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# محاضرات إحصاء رياضي//المرحله الرابعه

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## Discrete probability destructions

### 1-Bernoulli distribution

If the random experiment being repeated has only two outcomes such as ( success, failure) for example (Male, female , (yes , no) , (head. Tail ) and so on , the we have a particularly important case of repeated trials, known as Bernoulli trials

**Def:** The discrete , r .v x is said to have a Bernoulli distribution with parameter p denoted as X ~ Ber (1,p) if its probabity mass funetion (p.m.f) is given as:

$$f(x) = \begin{cases} p^x(1-p)^{1-x} & ,x=0,1 \\ 0 & 0.w \end{cases}$$

**Properties :** (1)The mean $M_x$= E(x) = p

**<u>Proof :</u>** E(x) = $\sum_{x=0}^{1} x\, f(x) = 0\, f(0) + 1.\, f(1)$

= 0+p(1-p)$^0$=p

(2) The variance $d^2x$= p(1-p)

**<u>Proof :</u>** E(x$^2$) = $\sum_{x=0}^{1} x^2 f(x) = (0)^2\, f(0) + p(1)^2 = p(1-p)^0 = p$

$\therefore d_x^2$ = E(x$^2$) - (E(x))$^2$ = p-p$^2$=p( 1-p)

(3) The m.g.f of x is $M_x$(t)=(1-p+pe$^t$)

**<u>Proof :</u>** $M_x$(t) =E($e^{tx}$)= $\sum_{x=0}^{1} e^{tx} p(x) = e^0\, f(0) + e^t f(1)$

= 1.(1-p)+e$^t$p=(1-p+pe$^t$).

### 1- Binomial distribution

A random variable x that has a p.m.f.

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & ,x=0,1,\dots,n \\ 0 & 0.w \end{cases}$$

هذه الدالة تستخدم الحاجات او الحوادث المتناقصة

Is said to have a binomial distribution denoted X~b (n, p), where n is positive integer and 0< p<1 are the parameters of the distribution

**Ex:** verify that f(x) given above is a p.m.f (1)f(x) > 0 (2)$\sum f(x) = 1$

**SoL:** Two conditions must be satisfied

1- f(x) >0 and (2) )$\sum f(x) = 1$

It is elear that the first condition is satisfied Since 0<p<1 and n is positive integer for the Second condition we have

$$\sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = [p + (1-p)]^n = 1 \quad ([\mathbf{a} + \mathbf{b})^{\mathbf{n}}$$
$$= \sum \binom{\mathbf{n}}{\mathbf{x}} \mathbf{a}^{\mathbf{x}} \mathbf{b}^{\mathbf{n-x}})$$

**Properties**

(1) $M_x$ = E(x)= np

**Proof :** $M_x =$ E(x) = $\sum_{x=0}^{n} x\, f(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

$\sum_{x=0}^{n} x \frac{n!}{x!(n-x)!}\ p^x (1-p)^{n-x}$

= $\sum x \frac{n(n-1)!}{x(x-1)!(n-x)!} p\, p^{x-1} (1-p)^{n-x}$

= np$\sum_{x=1}^{n-1} \frac{(n-1)!}{(x-1)!(n-x)!}\ p^{x-1}(1-p)^{n-x}$

Putting $M$=n-1 , y=x-1 ⇒ then $M$-y = n-x

$M_x$= E(x) np $\sum_{y=0}^{M} \frac{M!}{y!(M-y)!}\ p^y (1-p)^{M-y}$

= np $\sum_{y=0}^{M} \binom{M}{y}\ p^y\ (1-p)^{M-y}$= np(1)= np

(2) var(x) = $\delta_x^2$= np(1-p)

**Proof :** var (x) = E($x^2$)-[E(x)]$^2$

Writing E($x^2$ as E (x(x-1))+E(x)

E(x(x-1)) – $\sum x(x-1) f(x) = \sum x(x-1) \binom{n}{x}\ p^x\ (1-p)^{n-x}$

$$= \sum x(x-1) \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum x(x-1)\frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!}p^2 p^{x-2}(1-p)^{n-x}$$

