الفصل الثالث Chapter Three

اختبار الفرضيات Testing of Hypothesis

مفاهيم أساسية Basic Concepts

الفرضية الإحصائية Statistical Hypothesis

هي عبارة عن إدعاء قد يكون صائباً (true) أو خاطئاً (false) حول معلمة أو أكثر لمجتمع (أو عدة مجتمعات). إن الفرضية التي يضعها الباحث تدعى بالفرضية الصفرية أو فرضية العدم Null Hypothesis ويرمز لها H_o. إن رفض فرضية العدم يقودنا إلى قبول فرضية أخرى بديلة عنها تسمى بالفرضية البديلة Alternative Hypothesis ويرمز لها H_1.

الفرضية البسيطة والمركبة Simple and Composite Hypothesis

يقال عن الفرضية الإحصائية أنها بسيطة إذا افترضنا أن معلمة (أو معالم) المجتمع تساوي قيمة محددة، ويقال عنها أنها مركبة إذا افترضنا أن معلمة (أو معالم) المجتمع هي أكبر (أو أصغر) من قيمة محددة.

For ex.) H_o: $\mu \le 2, \delta^2 > 0$ Composite

H_o: $\mu = 3, \delta^2 = 10$ Simple

H_o: $\mu = 3, 0 < \delta^2 < 10$ Composite

إحصاءة الاختبار Test Statistic

عبارة عن متغير عشوائي بدلالة مفردات العينة وله توزيع احتمالي معلوم ويصف العلاقة بين القيم النظرية للمجتمع والقيم المحسوبة عن العينة (مثلا $\bar{x}$ أو $\sum x_i^2$ ... الخ).

منطقة الرفض أو المنطقة الحرجة Critical Region

هي تلك المنطقة التي إذا وقعت فيها قيمة إحصاءة الاختبار وأدت إلى رفض فرضية العدم H_o ويرمز لها بالرمز C.

الخطأ من النوع الأول والثاني Type I and II Errors

هنالك نوعان من الأخطاء التي قد يقع فيها الباحث عن اختيار الفرضيات وهما:

الخطأ من النوع الأول: ويتمثل برفض فرضية العدم H_o عندما تكون هي الصحيحة.

والخطأ من النوع الثاني: ويتمثل بقبول فرضية العدم H_o عندما تكون خاطئة.

حجم الاختبار أو مستوى المعنوية

The Size of the Test or Level of Significant

وهو احتمال الوقوع في الخطأ من النوع الأول ويرمز له α، أي انه:

$$\alpha = \Pr[\text{type I error}] = \Pr[\text{reject } H_o \mid H_o \text{ is true}]$$

$$= \Pr[x_1, x_2, ..., x_n \in C \mid H_o]$$

حجم الخطأ من النوع الثاني The Size of Type II Error

ويعرف كالآتي:

$$\beta = \Pr[\text{type II error}] = \Pr[\text{accept } H_o \mid H_1]$$

$$= \Pr[x_1, x_2, ..., x_n \in C^c \mid H_1]$$

حيث أن C^c تمثل المنطقة المكملة إلى C (C complement) وهي تمثل منطقة قبول فرضية العدم H_o.

قوة الاختبار The Power of the Test

هو احتمال رفض فرضية العدم H_o علما أنها خاطئة (أي أن البديلة H_1 هي الصحيحة) ويرمز له K(θ) أي أن:

$$K(\theta) = \Pr[\text{reject } H_o \mid H_1]$$

$$= \Pr[x_1, x_2, ..., x_n \in C \mid H_1] = 1 - \Pr[x_1, x_2, ..., x_n \in C^c \mid H_1]$$

$$K(\theta) = 1 - \beta$$

Ex.

Let the r.v x has the p.d.f

$$f(x,\theta) = \begin{cases} \theta^x (1-\theta)^{1-x} & , x = 0, 1 \\ 0 & , o.w \end{cases}$$

To test the simple hypothesis H_o: $\theta = \frac{1}{4}$

Against the alternative hypothesis H_1: $\theta < \frac{1}{4}$

Suppose that critical region is

$C = \{x_1, x_2, ..., x_{10}, \sum_{i=1}^{10} x_i \le 1\}$ find:

a) The power of the test $K(\theta)$.

b) The power of the test at $\theta = \frac{1}{16}$.

c) Pr [type II error] at $\theta = \frac{1}{16}$.

d) The significant level α.

