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محاضرات إحصاء رياضي//المرحله الرابعه

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1- Continuous probability Distributions

1- The uniform distribution: -

A continuous r.v X is said to follow a uniform distribution denoted as $x \sim u$ (a, b) if the p.d.f of x is

$$f(x) = \begin{cases} \frac{1}{b-a} & , a \leq x \leq b \\ 0 & 0.w \end{cases}$$

The real numbers a, b are the parameters of the distribution

It can be shown that f(x) is actually a p.d .f sine

1- $f(x) = \frac{1}{b-a} > 0$ (because a < b)

2- $\int_a^b f(x)dx = \frac{1}{b-a} \int_a^b dx = 1$

<u>Properties:</u>

1- The mean $M_x = \frac{b+a}{2}$

Proof: $M_x = E(x) = \int_a^b xf(x)dx = \frac{1}{b-a} \int_a^b xdx$

$= \frac{1}{b-a} \frac{x^2}{2} \Big|_a^b = \frac{1}{2(b-a)} (b^2 - a^2) = \frac{b+a}{2}$

2- var (x) $\delta_x^2 = \frac{(b+a)^2}{12}$

Proof: $\delta_x^2 = E(x^2) - [E(x)]^2$

$E(x^2) = \int_a^b x^2 f(x)\, dx = \int_a^b \frac{1}{b-a} x^2\, dx = \frac{1}{b-a} \frac{x^3}{3} \Big|_a^b$

$= \frac{1}{3(b-a)} (b^3 - a^3) = \frac{1}{3(b-a)} (b-a)(b^2+ab+a^2)$

$= \frac{b^2+ab+a^2}{3}$

$\delta_x^2 = \frac{b^2+ab+a^2}{3} - (\frac{b+a}{2})^2 = \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4}$

$=\frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} = \frac{b^2-2ab+a^2}{12}$

$= \frac{(b+a)^2}{12}$

3- The m.g.f of the distribution is

$M_x(\text{t}) = \frac{e^{bt}-e^{at}}{b(b-a)}$, t > 0 (prove it) M(t)=E(e^{tx})=$\int_a^b e^{tx}.f(x)dx$

4- The kth moment about origin is :

E (x^k) = $\frac{b^{k+1}-a^{k+1}}{(k+1)(b-a)}$

Proof: E(x^k) = $\int_a^b x^k f(x)dx = \frac{1}{b-a}\int_a^b x^k\,dx$

$= \frac{1}{(k+1)(b-a)} x^{k+1}\Big|_a^b = \frac{b^{k+1}-a^{k+1}}{(k+1)(b-a)}$

Ex: let x~u (-a , a) , a >0 find the value of a if it is known that p_r (x$>1) = \frac{1}{3}$

Solution : f (x) = $\frac{1}{a-(-a)} = \frac{1}{2a}$

p_r (x$>1) = \int_1^a \frac{1}{2a}\quad dx = \frac{1}{2a}\int_1^a dx = \frac{1}{2a}\,x\Big|_1^a = \frac{a-1}{2a} = \frac{1}{3}$

2a= 3a-3 ⇒ $a = 3$

2- Gamma distribution

Def : If $\propto>0$, we define the gamma function $\Gamma_\propto = \int_0^\propto y^{\propto-1}e^{-y}dy$

Properties of gamma function :

1- $\Gamma_{(a+1)} = \propto\Gamma_\propto$
2- $If\ \propto is\ positive\ integer\ then\ \Gamma_{(\propto+1)} = \propto!$
3- $\Gamma_\propto = 2\int_0^\infty x^{2\propto-1}\,e^{-x2}dx$
4- $\Gamma_{\frac{1}{2}} = \sqrt{\pi}$

Note :

1- If $\propto = 1 \Rightarrow \Gamma(\propto) = \int_0^{\propto} y^{\propto-1} e^{-y}\, dy = 1$

2- If $\propto > 1 \Rightarrow \Gamma_{(x)} = (a-r)\int_0^{\propto} y^{\propto-2} e^{-y} dy = (\propto -1)\pi^{(\propto-1)} = \pi(\propto) =$
$(\propto -1)!$

Def : the continuous r.v x is said to have a gamma distribution with parameters $\propto$, $\beta > 0$ denoted as x~G $(\propto, \beta)$ if the p. d. f of x is $f(x) = \begin{cases} \frac{1}{\Gamma_{\propto}\beta^{\propto}} x^{\propto -1}\ e^{-x/\beta}, x > 0 \\ 0 \qquad\qquad 0.w \end{cases}$

Properties of gamma distribution: -

1- The m. g. f of the distribution is $M_x(t) = (1 - \beta t^{-\propto})$, $t < \frac{1}{\beta}$

Proof : $M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx = \int_0^{\infty} e^{tx} \frac{1}{\Gamma_{\propto}\beta^{\propto}} x^{\propto -1} e^{\frac{-x}{\beta}} dx$

$= \frac{1}{\Gamma_{\propto}\beta^{\propto}} \int_0^{\infty} x^{\propto-1} e^{tx-\frac{x}{\beta}} dx = \frac{1}{\Gamma_{\propto}\beta^{\propto}} \int_0^{\infty} x^{\propto-1}\ e^{\frac{\beta tx - x}{\beta}}\ dx$

$= \frac{1}{\Gamma_{\propto}\beta^{\propto}} \int_0^{\infty} x^{\propto-1}\ e^{-x(\frac{1-\beta t}{\beta})}\ dx$

Patting $y = x(\frac{1-\beta t}{\beta}) \Rightarrow x = \frac{\beta}{1-\beta t} y, dx = \frac{\beta}{1-\beta t} dy$

$M_x(t) = \frac{1}{\Gamma_{\propto}\beta^{\propto}} \int_0^{\infty} \left[\frac{\beta y}{1-\beta t}\right]^{\propto-1}\ e^{-y}\ \frac{\beta}{1-\beta t}\ dy$

$= \frac{1}{\Gamma_{\propto}\beta^{\propto}}\ \frac{\beta^{\propto}}{(1-\beta t)^{\propto}}\ \int_0^{\infty} e^{-y}\, y^{\propto -1}\ dy$

$= \frac{1}{\Gamma_{\propto}\beta^{\propto}} \left(\frac{\beta}{1-\beta t}\right)^{\propto} \Gamma_{\propto} = \frac{1}{(1-\beta t)^{\propto}} = (1-\beta t)^{-\propto}$

2- $M_x = \mathrm{E(x)} = \propto \beta$

3- $\delta \mathrm{x}^2 = \mathrm{ver\ (x)}. \propto \mathrm{B}^2$ prove it

4- the k^{th} *moment about origin is* $E(x^k) = \frac{\beta^k\, \Gamma_{\propto + k}}{\Gamma_{\propto}}$, $k = 1,2,3, \ldots$

proof : $\mathrm{E}(x^k) = \int_0^{\infty} x^k f(x) dx = \int_0^{\infty} x^k \frac{1}{\Gamma_{\propto}\beta^{\propto}} x^{\propto -1} e^{\frac{-x}{\beta}} dx$

$= \frac{1}{\Gamma_{\propto}\beta^{\propto}} \int_0^{\infty} x^{k+\propto -1}\ e^{-x/\beta}\, dx$

Let $y = \frac{x}{\beta} \Rightarrow x = \beta y \Rightarrow dx = \beta\, d\, y$

$E(x^k) = \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} \int_0^{\infty} (\beta y)^{\propto + k - 1}\, e^{-y}\, (\beta\, d\, y)$