$= n\,(n-1)p^2\sum_{x=2}^{n-2}\frac{(n-2)!}{(x-2)!(n-x)!}\;p^{x-2}(1-p)^{n-x}$

Putting y=x-2, $m$=n-2 then $m$-y=n-x, if x= 2 then y=0 and

$Ex(x-1) = n(n-1)p^2\sum_{y=0}^{m}\frac{m!}{y!(m-y)!}\,p^y\,(1-p)^{m-y}$

$= n(n-1)p^2\sum_{y=0}^{m}\binom{m}{y}p^y\;\;(1-p)^{m-y}$

$= n\,(n-1)p^2(1)$ (binomial formula)

$= n\,(n-1)p^2$

$E(x^2)=n(n-1)p^2+E(x)= n\,(n-1)p^2+np=n^2p^2 - np2+np$

$Var\,(x)= np^2 - np^2 +np-n^2p^2-np-np^2$

$\therefore$ var $(\alpha) = \delta = 2$ np(1-p)

3-the moment generating function is $\mu_x(t) = [1-p+pe^t]^n$

<u>**Proof :**</u> $M_x\,(t)= E(e^{tx}) = \sum e^{tx}f(x) = \sum_{x=0}^{n} e^{tx}\binom{n}{x}\,p^x(1-p)^{n-x}$

$= \sum\binom{n}{x}\;(pe^t)^x\,(1-p)^{n-x} = [1-p+pe^t]^n[[a+b]^n = \sum\binom{n}{x}\;{}^{x}_{a}\;b^{n-\alpha})]$

**Ex:** Find E(x) and var (x) by using the m.g.f

<u>Hint :</u> $E(x) = M_x'\,(0)$

$\delta_x^2 = M_x''(0) = [M_x'(0)]^2$

<u>**Proof :**</u> we have $M_x(t) = [1-p+pet]^n$

$M_x'(t) = n[1-p+pe^t]^{n-1}\,(pe^t]$

$M_x'(0) = np$

$M_x''(t) = n(n-1)\,[1-p+pe^t]^{n-2}(pe^t)^2+n\,[1-p=pe^t]^{n-1}pe^t$

$M_x''(0) = n(n-1)\,p^2+np=n^2p^2-np^2+np$

$\delta_x^2 = \mu_x''(0) = [\mu_x'(0)]^2 = n^2 p^2 - np^2 + np - n^2p^2$

$= np - np^2 = np(1-p)$

**Ex:** if $x \sim b(n,p)$, show that : $E\left(\frac{x}{n}\right) = p$ and

$E\left(\left(\frac{x}{n} - p\right)^2 = \frac{p(1-p)}{n}\right.$

**Sol:** $E\left(\frac{x}{n}\right) = \frac{1}{n} E(x) = \frac{1}{n}(np) = p$ [since $x \sim b(n, p)$, $E(x) = np$]

Let $\frac{x}{n} - p = y$ then $E\left(\left(\frac{x}{n} - p\right) 2 = E(y^2)\right.$

But $E(y^2) = var(y) + [E(y)]^2$

$E\left(\frac{x}{n} - p\right)^2 = var\left(\frac{x}{n} - p\right) + \left[E\left(\frac{x}{n} - p\right)\right]^2$

$= var\left(\frac{x}{n}\right) + \left[\frac{1}{n} E(x) - p\right]^2$

$= \frac{1}{n^2} var(x) + \left[\frac{1}{n} np - p\right]^2 = \frac{1}{n^2} np(1-p) + 0 = \frac{p(1-p)}{n}$

**Ex:** let the independent r. vs. $x_1$ , $x_2$ , $x_3$ have the same p.d.f. $f(x) = 3x^2$, $0<x<1$, Find the probability that exeatly two of these three variable exceed $\frac{1}{2}$

**Solution:** At the fist we have to find the probability that any one of these three variable exeeced $\frac{1}{2}$ as follows :

$P = \int_{1/2}^{1} 3x^2 dx = x^3 \int_{1/2}^{1} = 1 - \frac{1}{8} = \frac{7}{8}$

The probability of exactly two of these three variables exceed $\frac{1}{2}$ is

$f(2) = pr(x=2) = \binom{3}{2} \left(\frac{7}{8}\right)^2 \left(\frac{1}{8}\right) = \frac{147}{512}$

**Ex:** let $x_1$, $x_2$, ….. $x_k$ loeindependent r.vs such that x: $\sim$ , i=1,2,…k

Show that $\sum_{i=1}^{k} x_i \sim b\left(\sum_{i=1}^{k} n_i, p\right)$

**Proof :** let $y = \sum_{i=1}^{k} x_i$ by using the m.g.f

$M_y$ (t) = E ($e^{ty}$) = E($e^{t\Sigma xi}$) $= E(e^{t(x_1+x_2+..+x_k)})$

= E($e^{tx_1}$ $e^{tx_2}$... $e^{tx_k}$)