Solution:

Since x ~ Bernoulli $(1, \theta)$ then

$y = \sum_{i=1}^{10} x_i$ ~ binomial $(10, \theta)$

$$f(y,\theta) = \begin{cases} C_y^{10}\ \theta^y (1-\theta)^{10-y} & y = 0, ..., 10 \\ 0 & o.w \end{cases}$$

a) $K(\theta) = \Pr[y \le 1 \mid H_1]$

أي انه يجب علينا إيجاد $p(y \le 1)$ تحت ظروف H_1

$\Pr(y \le 1) = \Pr(y = 0) + \Pr(y = 1)$

$= C_0^{10}\ \theta^0 (1-\theta)^{10} + C_1^{10}\ \theta(1-\theta)^9 = (1-\theta)^{10} + 10\theta(1-\theta)^9$

$$= (1-\theta)^9[1-\theta+10\theta] = (1-\theta)^9(1+9\theta),\ 0<\theta<\frac{1}{4}$$

b) $K(\frac{1}{16}) = (1-\frac{1}{16})^9(1+\frac{9}{16}) = (\frac{15}{16})^9(\frac{25}{16})$

c) $\beta(\frac{1}{16}) = 1-K(\frac{1}{16}) = 1-(\frac{15}{16})^9(\frac{25}{16})$

d) α = Pr[type I error] = Pr[reject H_o | H_o]

= Pr[$y \le 1$ | H_o]

أي علينا إيجاد $P(y \ge 1)$ تحت ظروف H_o (أي عندما تكون H_o صحيحة)

$P[y \le 1 \mid H_o] = P(y=0) + P(y=1)$ عندما $\theta = \frac{1}{4}$

$$= (1-\theta)^9(1+9\theta)\Big]_{\theta=\frac{1}{4}} = (\frac{3}{4})^9(\frac{13}{4})$$

Ex.

Suppose that $x_1, x_2, ..., x_n$ is a random sample from $N(\theta, 1)$, to test the simple hypothesis H_o: $\theta = 0$ against the simple alternative hypothesis H_1: $\theta = 1$, the critical region is:

$C = \{x_1, x_2, ..., x_n: \bar{x} \ge k\}$ find n and k such that $\alpha = \beta = 0.01$.

Solution

α = Pr[reject H_o | H_o is true]

$\alpha = \Pr[\bar{x} \ge k \mid \theta = 0]$

$$\bar{x} \sim N(\theta, \frac{1}{n}) \Rightarrow \bar{x} \mid H_o \sim N(0, \frac{1}{n})$$

$$\alpha = \Pr[\bar{x} \ge k \mid \theta = 0] = \Pr[\frac{\bar{x}-m}{\delta/\sqrt{n}} \ge \frac{k-m}{\delta/\sqrt{n}}] = \Pr[Z \ge \frac{k-0}{1/\sqrt{n}}]$$

$$= \Pr[Z \ge \sqrt{n}\ k]$$

$$0.01 = \Pr[Z \ge \sqrt{n}\ k] = 1 - \Pr[Z < \sqrt{n}\ k]$$

$\Pr[Z < \sqrt{n}\, k] = 1 - 0.01 = 0.99$

$\boxed{\sqrt{n}\, k] = 2.33}$ .... (1) from tables of standard normal dist.

Also

$\beta = \Pr[\text{accept } H_o \mid H_1] = \Pr[\bar{x} < k \mid \theta = 1]$

$$\bar{x} / H_1 \sim N(1, \frac{1}{n})$$

$$\beta = \Pr[\frac{\bar{x} - m}{\delta / \sqrt{n}} < \frac{k - m}{\delta / \sqrt{n}}] = \Pr[Z < \frac{k - 1}{1 / \sqrt{n}}] = \Pr[Z < \sqrt{n}(k - 1)]$$

$$0.01 = \Pr[Z < \sqrt{n}(k - 1)]$$

$\boxed{\sqrt{n}(k - 1) = -2.33}$ .... (2) from tables

Solving equation (1) and (2)

By (2) $\Rightarrow \sqrt{n} = \dfrac{-2.33}{k - 1}$

By (1) $\Rightarrow \dfrac{-2.33}{k - 1} k = 2.33$

-2.33k = 2.33k - 2.33

4.66k = 2.33

$k = \dfrac{2.33}{4.66} = 0.5$

Now by (2) $\Rightarrow \sqrt{n}\ (0.5 - 1) = -2.33$

$\sqrt{n}\ (-0.5) = -2.33 \Rightarrow \sqrt{n} = 4.66$

n = 21.715 $\approx$ 22 (approximately)

Ex.