$= \frac{1}{\Gamma_{\propto}\, \beta^{\propto}}\, \beta^{\propto + k} \int_0^{\infty} y^{\propto + k - 1}\, e^{-y} dy$ $\int_0^{\infty} y^{\propto - 1}\, e^{-y} dy = \Gamma_{(\propto)}$

$= \frac{1}{\Gamma_{\propto}\, \beta^{\propto}}\, \beta^{\propto} \beta^{k} \Gamma_{(\propto + k)} = \frac{\beta^{k} \Gamma_{\propto + k}}{\Gamma_{\propto}}$ $, k = 1, 2, \ldots.$

Ex: use the formula of (x^k) , to find μ_x , δx^2

Solution : putting $k = 1$ *then*

$E(x) = M_x\ \frac{\beta \Gamma_{\propto + 1}}{\Gamma_{\propto}} = \frac{\beta \propto \Gamma_{\propto}}{\Gamma_{\propto}} = \propto \beta$

Putting $k = 2$, *we get*

$E(x^2) = \frac{\beta^2\, \Gamma_{(\propto + 2)}}{\Gamma_{\propto}} = \frac{\beta^2 (\propto + 1) \Gamma_{(\propto + 1)}}{\Gamma_{\propto}}$

$= \frac{\beta^2 (\propto + 1) \propto \Gamma_{\propto}}{\Gamma_{\propto}} = \beta^2 \propto (\propto + 1)$

$\delta x^2 = var\,(x) = E(x^2) - [E(x)]^2$

$= \beta^2\, (\propto + 1) \propto\ - \beta^2\ \propto^2$

$= \beta^2\, \propto^2 + \beta^2 \propto - \beta^2\, \propto^2 = \propto \beta^2$

3- the chi square distribution

The chi square distribute on is as pecial case of gamma distribution which $\propto = \frac{r}{2}$ and B=2 where ris positive integer . Hence the p.d.f of the r. v. x is

$$f(x) = \begin{cases} \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}}\ x^{\frac{r}{2} - 1}\ e^{\frac{-x}{2}}, & x > 0 \\ 0 & 0.w \end{cases}$$

and we write x~ $x^2\,(r)$ where r is the number of degrees freedom representing the parameter of the distribution .

properties :-

the properties of chi square dist are the same of properties of gamma dist when $\propto = \frac{r}{2}$, $\beta = 2$ that is :-

1) $M_x(t) = (1-2t)^{-r/2}$
2) $M_x = E(x) = \frac{r}{2} \cdot 2 = r$
3) Var(x) = $\delta x^2 = \left(\frac{r}{2}\right) 2^2 = 2r$
4) E(x^k) = $\frac{2^k \, \Gamma_{(\frac{r}{2}+k)}}{\Gamma_{\frac{r}{2}}}$, $k = 1,2,\ldots.$

4-Beta distribution

Def :- If $\propto > 0$, $\beta > 0$, we define the Beta function as

$\beta\,(\propto, \beta) = \int_0^1 x^{\propto-1}(1-x)^{\beta-1}\,dx$

It can be show that $\beta\,(\propto, \beta) = \frac{\Gamma_\propto \Gamma_B}{\Gamma_{\propto+rs}}$, $\propto , \beta > 0$

Def :- the continuous r. v. x is said to have a Beta distribution denoted as x ~ $\beta\,(\propto, \beta)$ If the p. d. f of x is $f(x) = \begin{cases} \frac{\Gamma_{\propto+\beta}}{\Gamma_\propto \Gamma_\beta} x^{\propto-1}\,(1-x)^{\beta-1} , 0 < x < 1 \\ 0 \qquad 0.w \end{cases}$

Properties :-

1) The k^{th} moment a bout origin is $E\,(x^k) = \frac{\Gamma(k+\propto)\Gamma(\propto+B)}{\Gamma_\propto \Gamma_{(k+\propto+B)}}$, $|k = 1, 2, \ldots$

Proof: $E\,(x^k) = \int_0^1 x^k\, f(x)dx = \frac{\Gamma_{\propto+B}}{\Gamma_\propto \Gamma_B} \int_0^1 x^{\propto 1-\,k-1}\,(1-x)^{\beta-1}dx$

$= \frac{\Gamma_{\propto+\beta}}{\Gamma_\propto \Gamma_\beta} \; \frac{\Gamma_{\propto+k}\Gamma_\beta}{\Gamma_{\propto+k+\beta}} \quad \left[\beta(\propto, \beta) = \frac{\Gamma_\propto \Gamma_\beta}{\Gamma_{\propto+\beta}}\right]$

$= \frac{\Gamma_{\propto+B}\, \Gamma_{\propto+k}}{\Gamma_\propto\, \Gamma_{k+\propto+\beta}}$

2) From the above formula , the mean and variance of the distribution can be derive as follows :- putting k=1 we obtain

$E(x) = M_x = \frac{\Gamma_{\propto+1}\Gamma_{\propto+\beta}}{\Gamma_{\propto}\Gamma_{\propto}+\beta+1} = \frac{\propto\Gamma_{\propto}\Gamma_{\propto+\beta}}{\Gamma_{\propto}(\propto+\beta)\Gamma_{\propto+\beta}}$ [$since\ \Gamma_{\propto+1} = \propto \Gamma_{\propto}$]

$M_x = \frac{\propto}{\propto+\beta}$

Putting k=2 we get $E(x^2) = \frac{\Gamma_{\propto+2}\Gamma_{\propto+\beta}}{\Gamma_{\propto}\Gamma_{\propto}+\beta+2} = \frac{(\propto+1)\propto\Gamma_{\propto}\Gamma_{\propto+}\beta}{\Gamma_{\propto}(\propto+\beta+1)(\propto+\beta)\Gamma_{\propto+\beta}}$

$= \frac{\propto(\propto+1)}{(\propto+\beta+1)(\propto+\beta)}$

Var (x) = $\int_x^2 = E(x^2) - [E(x)]^2$

$= \frac{\propto(\propto+1)}{(\propto+\beta+1)(\propto+\beta)} - \frac{\propto^2}{(\propto+\beta)^2} = \frac{\propto(\propto+1)(\propto+\beta)-\propto^2(\propto+\beta+1)}{(\propto+\beta)^2(\propto+\beta+1)}$

$= \frac{\propto^3+\propto^2\beta+\propto^2+\propto\beta-\propto^3-\propto^2\beta-\propto^2}{(\propto+\beta)^2(\propto+\beta+1)} = \frac{\propto\beta}{(\propto+\beta)^2(\propto+\beta+1)}$

$\therefore \int_x^2 = \frac{\propto\beta}{(\propto+\beta)^2(\propto+\beta+1)}$

5- The normal distribution

A continuous r.v x is said to have normal distribution with parameters M, δ^2 denote, as x~N(M, δ^2) if the p. d. f of x is $f(x) = \frac{1}{\sqrt{2\pi\,\delta^2}}\ e^{\frac{-1}{2}(\frac{x-M}{\delta})^2}$ $-\infty < x < \infty$

Proportion :-

1) The M. g. f of normal dist. Is $M_x(t) = e^{Mt\,\frac{\delta^2 t2}{2}}$

Proof:- $M_x(t) = E(e^{tx}) = \frac{1}{\sqrt{2\pi\delta^2}} \int_{-\infty}^{\infty} e^{tx}\, e^{\frac{-1}{2}\ (\frac{x-M}{\delta})^2}\, dx$