Since the variable are independent , then

$M_y$ (t) = E ($e^{tx_1}$) E($e^{tx_2}$) ... $E(e^{tx_k})$

= $M_{x1}$(t) $M_{x2}$(t) … $M_{xk}$(t)

= $(1\text{-}p\text{+}pe^t)^{n1}$ $[1\text{-}p\text{+}pe^t]^{n2}$ …. $[1\text{-}p\text{+}pe^t]^{nk}$ [since $x_1$~ b(ni,p)

= $(1\text{-}p\text{+}pe^t)\Sigma^{n1}$

y= $\sum_{i=1}^{k} \sim b\ (\sum_{i=1}^{k} n_i$, p)

**Ex:** let x~ b(n,p) show that

f(x+1) = $[\frac{n-x}{x+1}, \frac{p}{1-p}]$ f($\alpha$)

**solution :** f(x) = $(^n_x p^x$(1-p)$^{x-1}$ [since x~ b(n,p)]

f(x+1) = $(_{x+1}^{\ n}\ \ p^{x+1}$ (1-p)$^{n-x-1}$

$$\frac{f(x+1)}{f(x)} = \frac{(_{x+1}^{\ n} p^{x+1}(1-p)^{n-x-1}}{(^n_x p^x(1-p)^{n-x}}$$

$$= \frac{\frac{n!}{(x+1)!(n-x-1)!} p\, p^x(1-p)^{n-x-1}}{\frac{n!}{x(n-x)!}\ p^x(1-p)^{n-x}}$$

$$= \frac{n!}{(x+1)!(n-x-1)!}\ p\ (1-p)^{n-x-1} \text{ x } \frac{x!(n-x)!}{n!_{p}x(1-p)^{n-x}}$$

$$\frac{x!(n-x)(n-x-1)!}{(x+1)x!(n-x-1)!}\ \frac{p}{1-p} = \frac{n-x}{x+1}\ \frac{p}{1-p}$$

f(x+1) = $[\frac{n-x}{x+1} . \frac{p}{1-p}]\ f(x)$

**3-Poisson distribution :-**

Let x be a discrete r.v which can take on the values 0. 1, 2, …. Such that the p.m.f. of x is given by

$$f(x) = \begin{cases} \frac{e^{-\lambda}\ \lambda^{x}}{x!} & x = 0,1,2,\ldots. \\ 0 & 0.w \end{cases}$$

the distribution is called poisson distribution

denoted as $x \sim p(\lambda)$ where the positive constant $\lambda$ represent the parameter of the distribution

**properties**

1- the m.g.f of the distribution is $M_x(t) = e^{\lambda(e^t - 1)}$

**proof :** $M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} f(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!}$

$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)}{x!} = e^{-\lambda}\ e^{\lambda e^t} = e^{\lambda e^t - \lambda} = e^{\lambda(e^t - 1)}$

$\therefore M_x(t) = e^{\lambda(e^t - 1)}$

2- $M_x = \delta_x^2 = \lambda$

**Proof :** By using the m.g.f ($M_x(t) = e^{\lambda(e^t - 1)}$

$M_x(0) = e^{\lambda(e^0 - 1)} = 1$

$M_x'(t) = x\ e^t\ \ e^{\lambda(e^t - 1)} = \lambda\, e^t\ M_x(t)$

$M_x = E(x) = M_x'(0)\ \lambda\, e^0\ M_x(0) = \lambda(1)(1) = \lambda$

$M_x''(t) = \lambda e^t M_x'(t) + \lambda\, e^t M_x(t)$

$M_x''(0) \lambda e^e\ M_x'(0) + \lambda e^0 M_x(0) = \lambda^2 + \lambda$

Var(x) - $\delta_x^2 = M_x''(0) - [M_x'(0)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

3- **The poisson distribution is an approximation of binomial distribution as $\lambda = np$ and n approaches to infinity:**

**Proof :** the M.g.f of the binomial distribution is $M_x(t) = (1-p+pe^t)^n = [1+p(e^t-1)]^n$

Putting p= $\frac{\lambda}{n}$ , then

$M_x$ (t) = [1+$\frac{\lambda(e^t-1)}{n}$]$^n$

Using the well known result from calcnlus that $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$

$\lim_{n\to\infty}(1+\frac{\lambda(e^t-1)}{n})^n = e^{\lambda(e^t-1)}$

Which is the m.g.f of the poisson dist with parameter $\lambda$.

