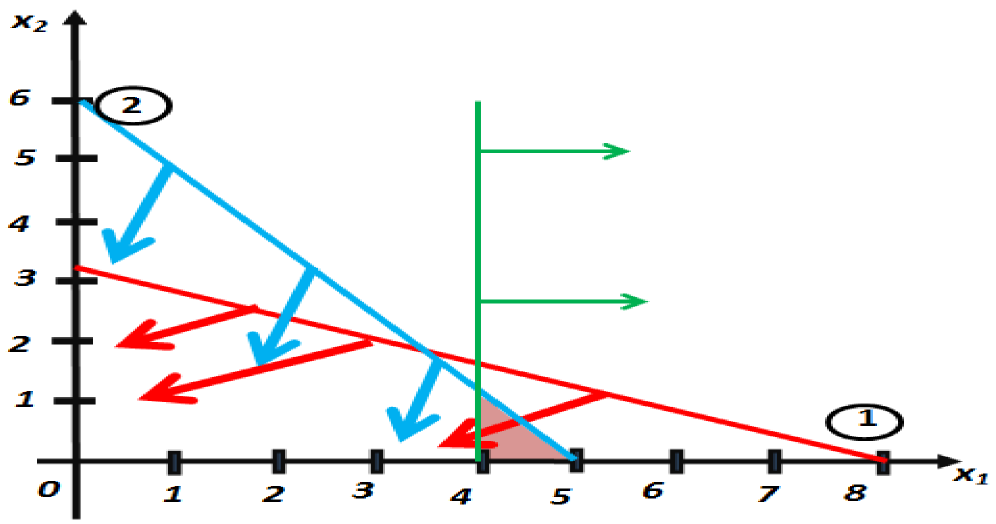


**Figure (3.8)**

The solution is $x_1 = 4, x_2 = \frac{6}{5} = 1.2$, and $Z_{max} = \frac{26}{5} = 5.2$.

Since the solution of subproblem $Z_1$ are integers, there is no need to branch subproblem $Z_1$(subproblem $Z_1$ is fathomed). The lower bound is now $Z_{max} = 5$. We branch from subproblem $Z_2$. Since $1 \leq x_2 \leq 2$, then choose $x_2$ as a branching variable. Construct two new problems by adding the constructs: $x_2 \leq 1, x_2 \geq 2$.