The consumption of electricity in a smalltown ship is assumed to be exponentially distributed with parameter θ. Determine the size of type I and type II errors if H_o: θ = 1000 kW is tested against H_1: θ = 2000 kW and if the test criterion is as follows:

Select any day at random. If the consumption of that day is 4000 kW or more, reject H_o, otherwise accept H_o.

Solution:

The critical region is

$C = \{x_1, x_2, ..., x_n: x_i \geq 4000\}$

$\alpha = \Pr[\text{reject } H_o \mid H_o \text{ is true}] = \Pr[x \geq 4000 \mid \theta = 1000]$

we have $f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}, \; x > 0, \theta > 1$

$$\therefore \alpha = \int_{4000}^{\infty} \frac{1}{1000} e^{-x/1000} dx = -e^{-x/1000}\Big|_{4000}^{\infty} = -e^{-\infty} + e^{-\frac{4000}{1000}} = e^{-4}$$

$\beta = \Pr[\text{accept } H_o \mid H_1 \text{ is true}] = \Pr[x < 4000 \mid \theta = 2000]$

$$\beta = \int_{0}^{4000} \frac{1}{2000} e^{-x/2000} dx = -e^{-x/2000}\Big|_{0}^{4000} = -e^{-\frac{4000}{2000}} + 1$$

$$= -e^{-2} + 1 = 1 - e^{-2}$$

Ex.

Let $x_1, x_2, \ldots, x_n$ be a r.s of size 25 from $N(\mu, 36)$. We shall reject H_o: $\mu = 75$ and accept H_1: $\mu = 80$ iff $\bar{x} > C$, where C is constant. Find the value of C at $\alpha = 0.01$ if it is known that $Z_{0.99} = 2.33$.

(Solve it) Ans.: C = 77.79

Exercises

1) Let $x_1, x_2, \ldots, x_n$ be a random sample from poisson (θ). The critical region for testing H_o: $\theta = \frac{1}{10}$ against H_1: $\theta > \frac{1}{10}$ is:

$$C = \{x_1, x_2, \ldots, x_n, \sum_{i=1}^{20} x_i \geq 1\}$$

a) find the power of the test $K(\theta)$.

b) find the level of significant α.

2) Consider a normal distribution $N(\theta, 4)$. The simple hypothesis H_o: $\theta = 0$ is rejected and the alternative composite hypothesis H_1: $\theta > 0$ is accepted if and only if the observed mean $\bar{x}$ of a random sample of size 25 os greater than or equal to $\frac{3}{5}$. Find the power of the test.

3) Let $x_1, x_2, ..., x_{25}$ denote a random sample of size 25 from a normal population with variance $\delta^2 = 4$. To test H_o: $x \sim N(0, 4)$ against H_1: $x \sim N(1, 4)$. The critical region is

$C = \{x_1, x_2, ..., x_{25}: \bar{x} > 0.04\}$ find the error sizes of this test.

Best Critical Region أفضل منطقة حرجة

يقال عن المنطقة الحرجة C أنها أفضل منطقة حرجة (BCR) بحجم α عند اختيار فرضية العدم البسيطة $H_o: \theta = \theta_o$ مقابل الفرضية البديلة البسيطة $H_1: \theta = \theta_1$ إذا وجدت منطقة حرجة أخرى مثل A بحيث أن:

1) $\Pr[x_1, x_2, ..., x_n \in A \mid H_o] = \Pr[x_1, x_2, ..., x_n \in C \mid H_o] = \alpha$

2) $\Pr[x_1, x_2, ..., x_n \in A \mid H_1] \leq \Pr[x_1, x_2, ..., x_n \in C \mid H_1]$

Ex.

Consider the r.v. X ~ binomial (5, θ) under $H_o: \theta = \frac{1}{2}$, $H_1: \theta = \frac{3}{4}$. The following table gives the density values of X under H_o and H_1.

X	0	1	2	3	4	5
$f(x, \frac{1}{2})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$
$f(x, \frac{3}{4})$	$\frac{1}{1024}$	$\frac{15}{1024}$	$\frac{90}{1024}$	$\frac{270}{1024}$	$\frac{405}{1024}$	$\frac{243}{1024}$

Assuming thet $\alpha = \frac{1}{32} = \Pr[\text{reject } H_o \mid H_o]$

We have two critical regions

$A = \{x: x = 0\}$, $C = \{x: x = 5\}$

$\Pr[x \in A \mid H_o] = \Pr[x \in C \mid H_o] = \frac{1}{32}$

Also

$\Pr[x \in A \mid H_1] = \frac{1}{1024}$

$\Pr[x \in C \mid H_1] = \dfrac{243}{1024}$

$\Rightarrow \Pr[x \in A \mid H_1] < \Pr[x \in C \mid H_1]$

$\Rightarrow$ C is the best critical region BCR.