Iet y=$\frac{x-M}{\delta} \Rightarrow x = M + \delta y \Rightarrow dx = \delta\ dy$

$M_x\ (t) = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(M+\delta y)}\quad e^{\frac{-y^2}{2}}\ (\delta\ dy)$

$= e^{Mt}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t\delta y}\quad e^{\frac{-y^2}{2}}\ dy$

$= e^{Mt}\ \frac{1}{\sqrt{2\pi}}\ \int_{-\infty}^{\infty} e^{\frac{-(y^2-2t\delta y)}{2}}\, dy$

نجري بإكمال المربع على الاس بإضافة وطرح مربع نصف معامل y ينتج :

$M_x\,(t) = e^{\mu t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-[(y^2-2\delta t y+\delta^2 t^2)-\delta^2 t^2]}{2}}\,dy$

$= e^{Mt}\,e^{\frac{\delta^2 t^2}{2}}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-(y-\delta t)^2}{2}}\,dy$

Let $\mathrm{Z} = \mathrm{y} - \delta \mathrm{t} \Rightarrow \mathrm{y} = \mathrm{Z} + \mathrm{dt} \Rightarrow \mathrm{dy} = \mathrm{dZ}$

$\mathrm{M_x(t)} = \mathrm{e}^{\mathrm{Mt}+\frac{\delta^2 t^2}{2}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{-z^2}{2}}\,\mathrm{dZ}\right]$

normal dist. $\therefore\ M_x(t) = e^{Mt+\frac{\delta^2 t^2}{2}}$

2) **The mean of the normal dist. is: -** $M_x = E(x) = M$

Proof:$M_x(t) = e^{Mt+\frac{\delta^2 t^2}{2}}$

$\mathrm{M}'_{x(t)} = (M + \frac{\delta^2 2t}{2})\ e^{Mt+\frac{\delta^2 t^2}{2}}$

$=(M+\delta^2 t)M_x(t)$

$M_x = E(x) = M_x(0) = (M + 0)M_x(0) = M.1 = M$

3) **The variance of the distribution is** $\delta x^2 = Var(x) = \delta^2$

Proof:-

$\mathrm{M}''_x(\mathrm{t}) = (\mathrm{M} + \delta^2 \mathrm{t})\mathrm{M}'_{x(t)} + \delta^2 \mathrm{M_x(t)}[\text{since } \mathrm{M_x(t)} = (\mathrm{M} + \delta^2 +)\mathrm{M_x(t)}]$

$\mathrm{M}''_x(0) = (\mathrm{M} + 0)\mathrm{M}'_x(0) + \delta^2 \mathrm{M_x}(0)$

$= (M+0)\ M+\delta^2 - 1 = \mathrm{M}^2 + \delta^2$

$\delta^2 = \mathrm{var(x)} = \mathrm{M}''_x(0) - [\mathrm{M}'_x(0)]^2$

$= \mathrm{M}^2 + \delta^2 - \mathrm{M}^2 = \delta^2$

Def :- If the r.v $Z \sim N(0,1)$, then we say that Z distributed as standard normal distribution with p. d. f. $\mathrm{f(Z)} = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\frac{-z^2}{2}}\ ,-\infty < \mathrm{Z} < \infty$

The mean , Variance and moment generating function of the r.v Z is

$\mathrm{M_Z} = 0, \delta_{\mathrm{Z}^2} = 1\,, \mathrm{M_Z(t)} = \mathrm{e}^{t^2/2}$

Theorem (1) : If the r.v $x \sim N(M, \delta^2)$ then $Z = \frac{x-\mu}{\delta} \sim N\ (0,1)$

Proof :- By using the transformation method we have $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-M}{\delta})^2}, -\infty < x < \infty$ the space of x denoted by A and the space of Z denoted by Bare defined as :-

$A = \{x = -\infty < x < \infty\}, \beta = \{Z = -\infty < Z < \infty\}$

$Z = u(x) = \frac{x-M}{\delta}$ *is* $(1-1)$ trans formation maps A onto β

$x = u^{-1}(Z) = M + \delta Z$ *is* $(1-1)$ transformation maps B onto A

$|J| = \left|\frac{dx}{dZ}\right| = \delta dz \Rightarrow g(Z) = f\,[u^{-1}(Z)]|J|$

$g(Z) = \frac{1}{\sqrt{2\pi}\delta}\ e^{\frac{-1}{2}Z^2}\ (\delta) = \frac{1}{\sqrt{2\pi}}\ e^{\frac{-1}{2}Z^2}$

$\therefore\ Z = \frac{x-M}{\delta} \sim N\ (0,1)$

Calculating the probabilities

The probabilities concerning the r.v.x which distributed as N (M, δ^2) can be expressed in terms of probabilities concerning $Z = \frac{x-M}{\delta}$ which distributed as N(0,1), However an integral like $\int_{-\infty}^{k} \frac{1}{\sqrt{2\pi}}\ e^{-Z2/2}\ \delta Z$ cannot be evaluated . Instead we use tables which approximate the value of this integral for differend values of k In general, the following rules are important.

1) $p_r(Z < 0) = p_r(Z > 0) = 0.5$
2) $p_r(Z < -Z_1) = 1 - p_r(Z < Z_1), Z_1 > 0$
3) $p_r(Z_1 < Z < Z_2) = p_r(Z < Z_2) - p_r(Z < Z_1) = N(Z_2) - N(Z_1)$

Ex:- Given that X~N (2,25) , find $p_r\ (0 < x < 10)$

Solution :-Type equation here.

$p_r\ (0 < x < 10) = p_r\left(Z < \frac{10-2}{5}\right) - p_r(Z < \frac{0-2}{5}),\ Z = \frac{x-M}{\delta}$

$= p_r\ (-0.4 < Z < 1.6)$

$= p_r\ (Z < 1.6) - p_r\ (Z < -0.4)$

$= p_r (Z < 1.6) - [1 - p_r (Z < 0.4)]$

$= N(1.6) - [1 - N(0.4)]$

$= 0.45 -[1 - 0.655] = 0.6$ [from tables]

Theorem (2) :- If the r.v $X \sim N (M, \delta^2)$ then the r.v $y = (\frac{x-M}{\delta})^2 \sim x^2(1)$

Proof :- By using the m. g. f. method $M_y(t) = E(e^{ty}) = E\left[e^{t(\frac{x-M}{\delta})^2}\right]$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\delta}\, e^{t(\frac{x-M}{\delta})^2}\, e^{\frac{-1}{2}(\frac{x-M}{\delta})^2}\, dx$

Putting $Z = \frac{x-M}{\delta} \Rightarrow x = M + \delta Z \Rightarrow dx = \delta dZ$

$M_x(t) = \frac{1}{\sqrt{2\pi}\,\delta} \int_{-\infty}^{\infty} e^{tZ^2 - \frac{1}{2}Z^2} (\delta d\, Z)$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{Z^2(t-\frac{1}{2})}\, dZ$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-Z^2}{2}(1-2t)} dZ$

Let $w = Z\sqrt{1-2t} \Rightarrow Z = \frac{1}{\sqrt{1-2t}}\, w\, , dZ = \frac{1}{\sqrt{1-2t}}\, dw$