**Subproblem $Z_3$ :**

$max \quad Z = x_1 + x_2$

$S.t \quad 2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1 \geq 4$

$x_2 \leq 1$

$x_1, x_2 \geq 0$ , $x_1$ and $x_2$ are integers

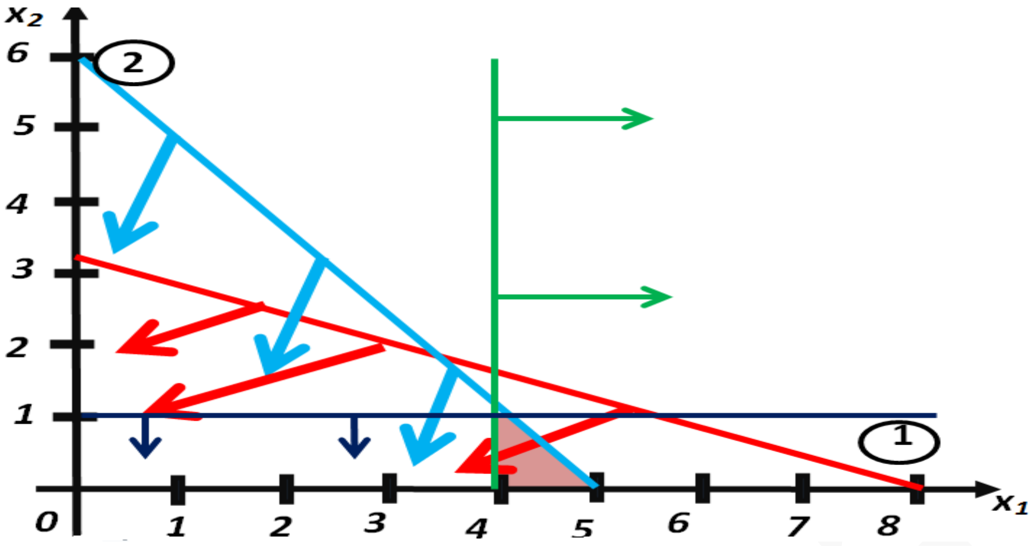


**Figure (3.9)**

The solution is: $x_1 = 25/6, x_2 = 1,$ and $Z_{max} = \frac{31}{6} = 5.17.$

**Subproblem $Z_4$:**

$$\begin{aligned}
&max \quad Z = x_1 + x_2 \\
&S.t \quad 2x_1 + 5x_2 \leq 16 \\
&\qquad\;\; 6x_1 + 5x_2 \leq 30 \\
&\qquad\;\; x_1 \geq 4 \\
&\qquad\;\; x_2 \geq 2 \\
&\qquad\;\; x_1, x_2 \geq 0 \text{ , } x_1 \text{ and } x_2 \text{ are integers}
\end{aligned}$$

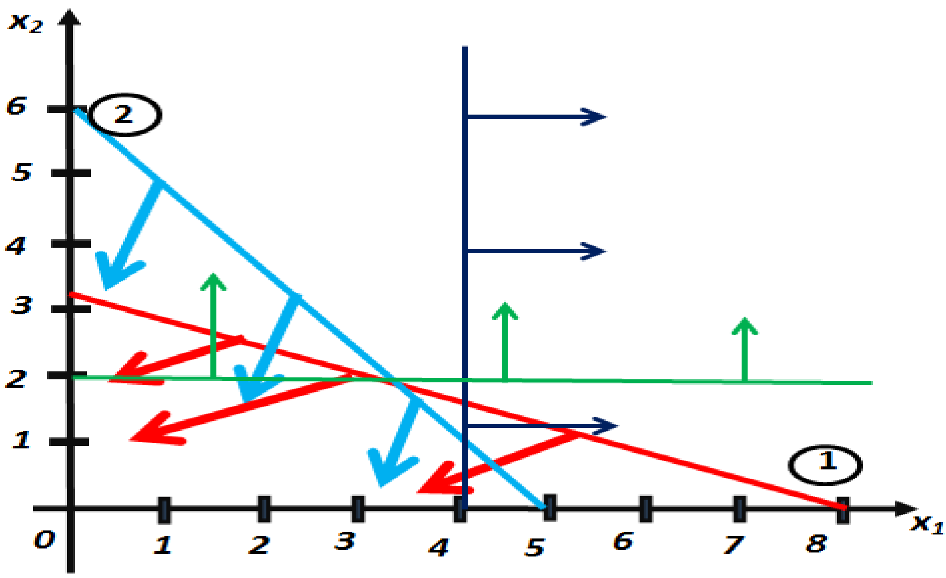


**Figure (3.10)**

There is no feasible solution.

Subproblem $Z_4$ is fathomed. Since the solution of subproblem $Z_3$ is $Z_{max} = \frac{31}{6} = 5.17$, which is not inferior to the lower bound. Therefore it can be branched from subproblem $Z_3$ into further subproblems. Since $x_1$ is the only fractional valued variable. Since $4 \leq x_1 \leq 5$ construct two new problems by adding the constructs: $x_1 \leq 4, x_1 \geq 5$.

**Subproblem $Z_5$:**

$$
\begin{aligned}
max \quad & Z = x_1 + x_2 \\
S.t \quad & 2x_1 + 5x_2 \leq 16 \\
& 6x_1 + 5x_2 \leq 30 \\
& x_1 = 4 \\
& x_2 \leq 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

$x_1$ and $x_2$ are integers

The solution is: $x_1 = 4$ , $x_2 = 1$, and $Z_{max} = 5$.