Ex.

For the above example let us use the r.v. x to test $H_o : \theta = \dfrac{1}{2}$ against $H_1 : \theta = \dfrac{3}{4}$ but by assuming $\alpha = \dfrac{6}{32}$.

In this case we will have four critical regions of size $\alpha = \dfrac{6}{32}$, these regions are:

$C_1 = \{x: x = 0, 1\}$, $C_2 = \{x: x = 0, 4\}$

$C_3 = \{x: x = 1, 5\}$, $C_4 = \{x: x = 4, 5\}$

One of these regions is BCR.

$\Pr[x \in C_1 \mid H_1] = \Pr[x = 0, 1 \mid \theta = \dfrac{3}{4}] = \dfrac{1}{1024} + \dfrac{15}{1024} = \dfrac{16}{1024}$

$\Pr[x \in C_2 \mid H_1] = \dfrac{406}{1024}$

$\Pr[x \in C_3 \mid H_1] = \dfrac{258}{1024}$

$\Pr[x \in C_4 \mid H_1] = \dfrac{648}{1024}$

$\Rightarrow C_4$ is BCR of size $\alpha = \dfrac{6}{32}$.

تمارين الفصل الثالث

Ex.

Let $x_1, x_2, ..., x_n$ be a r.s of size 25 from $N(\mu, 36)$. We shall reject $H_o: \mu = 75$ and accept $H_1: \mu = 80$ iff $\bar{x} > c$, where c is constant. Find the value of c at $\alpha = 0.01$ if it is known that $Z_{0.99} = 2.33$.

Solution:

$\alpha = Pr[\text{reject } H_o \mid H_o] = Pr[\bar{x} > c \mid H_o]$

$$x \sim N(\mu, 36) \Rightarrow \bar{x} \sim N(\mu, \frac{36}{25}) \Rightarrow \bar{x} \mid H_o \sim N(75, \frac{36}{25})$$

$$\alpha = Pr[\frac{\bar{x}-\mu}{\delta/\sqrt{n}} > \frac{c-\mu}{\delta/\sqrt{n}}] = Pr[Z > \frac{c-75}{6/5}]$$

$$0.01 = Pr[Z > \frac{5}{6}(c-75)]$$

$$0.01 = Pr[Z > \frac{5c-375}{6}]$$

$$0.01 = 1 - Pr[Z \le \frac{5c-375}{6}]$$

$$Pr[Z \le \frac{5c-375}{6}] = 1 - 0.01 = 0.99$$

$$\frac{5c-375}{6} = 2.33 \Rightarrow 5c - 375 = (2.33)6$$

$5c = 6(2.33) + 375$

$5c = 388.98 \Rightarrow c = 77.79$

Exercises

1) Let $x_1, x_2, ..., x_n$ be a random sample from Poisson (θ). The critical region for testing $H_o: \theta = \frac{1}{10}$ against $H_1: \theta > \frac{1}{10}$ is

$$C = \{x_1, x_2, ..., x_n, \sum_{i=1}^{20} x_i \ge 1\}.$$

a) Find the power of the test $K(\theta)$.

b) Find the level of significant α.

Solution:

a) $K(\theta) = \Pr[\text{reject } H_o \mid H_1] = \Pr[\sum_{i=1}^{20} x_i \geq 1 \mid H_1]$

Since $x \sim$ Poisson (θ), $y = \sum_{i=1}^{20} x_i \sim$ Poisson (20θ)

$$[\mu_y(t)] = E(e^{ty}) = E(e^{t\sum_{i=1}^{20} x_i}) = E(e^{tx_1 + tx_2 + \ldots + tx_{20}})$$

$$= \mu_{x_1}(t).\mu_{x_2}(t)\ldots..\mu_{x_{20}}(t) = e^{\theta(e^t - 1)}\ldots e^{\theta(e^t - 1)} = e^{20\theta(e^t - 1)}]$$

$$f(y, \theta) = e^{-20\theta}(20\theta)^y$$

$$K(\theta) = \Pr[y \geq 1 \mid \theta > \frac{1}{10}] = 1 - \Pr(y < 1) = 1 - \Pr[y = 0]$$

$$= 1 - \frac{e^{-20\theta}(20\theta)^0}{0!} = 1 - e^{-20\theta}, \theta > \frac{1}{10}$$

b) $\alpha = \Pr[\text{reject } H_o \mid H_o] = \Pr[y \geq 1 \mid \theta = \frac{1}{10}]$

$$= 1 - \Pr[y < 1]\Big|_{\theta = \frac{1}{10}} = 1 - e^{-20\theta}\Big|_{\theta = \frac{1}{10}} = 1 - e^{-\frac{20}{10}} = 1 - e^{-2}$$

2) Consider a normal distribution $N(\theta, 4)$. The simple hypothesis H_o: $\theta = 0$ is rejected and the alternative composite hypothesis H_1: $\theta > 0$ is accepted if and only if the observed mean $\bar{x}$ of a random sample of size 25 is greater than or equal to $\frac{3}{5}$. Find the power of the test.