$\Rightarrow M_y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-w^2}{2}}\, \frac{1}{\sqrt{1-2t}}\, dw$

$= \frac{1}{\sqrt{1-2t}} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-w^2}{2}}\, dw\right] = \frac{1}{\sqrt{1-2t}}\,(1)$

$M_y(t) = (1-2t)^{-1/2}$ which is the m. g. f. of chi-square dist. with 1 degree of free

Doue $y = (\frac{x-\mu}{\delta})^2 \sim x^2(1)$

Theorem(3) :- If xi , i= 1,2,…,n distributed as $x \sim N(0,1)$ then $\sum_{i=1}^{n} x_i^2 \sim x^2(n)$

Proof :- let

$y = \sum_{i=1}^{n} x_i^2$, then $\mu_y(t) = E(e^{ty})$

$M_y(t) = E\left(e^{t(y)}\right) = E\left(e^{t(x_1^2+x_2^2+\cdots+x_n^2)}\right)$

$= \mathrm{E}(e^{tx_1^2})\, E\left(e^{tx_2^2}\right) \ldots E(e^{tx_n^2})$

Since each of x_1, x_2,…,xn ~N(0,1) then each of $x_1^2, x_2^2, \ldots, x_n^2 \ \sim x^2(1)$ (theories 2)

$M_y(t) = \ (1-2t)^{-1/2}\,(1-2t)^{-1/2} \ldots \ (1-2t)^{-1/2}$ n-terms

$M_y(t) = \left[((1-2t)^{-1/2}\right]^n = \ (1-2t)^{-n/2}$

y= $\sum_{i=1}^{n} xi^2 \sim x^2(n)$

Summary of theorem:

1- x~N(M, δ^2) then $Z = \frac{x-\mu}{\delta}$ ~N (0,1)

2- X~N (M,δ^2) then the r.v z= $(\frac{x-M}{\delta})^2 \sim x^2(1)$

3- $x \sim N$ (0,1) then $\sum_{i=1}^{n} x_i^2 \sim x^2(n)$

(6) the students t distribution

Let the r.v W~N(0,1) and the r.v. V~$x^2(r)$, where W and V are stochastically independent . then T=$\frac{w}{\sqrt{\frac{v}{r}}}$ has students t distribution with p. d. f given by

$$g(t) = \frac{\Gamma_{(r+1)/2}}{\sqrt{\pi r}\,\Gamma_{\frac{r}{2}}} \quad \frac{1}{(1+\frac{t^2}{r})^{\frac{(r+1)}{2}}} \quad , -\infty < t < \infty$$

Proof :- the joint p. d. f. of W and V is $\phi(w,v) = \frac{1}{\sqrt{2\pi}}\, e^{-w^2/2}\, \frac{1}{\Gamma_{\frac{r}{2}}\, 2^{r/2}}\, (v)^{\frac{r}{2}-1}\, e^{\frac{-v}{2}}$

$-\infty < w < \infty , 0 < v < \infty$

Let : $t = \frac{w}{\sqrt{\frac{v}{r}}}$ and u=v define $a(1-1)$ transformation mapping the space $\{(w,v)_i - \infty < w < \infty , 0 < v < \infty\}$ onto the space $\{(t,u)_i - \infty < t < \infty , 0 < u < \infty\}$

$w = t\frac{\sqrt{u}}{\sqrt{r}}$, $v = u$, $J = \begin{vmatrix} \frac{dw}{dt} & \frac{dw}{du} \\ \frac{dv}{dt} & \frac{dv}{du} \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{u}}{\sqrt{r}} & \frac{t}{2\sqrt{ur}} \\ 0 & 1 \end{vmatrix} = \frac{\sqrt{u}}{\sqrt{r}}$

$$g(t,u) = \emptyset\left[\frac{t\sqrt{u}}{\sqrt{r}}, u\right] \ . \ |J|$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma_{\frac{r}{2}}\,2^{r/2}}\ u^{\frac{r}{2}-1}\, e^{\frac{-u}{2}}\, e^{\frac{-t^2u}{2r}}\, \frac{\sqrt{u}}{\sqrt{r}}$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma_{\frac{r}{2}}\,2^{r/2}}\ u^{\frac{r}{2}-1}\, e^{\frac{-u}{2}\left[1+\frac{t^2}{r}\right]} \frac{\sqrt{u}}{\sqrt{r}}$$

$-\infty < t < \infty\,, 0 < u < \infty$ Now $g(t) = \int_0^\infty g(t,u)du$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\,\Gamma_{\frac{r}{2}}\,2^{r/2}}\ u^{\frac{r}{2}-1}\, e^{\frac{-u}{2}\left[1+\frac{t^2}{r}\right]}\ \frac{\sqrt{u}}{\sqrt{r}}\, du$$

Putting $Z = \frac{u}{2}\left[1+\frac{t^2}{r}\right] \Rightarrow u\left[1+\frac{t^2}{r}\right] = 2Z$

$$u = \frac{2Z}{1+\frac{t^2}{r}} \Rightarrow du = \frac{2}{1+\frac{t^2}{r}}\, dZ$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\,\Gamma_{\frac{r}{2}}\,2^{r/2}\, r^{1/2}} \left(\frac{2Z}{1+\frac{t^2}{r}}\right)^{\frac{r}{2}-1} e^{-Z}\, \frac{\sqrt{2Z}}{\sqrt{1+\frac{t^2}{r}}}\, \frac{2}{1+\frac{t^2}{r}}\, dZ$$

$$= \frac{1}{\sqrt{2\pi r}\,\Gamma_{\frac{r}{2}}\,2^{r/2}}\ \frac{2^{\frac{r}{2}-1}}{(1+\frac{t^2}{r})^{\frac{r}{2}-1}}\ \frac{\sqrt{2}}{(1+\frac{t^2}{r})^{1/2}}\ \frac{2}{(1+\frac{t^2}{r})}\ \int_0^\infty Z^{\frac{r}{2}-1} Z^{1/2} e^{-Z} dZ$$

$$= \frac{1}{\sqrt{2\pi r}\,\Gamma_{\frac{r}{2}}}\ \frac{\sqrt{2}}{(1+\frac{t^2}{r})^{\frac{r+1}{2}}}\ \int_0^\infty Z^{\frac{r}{2}+\frac{1}{2}-1} e^{-Z} dZ$$

$$g(t) = \frac{\Gamma_{\frac{r+1}{2}}}{\sqrt{\pi r}\quad \Gamma_{\frac{r}{2}}}\ \frac{1}{(1+\frac{t^2}{r})^{\frac{r+1}{2}}} \quad , -\infty < t < \infty$$

The mean and variance of t distribution

The mean of students t , distribution is :- $E(t) = E\left(\frac{w}{\sqrt{\frac{v}{r}}}\right) = E(W)E\left(\frac{1}{\sqrt{\frac{v}{r}}}\right)$

[since ,w. v are indep.] but W~N(0,1) , E(W)=0 hence E(t)=0 , E $\left(\frac{1}{\sqrt{\frac{r}{r}}}\right) = 0$

the variance of distribution is derived as follows var (t) = E$(t^2) - [E(t)]^2 = E(t^2)$

var (t) = E$\left[\frac{w}{\sqrt{\frac{r}{r}}}\right]^2 = E\left[\frac{w^2}{\frac{v}{r}}\right] = E(w^2)E\left(\frac{1}{\frac{v}{r}}\right)$

since W~N(0,1) then E$(w^2) = Var(w) + [E(w)]^2$ = 1

var(t)= 1.E$\left(\frac{1}{\frac{v}{r}}\right) = r\ E\ (\frac{1}{V})$

but V~$x^2(r)$ then E$\left(\frac{1}{V}\right) = \int_0^\infty \frac{1}{V}\ \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}}\ v^{\frac{r}{2}-1} e^{\frac{-v}{2}} \delta v$