**Ex:** verify that the function f(x) = $\frac{e^{-\lambda}\lambda^x}{x!}$, x= 0,1,2,…is actuallg aprobaility function

$\begin{cases} f(x)>0 \\ \sum f(x)=1 \end{cases}$

**Solution :** first , we see that f(x)=> 0 for x= 0,1,2,…. given that $\lambda > 0$

Second , we have

$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}\lambda^x = 1$

**Ex:** let $x_1$, $x_2$, ……, $x_n$ be independent r.v such that $x_i \sim p(\lambda!)$ , i=1,2,…,n , then $\sum_{i=1}^{n} x_i \sim p(\sum_{i=1}^{n}\lambda i)$

**Proof :** let y= $\sum_{i=1}^{n} x_i$ then $M_y$(t) = E($e^{ty}$)

$M_y$(t) = E($e^{t\Sigma xi}$)= E($e^{t\,(x_1+\,x_2+\,\ldots\ldots+\,x_n)}$)= E($e^{tx_1+tx_2+\cdots+tx_n}$) since $x_1, x_2, .., x_n$ are independent then

$M_y$ (t) = E ($e^{tx_1}$) E ($e^{tx_2}$) … E ($e^{tx_n}$)

= $M_{x_1}$(t), $M_{x_2}$ (t)… $M_{x_n}$ (t)= $e^{\lambda 1(e^t-1)}$, $e^{\lambda 2(e^t-1)}$ …. $e^{\lambda n(e^t-1)} = e^{\Sigma\lambda 1(e^t-1)}$

y=$\sum_{i=0}^{n} x_i \sim p(\sum_{i=1}^{n}\lambda i)$

**4- Negative binomial distribution**

Consider an experiment of independent Bernoulli trials performed until we get a total of (r) successes and then stops . The probability of each individual trial resulting in a success is (p) where $0 < p < 1$ . let x denote the number of failures encountered before we get the first r successes, then the p.m.f of X is given by

$$f(x) = \begin{cases} \binom{x+r-1}{x} p^r (1-p)^x, & x=0,1,2,\ldots, \quad r=1,2,\ldots \\ 0 & 0.w \end{cases}$$

and we write X~N b(r,p) where the constants r.p are the parameters of dist .

**Ex:** show that f(x) is exactly a p.m.f .

**Solution :** 1) it is clear that f(x) >0 since each x, r are positive and $0 < p < 1$

2)Applying the rule $\sum_{j=0}^{\infty} \binom{n+j-1}{j} Z^j = (1-Z)^{-n}$

Then $\sum f(x) = \sum_{x=0}^{\infty} \binom{x+r-1}{x} p^r (1-p)^x = p^r \sum_{x=0}^{\infty} \binom{x+r-1}{x} (1-p)^x$

$= p^r [1-(1-p)]^{-r} = p^r p^{-r} = 1$

**Properties :-** the moment generating function $M_x(t) = [\frac{p}{1-(1-p)e^t}]^n$

**Proof :** $M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(x)$

$= \sum e^{tx} \binom{x+r-1}{x} p^r (1-p)^x = p^r \sum \binom{x+r-1}{x} [(1-p)e^t]^x$

$= p^r [1-(1-p)e^t]^{-r} = [\frac{p}{1-(1-p)e^t}]^r$

2) The mean of the distribution is given by $M_x = \frac{r(1-p)}{p}$

**Proof :** we have $M_x(t) = [\frac{p}{1-(1-p)e^t}]^r$

$M_x'(t) = r [\frac{p}{1-(1-p)e^t}]^{r-1} \frac{1-(1-p)e^t}{[1-(1-p)e^t]^2}$

$M_x = E(x) = M_x(0) = r [\frac{p}{1-(1-p)}]^{r-1} \frac{p(1-p)}{[1-(1-p)]^2}$

$= r[\frac{p}{p}]^{r-1} \frac{p(1-p)}{p^2} = \frac{r(1-p)}{p}$

3) the variance of the distribution is $\delta_x^2 = \frac{r(1-p)}{p^2}$

**Proof:** we have $\mu_x$ (t)= $[\frac{p}{1-(1-p)e^t}]^r \Rightarrow \mu_x$ (0)=1

$M_x$ (t) = r $[\frac{p}{1-(1-p)e^t}]^{r-1} \frac{p(1-p)e^t}{[1-(1-p)e^t]^2}$

= r $[\frac{p}{1-(1-p)}]^r [\frac{p}{[1-(1-p)]^2}]^{-1} \frac{p(1-p)e^t}{[1-(1-p)e^t]^2}$