Solution:

$$C = \{x_1, x_2, \ldots, x_{25} \mid \bar{x} \geq \frac{3}{5}\}$$

$$K(\theta) = \Pr[\text{reject } H_o \mid H_1] = \Pr[\bar{x} \geq \frac{3}{5} \mid \theta > 0]$$

Since x ~ N(θ, 4) $\Rightarrow \bar{x} \sim N(\theta, \frac{4}{25})$

$$K(\theta) = \Pr[\frac{\bar{x}-\mu}{\delta/\sqrt{n}} \geq \frac{\frac{3}{5}-\mu}{\delta/\sqrt{n}}] = \Pr[Z \geq \frac{\frac{3}{5}-\theta}{2/5}]$$

$$= \Pr[Z \geq \frac{5}{2}[\frac{3}{5}-\theta]] = \Pr[Z \geq [\frac{3-5\theta}{2}]]$$

$$= 1 - \Pr[Z < \frac{3-5\theta}{2}]$$

3) Let $x_1, x_2, ..., x_{25}$ denote a random sample of size 25 from normal population with variance $\delta^2 = 4$. To test H_o: x ~ N(0, 4) against H_1: x ~ N(1, 4). The critical region is C = {$x_1, x_2, ..., x_{25}$: $\bar{x} > 0.04$}, find the error sizes of this test.

Solution:

$$\alpha = \Pr[\bar{x} > 0.04 \mid H_o],$$

$$x \sim N(\mu, \delta^2) \Rightarrow \bar{x} \sim N(\mu, \frac{\delta^2}{n})$$

$$\bar{x} \sim N(\mu, \frac{4}{25}) \Rightarrow \bar{x} \mid H_o \sim N(0, \frac{4}{25})$$

$$\alpha = \Pr[\frac{\bar{x}-\mu}{\delta/\sqrt{n}} > \frac{0.04-\mu}{\delta/\sqrt{n}}] = \Pr[Z > \frac{0.04-0}{2/5}]$$

$$1 - \Pr[Z \leq \frac{5}{2}(0.04)] = 1 - \Pr[Z \leq 0.1]$$

= 1 - 0.540 = 0.46

$\therefore\ \alpha = 0.46$

Now to find β

$$\beta = \Pr[\text{accept } H_o \mid H_1] = \Pr[\bar{x} \leq 0.04 \mid H_1]$$

$$= \Pr[\frac{\bar{x}-\mu}{\delta/\sqrt{n}} \leq \frac{0.04-\mu}{\delta/\sqrt{n}}] = \Pr[Z \leq \frac{0.04-0}{2/5}]$$

$= \Pr[Z \le \frac{5}{2}(0.04 - 1)]$

$= \Pr[Z \le 2.4] = 1 - N(2.4)$

$= 1 - 0.992 = 0.008$

Let x be discrete random variable whose density values under H_o and H_1 are given by the following table

x	1	2	3	4	5	6	7
$f(x \mid H_o)$	0.01	0.02	0.03	0.05	0.05	0.07	0.77
$f(x \mid H_1)$	0.03	0.09	0.10	0.10	0.20	0.18	0.30

List all critical regions of size $\alpha = 0.10$ then find the BCR.

$C_1 = \{x: x = 1, 2, 6\}$, $C_2 = \{x: x = 2, 3, 4\}$, $C_3 = \{x: x = 2, 3, 5\}$,

$C_4 = \{x: x = 4, 5\}$, $C_5 = \{x: x = 3, 6\}$

Now to find the BCR

$\Pr[x \in C_1 \mid H_1] = 0.03 + 0.09 + 0.18 = 0.30$

$\Pr[x \in C_2 \mid H_1] = 0.09 + 0.10 + 0.10 = 0.29$

$\Pr[x \in C_3 \mid H_1] = 0.09 + 0.10 + 0.20 = 0.39$

$\Pr[x \in C_4 \mid H_1] = 0.10 + 0.20 = 0.30$

$\Pr[x \in C_5 \mid H_1] = 0.10 + 0.18 = 0.28$