$= \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}} \int_0^\infty v^{-1}\ v^{\frac{r}{2}-1} e^{\frac{-v}{2}}\ \delta v = \frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}} \int_0^\infty v^{\frac{r}{2}-2} e^{\frac{-v}{2}}\ \delta v$

Putting Z=$\frac{v}{2}$ $\Rightarrow v = 2Z \Rightarrow \delta v = 2dZ$ $E\left(\frac{1}{V}\right) =$
$\frac{1}{\Gamma_{\frac{r}{2}} 2^{r/2}} \int_0^\infty (2Z)^{\frac{r}{2}-2}\ e^{-Z}\ 2dZ$

$= \frac{2^{\frac{r}{2}-2}\quad 2}{\Gamma_{\frac{r}{2}} 2^{r/2}} \qquad \int_0^\infty Z^{\frac{r}{2}-1-1} \quad e^{-Z}\ dZ$

$= \frac{1}{2\Gamma_{\frac{r}{2}}} \quad \Gamma_{\frac{r}{2}-1} = \frac{1}{2(\frac{r}{2}-1)\Gamma_{\frac{r}{2}-1}} \quad \Gamma_{\frac{r}{2}-1}$

$= \frac{1}{2(\frac{r}{2}-1)} = \frac{1}{r-2}$

Var(t)= r E$(\frac{1}{v}) = \frac{r}{r-2}$

(7) The F distribution

Let $x_1 \sim x^2(n_1)$ is independent from $x_2 \sim x^2(n_2)$ then the ration f= $\frac{x_1/n_1}{x_2/n_2}$ is said to have an F distribution with n_1 , n_2 degrees of freedom the p. d. f of the

distribution is :- $g(f) = \frac{\Gamma\frac{n_1+n_2}{2}}{\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} \frac{f^{\frac{n_1}{2}-1}}{\left[1+\frac{n_1}{n_2}f\right]^{\frac{n_1+n_2}{2}}}$ $, f > 0$ and we say that f~F(n_1, n_2) the mean of the distribution is given by $\mu_f = E(f) = \frac{n_2}{n_2-2}$ $, n_2 > 2$

Proof :- $E(f) = E\left(\frac{x_1/n_1}{x_2/n_2}\right) = \frac{\frac{1}{n_1}}{\frac{1}{n_2}} E\left(\frac{x_1}{x_2}\right)$

$= \frac{n_2}{n_1}\ E(x_1)\ E(\frac{1}{x_2})$

Since $x_1 \sim x^2(n_1), x_2 \sim x^2(n_2)$, then $E(x_1) = n_1$

$E(f) = \frac{n_2}{n_1}\ n_1\ E\left(\frac{1}{x_2}\right) = n_2 E(\frac{1}{x_2})$ bydef.

$$E\left(\frac{1}{x_2}\right) = \int_0^{\infty} \frac{1}{\Gamma\frac{n_2}{2}\, 2^{n_2/2}}\ \frac{1}{x_2}\ x_2^{\frac{n_2}{2}-1}\, e^{\frac{-x_2}{2}}\ dx_2$$

$$= \frac{1}{\Gamma\frac{n_2}{2}\, 2^{n_2/2}} \int_0^{\infty} x_2^{\frac{n_2}{2}-1-1} e^{\frac{-x_2}{2}}\, dx_2$$

$$Z = \frac{x_2}{2} \Rightarrow 2Z = x_2 \Rightarrow 2dZ = dx_2$$

$$= \frac{1}{\Gamma\frac{n_2}{2}\, 2^{n_2/2}} \int_0^{\infty} (2Z)^{\frac{n_2}{2}-2}\ e^{-Z} 2dZ$$

$$= \frac{1}{\Gamma\frac{n_2}{2}\, 2^{n_2/2}}\ 2^{\frac{n_2}{2}-2} \quad .2 \int_0^{\infty} Z^{\frac{n_2}{2}-1-1}\ e^{-Z} dZ$$

$$= \frac{\Gamma\frac{n_2}{2}-1}{2\Gamma\frac{n_2}{2}}$$

$\Gamma\frac{n_2}{2}-1+1$, $\Gamma_{\alpha+1=}\ \ \alpha\Gamma_\alpha$ القانون

$$= \frac{\Gamma\frac{n_2}{2}-1}{2(\frac{n_2}{2}-1)\Gamma\frac{n_2}{2}-1} \qquad = \frac{1}{n_2-2}$$

$$E(f) = n_2\ \ E\left(\frac{1}{x_2}\right) = \frac{n_2}{n_2-2}\ , n_2 > 2$$

The variance of the distribution is var(f)=2 $M_f^2 \frac{n_1+n_2-2}{n_1(n_2-4)}$ $, n_2 > 4$

Problems: -

(1) Let t be a r.v with p. d. f $g(t) = c\left[1+\frac{1}{5}t^2\right]^{-3}$ find the value of c such that the r. v. follows t distribution

Sol: $g(t) = \frac{\Gamma_{(r+1)/2}}{\sqrt{\pi r}\,\Gamma_{\frac{r}{2}}} \quad \frac{1}{(1+\frac{t^2}{r})^{\frac{(r+1)}{2}}}$

$\frac{(r+1)}{2}=3$,r=5 $\quad \frac{\Gamma_3}{\sqrt{\pi 5}\,\Gamma_{\frac{5}{2}}} = \frac{2.1}{\sqrt{\pi 5}\;3/2)+1}$

L $g(t) = \frac{\Gamma_{(r+1)/2}}{\sqrt{\pi r}\,\Gamma_{\frac{r}{2}}} \quad \frac{1}{(1+\frac{t^2}{r})^{\frac{(r+1)}{2}}}$

(2) et the r. v. t~$t(1)$show that the C.D.F of t is F(t)=0.5 $+\frac{1}{\pi}\tan^{-1}t$

Sol:$g(t) = \frac{\Gamma_{(r+1)/2}}{\sqrt{\pi r}\,\Gamma_{\frac{r}{2}}} \quad \frac{1}{(1+\frac{t^2}{r})^{\frac{(r+1)}{2}}}$

$$g(t) = \frac{\Gamma_{(1+1)/2}}{\sqrt{\pi}\,\Gamma_{\frac{1}{2}}} \quad \frac{1}{(1+\frac{t^2}{1})^{\frac{(1+1)}{2}}}$$

$$g(t) = \frac{1}{\pi}\,\frac{1}{(1+t^2)}$$

$\int_{-\omega}^{t} \frac{1}{\pi}\,\frac{1}{(1+t^2)}\,dt = \frac{1}{\pi}\tan^{-1}t$

(3) Let t=$\frac{w}{\sqrt{v/r}}$, where w~$N(0,1)$ and v~$x^2(r)$ show that T^2 has an F distributed with parameters $r_1 = 1, r_2 = r$

f= $\frac{x_1/n_1}{x_2/n_2}$

$t^2 = \frac{w^2}{v/r}$ $\quad$ w~$N(0,1)$ $\quad$ $w^2 \sim x^2(1)$ $\quad$ $t^2 = \frac{w^2/1}{v/r}$

v~$x^2(r)$

(4) Let f has F distribution with parameters r_1, r_2 prove that $\frac{1}{f}$ has an F distribution with parameters r_2 and r_1