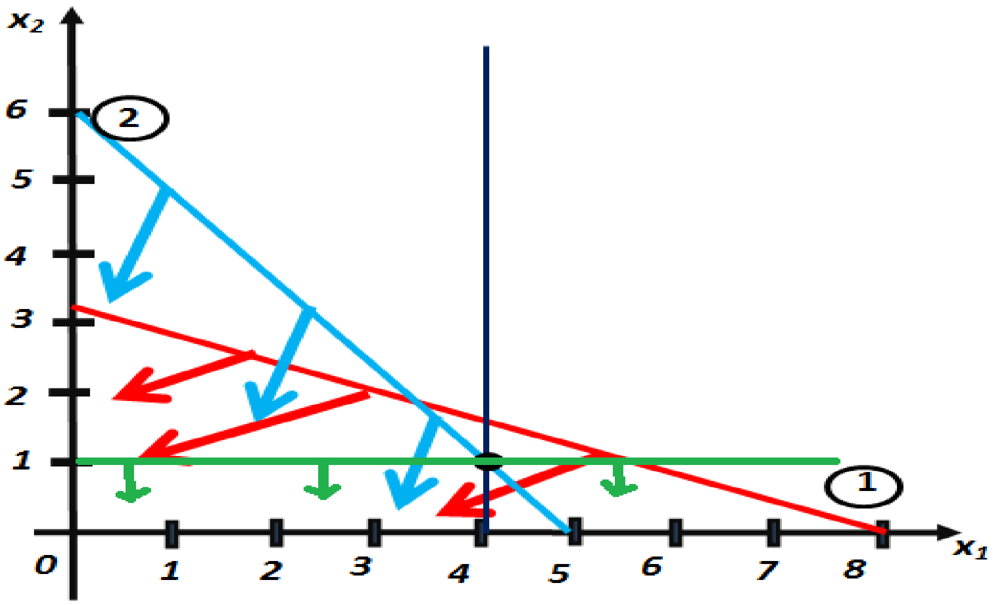


**Figure (3.11)**

**Subproblem $Z_6$:**

$$
\begin{aligned}
max \quad & Z = x_1 + x_2 \\
S.t \quad & 2x_1 + 5x_2 \leq 16 \\
& 6x_1 + 5x_2 \leq 30 \\
& x_1 \geq 5
\end{aligned}
$$

$$x_2 \le 1$$
$$x_1, x_2 \ge 0$$

$x_1$ and $x_2$ are integers

The solution is: $x_1 = 5$, $x_2 = 0$, and $Z_{max} = 5$.

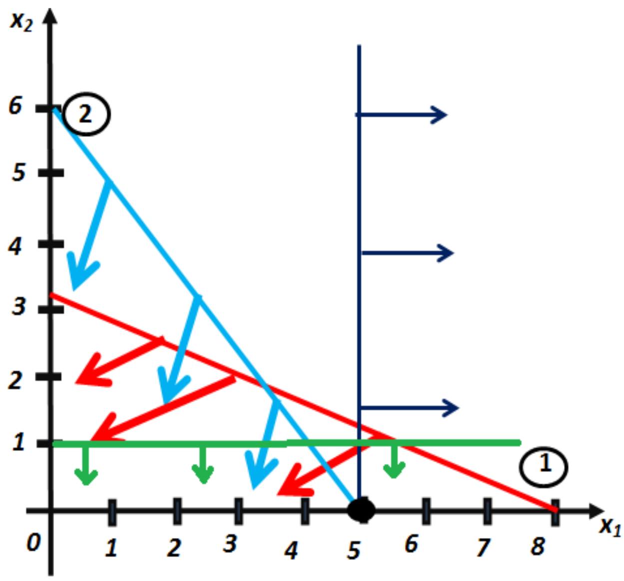


**Figure (3.12)**

There is more than one solution to this problem, they are:

$x_1 = 3, x_2 = 2,$ and $Z_{max} = 5$

$x_1 = 4, x_2 = 1,$ and $Z_{max} = 5$

$x_1 = 5, x_2 = 0,$ and $Z_{max} = 5$

Figure (3.13) summarize the generated subproblems in the form of a tree.