It can be written as

$M'_x$(t)= r $M_x$(t) $\frac{(1-p)e^t}{1-(1-p)e^t}$,

$M'_x(0) = r\, M_x\,(0) \frac{1-p}{p} = \frac{r(1-p)}{p}$

Putting u= (1-p) $e^t$ , $\frac{du}{dt}$ = (1-p)$e^t$ = u

$M'_x$(t) = r $M_x$(t) $\frac{u}{1-u}$

$M''_x$(t) = r $M_x$(t) $\frac{1-u+u}{(1-u)^2} \frac{du}{dt} + \frac{u}{1-u}$ r $M'_x$(t)

$= rM_x(t) \frac{1}{(1-u)^2}\, u + \frac{u}{1-u}\, rM'_x(t)$

$= \frac{ru}{1-u} [M_x$ (t) $\frac{1}{1-u} + M'_x$(t)]

$M''_x$(0) = r$\frac{1-p}{p}$ [ $\frac{1}{p} + \frac{r(1-p)}{p}$ ]

$\delta_x^2$ $M''_x$(0) = [ $M'_x$(0)]$^2$ = $\frac{r(1-p)}{p^2} + \frac{r^2(1-p)^2}{p^2} - \frac{r^2(1-p)^2}{p^2}$

$\delta_x^2$ =var (x) = $\frac{r(1-p)}{p^2}$

## 5) Geometric distribution

The geometric distribution is special case of negative binomial distribution when r=1 . hence :-

f(x) =$\begin{cases} p(1-p)^x, x=0,1,2,\dots \\ 0 \qquad\qquad 0.w \end{cases}$

The properties of geometric distiribtion can be obtained from the corresponding propertics of negative binomial dist by putting r=1 it followsthat:

$M_x(t) = \frac{p}{1-(1-p)e^t}$ , $M_x = \frac{1-p}{p}$ , $\delta_x^2 = \frac{1-p}{p^2}$

**<u>Ex:</u>** A fair die is thrown is successive independent . trials until the second three is observed . let x be a r.v that denotes the number of failures before the second three is observed .

i) Find the distribution of x.
ii) Find the probability of observing 10 no three is before the second three is observed .
iii) Find the mean, variance, and m.g.f of the distribution .

**<u>Solution :</u>** x $\sim Nb(2, \frac{1}{6})$ , that is r = 2 , p = $\frac{1}{6}$

f(x) = $\binom{x+r-1}{x} (\frac{1}{6})^2 (\frac{5}{6})^x$

ii) $p_r$(x=10)=f(10) = $\binom{11}{10} (\frac{1}{6})^2 (\frac{5}{6})^{10}$

iii)$M_x$=E$(x)$ = r $\frac{(1-p)}{p} = 2\frac{5/6}{1/6} = 10$

var (x) = $\delta_x^2$ = r $\frac{(1-p)}{p^2} = 2 \frac{5/6}{(1/6)^2}$

$M_x$ (t)=$[\frac{p}{(1-p)e^t}]^r = [\frac{1/6}{1-\frac{5}{6}e^t}]^2$

**Ex:** suppose we flip affair coin until we get ahead. Let x be the number of tails before we get ahead.

i- Find the p.m.f of x .
ii- Find the mean, variance , and m.g.f of x .

**Solution :** Sine r=1 (first head) the we have ageometric distribution with p= $\frac{1}{2}$ and hence f(x) = p (1-p)$^x$ = $(\frac{1}{2})\,(\frac{1}{2})^x$

ii)E(x) = $M_x = \frac{1-p}{p} = \frac{1/2}{1/2} = 1$

var (x) = $\delta_x^2 = \frac{1-p}{p^2} = \frac{1/2}{1/4} = 2$

$M_x$ (t) = $\frac{p}{(1-p)e^t} = \frac{1/2}{1-\frac{1}{2}e^t}$