$1/f = 1/\frac{x_1/n_1}{x_2/n_2}$ $\quad f = \frac{x_2/n_2}{x_1/n_1}$

3)Distribution of sample mean and sample variance

(1) the distribution of$\bar{x}$:- let $x_i, i = 1,2,\dots,n$ be a r.vs from N (M,δ^2) the distribution $\bar{x}$ is derived by using the m.g.f as follows : $M_{\bar{x}}(t) = E(e^{t\bar{x}}) = E\left(e^{t\frac{\sum x_i}{n}}\right) = E(e^{\frac{t}{n}(x_1+x_2+\cdots+x_n)})$

$= \text{E}(e^{\frac{t}{n}x_1})\, E\left(e^{\frac{t}{n}x_2}\right) \dots E(e^{\frac{t}{n}x_n})$

$= M_{x_1}\left(\frac{t}{n}\right) \mu_{x_2}\left(\frac{t}{n}\right) \dots M_{x_n}\left(\frac{t}{n}\right)$

Since each of $x_1, x_2, \dots, x_n$ distributed N (M,δ^2) then $M_x{}^{(t)} = e^{M\,t+\frac{\delta^2 t^2}{2}}$ $M_{\bar{x}}(t) = \left[M_x(\frac{t}{n})\right]^n = \left[e^{M\frac{t}{n}+\frac{\delta^2(\frac{t}{n})^2}{2}}\right]^n = e^{Mt+\frac{\frac{\delta^2}{n}t^2}{2}}$ which is similar to the m. g. f of normal dist. With mean M and variana $\frac{\delta^2}{n}$ $\quad \bar{x} \sim N(M, \frac{\delta^2}{n})$ $\quad f(\bar{x}) = \frac{1}{\sqrt{2\pi\frac{\delta^2}{n}}}\, e^{-1/2\ \frac{(\bar{x}-M)^2}{\delta^2/n}}, -\infty < \bar{x} < \infty$

for example if $x_1, x_2, \dots, x_n$ be a r.s from N (1,2) then $\bar{x} \sim N(1, \frac{2}{n})$ and

$\text{f}(\bar{x}) = \frac{1}{\sqrt{2\pi\,\frac{2}{n}}}\ e^{\frac{-(\bar{x}-1)^2}{2\frac{2}{n}}}, -\infty < \bar{x} < \infty$

properties :- According to theorems (1) , (2) , (3) if $\bar{x} \sim N(M, \frac{\delta^2}{n})$, then :-

1) $\frac{\bar{x}-M}{\delta/\sqrt{n}} \sim N(0,1)$
2) $(\frac{\bar{x}-M}{\delta/\sqrt{n}})^2 \sim x^2(1)$
3) $\sum_{i=1}^{k}(\frac{\bar{x}-M}{\delta/\sqrt{n}})^2 \sim x^2\ (k)$

2- Distribution of sample variance S^2

the sample variance S^2 is defined as : $S^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$ consider the sum of squares $\sum(x_i - M)^2 = \sum[(x_i - \bar{x}) + (\bar{x} - M)]^2$

$=\sum(x_i - \bar{x})^2 + 2\,(\bar{x} - M)\sum(x_i - \bar{x}) + n(\bar{x} - M)^2$
$\sum(x_i - M)^2 = \sum(x_i - \bar{x})^2 + n\,(x - M)^2$

dividing both sides by δ^2 we get , $\sum(\frac{x_i - M}{\delta})^2 = \frac{n\,s^2}{\delta^2} + \left(\frac{\bar{x} - M}{\delta/\sqrt{n}}\right)^2$

since $\sum(\frac{x_1 - M}{\delta})^2 \sim x^2(n)$ [theorem (3)]

Also $(\frac{\bar{x} - M}{d/\sqrt{n}})^2 \sim x^2(1)$ [property (2)]

it follows that $\frac{ns^2}{\delta^2} \sim x^2(n-1)$ (additive property of x^2 dist.)

Hence E$(\frac{ns^2}{\delta^2}) = n - 1 \Rightarrow \frac{n}{\delta^2}\,E(s^2) = n - 1$
E$(s^2) = \delta^2\,\frac{n-1}{n}$
var$(\frac{ns^2}{\delta^2}) = 2(n-1) \Rightarrow \frac{n^2}{\delta^4} var\,(s^2) = 2(n-1)$
var$(s^2) = \frac{2(n-1)}{n^2}\quad \delta^4$

Ex:- If $\bar{x}$ is the mean of r. v of size n from N(M,100) . find n such that $p_r(M - 5 < \bar{x} < M + 5) = 0.9\,s4$

Solution :- let $x_1 = M - 5\,, x_2 = M + s$
$p_r[M - s < \bar{x} < M + 5) = p_r[\,x_1 < \bar{x} < x_2]$

$=p_r[\frac{x_1 - M}{\delta/\sqrt{n}} < \frac{\bar{x} - M}{\delta/\sqrt{n}} < \frac{x_2 - M}{\delta/\sqrt{n}}]$

$= p_r\left[\frac{-5}{10/\sqrt{n}} < Z < \frac{5}{10/\sqrt{n}}\right] = p_r\left[\frac{-1}{2}\sqrt{n} < Z < \frac{1}{2}\sqrt{n}\right]$

$= p_r\left[Z < \frac{1}{2}\sqrt{n}\right) - [1 - p_r\left(Z < \frac{1}{2}\sqrt{n}\right)]$

$= \qquad 2p_r\left[Z < \frac{1}{2}\sqrt{n}\right] - 1 = 0.9\,s4$

$p_r\left(Z < \frac{1}{2}\sqrt{n}\right) = \frac{1.9\,54}{2} = 0.9\,77$

$\text{N}(\frac{1}{2}\sqrt{n}) = 0.9\,77$

$\frac{1}{2}\sqrt{n} = 2$ (approximately) from tables $\sqrt{n} = 4 \;\Rightarrow n = 16$

Ex:- let S^2 be the variance of ar.s of size six from N(μ,12) find $p_r(2.30 < S^2 < 22.2)$

Solution :- since $\frac{ns^2}{\delta^2} \sim x^2(n-1)$, then $\frac{6\,s^2}{12} = \frac{s^2}{2} \sim x^2\,(s)$

$p_r(2.30 < s^2 < 22.2) = \;p_r(1.1s \;< \frac{s^2}{2} \;< 11.1)$

$= p_r\left(\frac{s^2}{2} < 11.1\right) - \;p_r(\frac{s^2}{2} < 1.1\,s)$

= 0.9 50-0.050=0.90 (from x^2 tables)

Ex:- let $x_1, x_2, \dots, x_n$ be ar. S. from N(M,δ^2) let $\bar{x}$ and s^2 denote the mean and variance of the sample where $\bar{x}$ has s^2 are in dependent and E(s^2) $= \delta^2$, show that $t = \frac{\sqrt{n}\,(\bar{x}-M)}{s} \sim t(n-1)$ تتوزع بشكل تقربي عندما $n \to \infty$