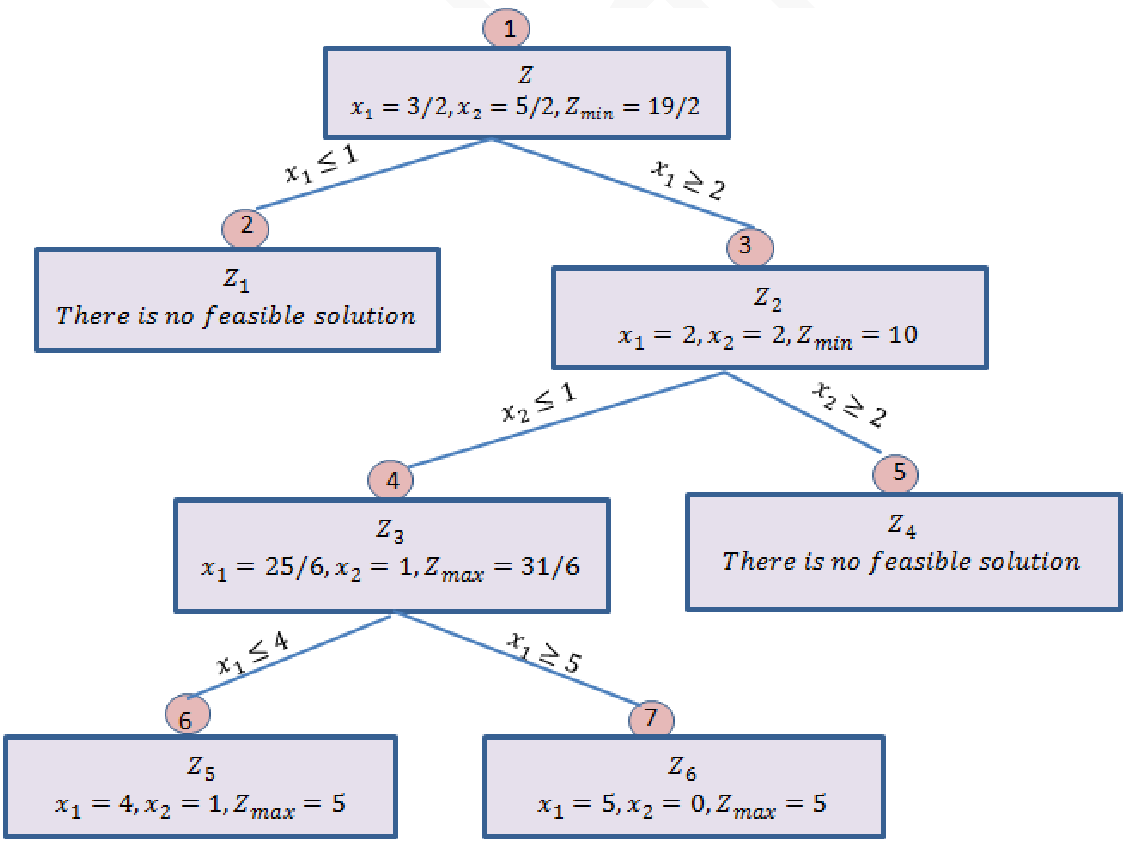


**Figure (3.13)**

## Example (3.13):

Find the optimal solution of the following ILPP:

$min \quad Z = 3x_1 + 2x_2$

$S.t \quad x_1 + x_2 = 4$

$x_1 + 3x_2 \geq 6$

$5x_1 + 3x_2 \geq 15$

$x_1, x_2 \geq 0, x_1$ and $x_2$ are integers

**Solution:**

The standard form of the ILPP is:

$min \quad Z = 3x_1 + 2x_2 + MR_1 + MR_2 + MR_3$

$S.t \quad x_1 + x_2 + R_1 = 4$

$x_1 + 3x_2 - s_1 + R_2 = 6$

$5x_1 + 3x_2 - s_2 + R_3 = 15$

$x_1, x_2, s_1, s_2, R_1, R_2, R_3 \geq 0, x_1$ and $x_2$ are integers

From the constraints:

$R_1 = 4 - x_1 - x_2$

$R_2 = 6 - x_1 - 3x_2 + s_1$

$R_3 = 15 - \ 5x_1 - 3x_2 + s_2$

Substitute $R_1, R_2,$ and $R_3$ in the Z=equation and rearrange Z-equation, the standard form will be:

$min \quad Z + (-3{+}7M)x_1 + (-2 + 7M)x_2 - Ms_1 + Ms_2 = 25M$

$S.t \quad x_1 + x_2 + R_1 = 4$

$x_1 + 3x_2 - s_1 + R_2 = 6$

$5x_1 + 3x_2 - s_2 + R_3 = 15$

$x_1, x_2, s_1, s_2, R_1, R_2, R_3 \geq 0, x_1$ and $x_2$ are integers

Let $x_1 = x_2 = s_1 = s_2 = 0, then\ R_1 = 4, R_3 = 6, R_3 = 15, Z = 25M.$

| B.V. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $R_1$ | $R_2$ | $R_3$ | b |
|---|---|---|---|---|---|---|---|---|
| $R_1$ | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 4 |
| $R_2$ | 1 | 3 | −1 | 0 | 0 | 1 | 0 | 6 |
| $R_3$ | $\boxed{5}$ | 3 | 0 | −1 | 0 | 0 | 1 | 15 |
| Z | −3+7M | −2+7M | −M | −M | 0 | 0 | 0 | 25M |

| B.V. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $R_1$ | $R_2$ | b |
|---|---|---|---|---|---|---|---|
| $R_1$ | 0 | 2/5 | 0 | 1/5 | 1 | 0 | 1 |
| $R_2$ | 0 | $\boxed{12/5}$ | −1 | 1/5 | 0 | 1 | 3 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 3/5 | 0 | $-1/5$ | 0 | 0 | 3 |
| **Z** | 0 | −1/5+14/5M | −M | $-\frac{3}{5}+2/5M$ | 0 | 0 | 4M+9 |

| B.V. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $R_1$ | b |
|---|---|---|---|---|---|---|
| $R_1$ | 0 | 0 | $\boxed{1/6}$ | 1/6 | 1 | 1/2 |
| $x_2$ | 0 | 1 | $-5/12$ | 1/12 | 0 | 5/4 |
| $x_1$ | 1 | 0 | 1/4 | $-1/4$ | 0 | 9/4 |
| **Z** | 0 | 0 | $-\frac{1}{12}+1/6M$ | $-\frac{7}{12}+1/6M$ | 0 | 1/2M+37/4 |

| B.V. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | b |
|---|---|---|---|---|---|
| $s_1$ | 0 | 0 | 1 | 1 | 3 |
| $x_2$ | 0 | 1 | 0 | 1/2 | 5/2 |
| $x_1$ | 1 | 0 | 0 | $-1/2$ | 3/2 |
| **Z** | 0 | 0 | 0 | $-1/2$ | 19/2 |

$\therefore\ x_1 = 3/2, x_2 = 5/2, and\ Z_{min} = 19/2$. The solution is not optimal, we need two constraints $x_1 \leq 1\ and\ x_1 \geq 2$.

**Subproblem $Z_1$:**

$min\quad Z = 3x_1 + 2x_2$

$S.t \quad x_1 + x_2 = 4$

$x_1 + 3x_2 \geq 6$

$5x_1 + 3x_2 \geq 15$

$x_1 \leq 1$

$x_1, x_2 \geq 0, x_1$ and $x_2$ are integers

Either we solve the above problem in the usual way from the beginning or by using the last table. The additional constraint is written as: $x_1 + s_3 = 1$

$x_1$ is a basic variable, so we substitute $x_1$ from the table:

$x_1 + 0.x_2 + 0.s_1 - \frac{1}{2}s_2 = \frac{3}{2}$, then: $x_1 = \frac{3}{2} + \frac{1}{2}s_2$

$\Rightarrow\ \frac{3}{2} + \frac{1}{2}s_2 + s_3 = 1 \quad \Rightarrow\ \frac{1}{2}s_2 + s_3 = \frac{-1}{2}$

| B.V. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | b |
|---|---|---|---|---|---|---|
| $s_1$ | 0 | 0 | 1 | 1 | 0 | 3 |
| $x_2$ | 0 | 1 | 0 | 1/2 | 0 | 5/2 |
| $x_1$ | 1 | 0 | 0 | $-1/2$ | 0 | 3/2 |

| | | | | | | |
|---|---|---|---|---|---|---|
| $s_3$ | 0 | 0 | 0 | $\boxed{1/2}$ | 1 | $-1/2$ |
| **Z** | 0 | 0 | 0 | $-1/2$ | 0 | $19/2$ |

There is no feasible solution

**Subproblem $Z_2$:**

$min \quad Z = 3x_1 + 2x_2$

$S.t \quad x_1 + x_2 = 4$

$x_1 + 3x_2 \geq 6$

$5x_1 + 3x_2 \geq 15$

$x_1 \geq 2$

$x_1, x_2 \geq 0, x_1$ and $x_2$ are integers

Either we solve the above problem in the usual way from the beginning or by using the last table. The additional constraint is written as: $-x_1 \leq -2$ $\Rightarrow -x_1 + s_3 = -2$

$x_1$ is a basic variable, so we substitute $x_1$ from the table:

$x_1 - \frac{1}{2}s_2 = \frac{3}{2}$, then: $\quad x_1 = \frac{3}{2} + \frac{1}{2}s_2$

$\Longrightarrow \quad -\frac{3}{2} - \frac{1}{2}s_2 + s_3 = -2 \quad \Longrightarrow -\frac{1}{2}s_2 + s_3 = \frac{-1}{2}$

| B.V. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | b |
|---|---|---|---|---|---|---|
| $s_1$ | 0 | 0 | 1 | 1 | 0 | 3 |
| $x_2$ | 0 | 1 | 0 | $1/2$ | 0 | $5/2$ |
| $x_1$ | 1 | 0 | 0 | $-1/2$ | 0 | $3/2$ |
| $s_3$ | 0 | 0 | 0 | $\boxed{-1/2}$ | 1 | $-1/2$ |
| **Z** | 0 | 0 | 0 | $-1/2$ | 0 | $19/2$ |
| $s_1$ | 0 | 0 | 1 | 0 | 2 | 3 |
| $x_2$ | 0 | 1 | 0 | 0 | 1 | 2 |
| $x_1$ | 1 | 0 | 0 | 0 | $-1$ | 2 |
| $s_2$ | 0 | 0 | 0 | 1 | $-2$ | 1 |
| **Z** | 0 | 0 | 0 | 0 | $-1$ | 10 |

$\therefore$ the optimal solution is $x_1 = 2, x_2 = 2, and \ Z_{min} = 10.$

Figure (3.14) summarize the generated subproblems in the form of a tree.

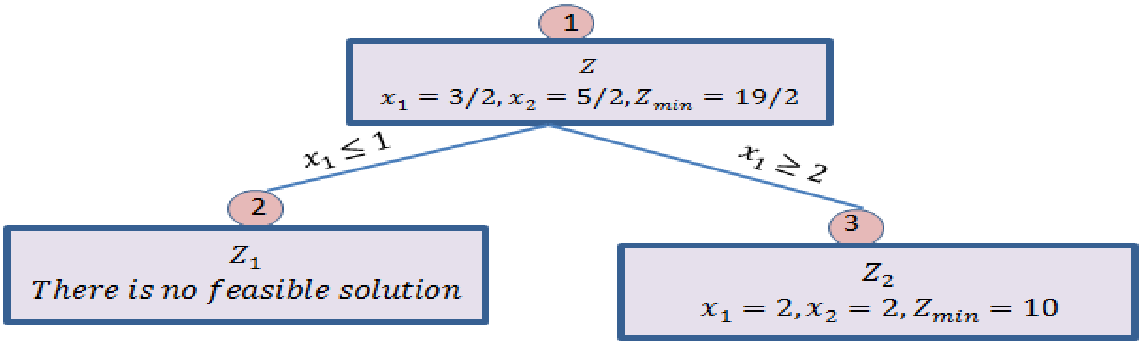


**Figure (3.14)**

## Exercises 3.4 (In addition to the text book exercises)

Use B & B method to solve the following ILPP:

1. $$\begin{aligned} &max \quad Z = 9x_1 + 3x_2 + 9x_3 \\ &S.t \quad -3x_1 + 6x_2 + 3x_3 \leq 12 \\ &\quad 12\,x_1 - 9x_3 \leq 18 \\ &\quad 3\,x_1 - 9x_2 + 6x_3 \leq 9 \\ &\quad x_1, x_2, x_3 \geq 0, x_1 \text{ and } x_3 \text{ are integers} \end{aligned}$$

**2.** $$\begin{aligned} &min \quad Z = 5x_1 + 4x_2 \\ &S.t \quad x_1 + x_2 \leq 5 \\ &\quad 10x_1 + 6x_2 \leq 45 \\ &\quad x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers} \end{aligned}$$