Proof :- since x$\sim N(M,\delta^2) \Rightarrow \bar{x} \sim N(M,\frac{\delta^2}{n})$

let $Z_1 = \frac{\bar{x}-M}{\delta/\sqrt{n}} \;\Rightarrow Z_1 \sim N(0,1)$

let $\qquad Z_2 = \frac{(n-1)s^2}{\delta^2} \;\Rightarrow Z_2 \sim x^2(n-1)$

$\frac{Z_1}{\sqrt{\frac{Z_2}{n-1}}} \sim t\;(n-1)$

but $\qquad \frac{Z_1}{\sqrt{\frac{Z_2}{n-1}}} = \frac{\frac{\bar{x}-M}{\delta/\sqrt{n}}}{\sqrt{\frac{(n-1)s^2}{(n-1)\delta^2}}} = \frac{\frac{\bar{x}-M}{\delta/\sqrt{n}}}{\frac{\sqrt{s^2}}{\sqrt{\delta^2}}} = \frac{\sqrt{n}\,(\bar{x}-M)}{\delta}\;\frac{\delta}{s}$

t$=\frac{\sqrt{n}\,(\bar{x}-M)}{s} \sim t\;(n-1)$

تمارين التوزيعات

Problems :-

1) Given that $x\sim$Ber $(1,\frac{1}{3})$ find

   (i) the p. m. f of x, (ii)$M_x, \delta_x{}^2$ $and\ \mu_x(t)$

2) The m. g. f of a r.v x is $M_x(t) = (\frac{2}{3}+\frac{1}{3}e^t)^9$ $=(1-p+pe^t)^n$

   (i) find the p.m.f of x (ii) find the mean M_x and variance $\delta_x{}^2$

   (iii)show that $p_r(M_x - 2\delta_x < x < M_x + 2\delta_x) = \sum_{x=1}^{5}(x^9)(\frac{1}{3})^x\,(\frac{2}{3})^{9-x}$

3) Let $x\sim b(2,p)$ and y$\sim b(4,p)$ if $p_r(x^{\sim} \geq 1) = 1 - p_r(x<1) = \frac{5}{9}$ find $p_r(y \geq 1)$

   =1-f(0)=1-$(x^2)px(1-p)^{2-x}$

4) Let x_1, x_2 be independent r. v. s such that $x_1\sim p(4)$ and $x_2\sim p(6)$ let $y = x_1 + x_2$ (i) find the p.m.f of y , (ii) find $M, \delta^2 y$ (iii) find $p_r(y \leq 1)$

5) given that $x\sim p(t)$ find the value of λ it is known that $f(x) = \frac{4}{x}\,f(x-1)$, x=1,2,….

6) Let $x_i\sim Nb(ri,p), i = 1,2,\dots,n$ show that $\sum_{i=1}^{n} x_i\sim Nb(\sum_{i=1}^{n} ri\,, p)$

7) Let $x\sim Nb(4,0.3)$ (i) find the p.m.f of x (ii) find $M_x, \delta x^2, M_x(t)$ (iii)let y=4+5x find $M_y, \delta_y{}^2$

تمارين التوزيعات المستمرة

Problems:-

1) Let $x\sim u(0,1)$, use the transformation method to find the distribution of $y = -2Ln$ x then find $M_y, \delta y^2$

2) Given that $x_1, x_2, \dots, x_n$ are independent random variables where $x_i\sim G(\propto i, \beta), i = 1\,,\dots,n$ show that $y = \sum_{i=1}^{n} x_i\sim G(\sum_{i=1}^{n} \propto i\,, \beta)$

3)

4) If $x_1 \sim G(2,3)$ and $x_2 \sim G(1,3)$ are two independent r.v.s (i) find the distribution of $y = x_1 + x_2$ (ii) find the mean and variance of y

$$y \sim G(3,3)$$

5) Let $x \sim \beta(\propto, \beta)$, let $y = Ln\frac{x}{1-x}$ find $M_y(t)$
6) In each case the r. v. x follow Beta distribution , find the value of the constant c. (i) $f(x) = cx^2(1-x)^5$ (ii) $f(x) = c\,(x - x^2)^{0.5}$

7) If $x \sim \beta\,(\propto, \beta)$. show that $y = (1-x) \sim \beta(\beta, \propto)$ $= c\,(x(1-x))^{0.5}$
8) If $x \sim N(0,1)$ find $E(x^{2k}), k \in \mathrm{I}^+$, then find $E(x^2)$ and $E(x^4$
9) If $x_1, x_2, \dots, x_n$ are independent r.v.s where $x_i \sim N(M_i\,, \delta^2), i = 1\,, 2, \dots, n$ show that $y = \sum_{i=1}^{n} x_i \sim N(\sum_{i=1}^{n} m_i, \sum_{i=1}^{n} {\delta_1}^2)$
10) If $x_1 \sim N\left(M_1, {\delta_1}^2\right), x_2 \sim N(M_2, {\delta_2}^2)$, where x_1, x_2 are independent r.vs show that $y = x_1 - x_2 \sim N(M_1 - M_2, {\delta_1}^2 + {\delta_2}^2)$
11) If $x \sim x^2(n)$ and (x+y) $\sim x^2(n+m)$ where x,y are inderendent r.vs . use the m.g.f to find the distribution of y
12) If $x \sim N(0,2)$ find $E(x^{k/2})$, where k is even positive number , then find $E(x^2)$
13) If the m.g.f of the r.v. x is $M_x(t) = e^{3t+8t^2}$ (i) find the distribution of x (ii) find the mean and variance of

if $\bar{x} \sim N(M, \frac{\delta^2}{n})$, then :- $\bar{x} = \frac{\sum x_i}{n}$

1) $\frac{\bar{x}-M}{\delta/\sqrt{n}} \sim N(0,1)$
2) $(\frac{\bar{x}-M}{\delta/\sqrt{n}})^2 \sim x^2(1)$
3) $\sum_{i=1}^{k}(\frac{\bar{x}-M}{\delta/\sqrt{n}})^2 \sim x^2\,(k)$

$s^2\bar{x}$ تمارين

Problems :- $\bar{x} \sim N(M, \frac{\delta^2}{n})$, 1- Let $\bar{x}$ be the mean of r.v of size (5) from N(0,125) find the value of c if $p_r(\bar{x} < c) = 0.90$, and from tables it is known that N(1.282) = 0.90

Sol:

$p_r[\bar{x} < c] = 0.90$

$= p_r\left[\frac{\bar{x}-M}{\delta/\sqrt{n}} < \frac{c-M}{\delta/\sqrt{n}}\right] = 0.90$ $\sqrt{125} = \sqrt{25.5} = 5\sqrt{5}$

$= p_r[\frac{\bar{x}-0}{5\sqrt{5}/\sqrt{5}} < \frac{c-0}{5\sqrt{5}/\sqrt{5}}\} = 0.90$

$$p_r\left(Z < \frac{c}{5}\right) = 0.9$$

$$N(\frac{c}{5}) = 0.9\ 0$$

N(1.282) = 0.90 $\frac{c}{5} = 1.282$

C=6.1

2- Let $x_1, x_2, \dots, x_n$ be a r.s from G $(\propto, \beta)$. show that $\bar{x} \sim G(\propto n, \frac{\beta}{n})$ then show that $E(\bar{x}) = \propto \beta$, $var\ (\bar{x}) = \frac{\propto\beta^2}{n}$

Sol:

$M_{\bar{x}}(t) = E(e^{t\bar{x}}) = E(e^{\frac{t}{n}(x_1+x_2 \dots + x_n)})$

$$= E(e^{\frac{t}{n}(x_1+x_2 \dots + x_n}$$

$= E\left(e^{\frac{t}{n}(x_1}\right) E\left(e^{\frac{t}{n}(x_2}\right) \dots \dots E(e^{\frac{t}{n}(x_n})$

$(1-\beta\frac{t}{n})^{-\propto}(1-\beta\frac{t}{n})^{-\propto} \dots (1-\beta\frac{t}{n})^{-\propto}$ n-terms

$(1-\beta\frac{t}{n})^{-\propto n}$

$\bar{x} \sim G(\propto n, \frac{\beta}{n})$

$E(\bar{x}) = \propto \beta = . \propto n\frac{\beta}{n} = \propto \beta$

ver $(\bar{x}) = \propto B^2 = \propto \frac{n\beta^2}{n^2} = \frac{\propto\beta^2}{n}$

3- Let $x_1, x_2, \dots, x_n$ be a r.s from G (3,1) . find the p.d.f of $\bar{x}$, then find E($\bar{x}$) and var($\bar{x}$)

Sol:

$$M_{\bar{x}}(t) = E(e^{t\bar{x}}) = E(e^{\frac{t}{n}(x_1+x_2 \dots +x_n)})$$

$$= E(e^{\frac{t}{n}(x_1+x_2 \dots +x_n}$$

$$= E\left(e^{\frac{t}{n}(x_1}\right) E\left(e^{\frac{t}{n}(x_2}\right) \dots \dots E(e^{\frac{t}{n}(x_n})$$

$(1 - 1\frac{t}{n})^{-3}\ (1 - \frac{t}{n})^{-3} \dots\ (1 - \frac{t}{n})^{-3}$ WHERE $(1 - \beta t)^{-\propto}$ n-terms

$(1 - \frac{1}{n}t)^{-3n}$

$\bar{x} \sim G(3n, \frac{1}{n})$

$$f(x) = \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} x^{\propto -1}\, e^{\frac{-x}{\beta}}$$

$$f(\bar{x}) = \frac{1}{\Gamma_{3n} \left(\frac{1}{n}\right) 3n} x^{3\propto -1}\, e^{\frac{-x}{1/n}}$$

E($\bar{x}$) $= \propto \beta =. 3n\frac{1}{n} = 3$

ver ($\bar{x}$) $= \propto B^2$ $= 3n\frac{1}{n^2}$ =3/n

G (5,2) . show that $\bar{x} \sim G(5n, \frac{2}{n})$

(4)Distribution of order statistics

Let $x_1, x_2, \dots, x_n$ denote a r.s from a dist of continuous type having p.d.f (x) which is positive provided $a < x < b$ let y_1 be the smallest of these x_i, y_2 the next x_i in order of magnitude , … and y_n the largest x_i , that is $y_1 < y_2 < \cdots < y_n$ represent

$x_1, x_2, \dots, x_n$ when the latter are arranged in ascending order of magnitude , then $y_i, i = 1,2, \dots n$ is called the i^{th} order statistic of the r.s $x_1, x_2, \dots, x_n$

It can be shown that the joint p.d.f of $y_1, y_2, \dots, y_n$ is $g(y_1, y_2, \dots, y_n) = n!\, f(y_1) f(y_2) \dots f(y_n) \dots .(1)$ the marginal distribution of y_k is

$$\mathrm{g}(y_k) = \frac{n!}{(k-1)!(n-k)!} [F(y_k)]^{k-1} [1 - F(y_k)]^{n-k} f(y_k) \dots .(2) \quad \text{c.d.f}$$

NOTE: F$\dots .\partial \quad \dots .f$

f$\dots\dots\int \quad \dots .F$

The joint p.d.f for any two order statistics y_i, y_j is

$$g(y_i, y_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} [F(y_i)]^{i-1} [F(y_j) - F(y_i)]^{j-i-1} [1 - (y_j)]^{n-j} f(y_i) f(y_j) \qquad \dots (3)$$

Ex:- let $y_1 < y_2 < y_3 < y_4$ denote the order statistics on of a r.s of size 4 from a distribution having p.d.f $f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & o.w \end{cases}$ n=4,,,,k=3

Find the p.d.f of y_3 then find $p_r(\frac{1}{2} < y_3)$

Solution :- Applying formula (2) with n=4 and $k = 3$ we get

$$\mathrm{g}(y_k) = \frac{n!}{(k-1)!(n-k)!} [F(y_k)]^{k-1} [1 - F(y_k)]^{n-k} f(y_k) \dots .$$

$$g(y_3) = \frac{4!}{2! \quad 1!} [F(y_3)]^2 [1 - F(y_3)]^1 f(y_3)$$

Since $\mathrm{F}(y_3 = \int_0^x f(u)du = \int_0^x 2u du = \frac{2u^2}{2} \Big|_0^x = x^2$ then $\mathrm{F}(y_3) = {y_3}^2$

$$g(y_3) = 12[{y_3}^2]^2 [1 - {y_3}^2] 2y_3$$

$$= 24\, {y_3}^4 (1 - {y_3}^2) y_3 = 24 {y_3}^5 (1 - {y_3}^2) \qquad 0 < y_3 < 1$$

$$p_r\left(\frac{1}{2} < y_3\right) = \int_{1/2}^1 24\, {y_3}^5\, (1 - {y_3}^2) d\, y_3$$

$$= \int_{1/2}^1 24\, ({y_3}^5 - {y_3}^7) d\, y_3 = 24[\frac{{y_3}^6}{6} - \frac{{y_3}^8}{8} \Big|_{1/2}^{1} \qquad ===$$

Distribution of functions of order statistic :-

(1) The median : when the observations are arranged in ascending order of magnitude , then

(a) if n is odd , the median is the observation of orders $\frac{n+1}{2}$

(b) if n is even , the median is the average observation of order $\frac{n}{2}, \frac{n+1}{2}$

if x is ar.v with p.d.f f(x) and distribution function f(x), then the value of the median is the value of x that satisfies the equation F(x) $= \frac{1}{2}$

Ex:- let x_1, x_2, x_3 be ar.s from the dist. $f(x) = e^{-x}, x > 0$

(i) find the p.d.f of the smallest value of the sample
(ii) find the joint p.d.f of the largest and smallest value of the sample
(iii) find the p.d.f of the median and the value of the median

solution : let y_1 is the smallest value of the sample followed by y_2 the y_3

$f(x) = e^{-x} \Rightarrow F(x) = \int_0^x f(u)du = \int_0^x e^{-u}du = -e^{-u}\,\Big|_0^x$

$= -e^{-x} + e^0 = 1 - e^{-x}$

from formula (2) with $k = 1$ and n=3

$$g(y_k) = \frac{n!}{(k-1)!(n-k)!}\,[F(y_k)]^{k-1}\,[1 - F(y_k)]^{n-k}\,f(y_k) \ldots .(2)$$

we get

$$g(y_1) = \frac{3!}{0!\;\;2!}\quad [F(y_1)]^0\,[1 - F(y_1)]^2\,f(y_1)$$

$g(y_1) = 3\,(1 - 1 + e^{-y_1})^2 e^{-y_1} = 3\,e^{-2y_1}\,e^{-y_1} \;= 3e^{-3y_1}$

(ii) from formula (3) with i=1, j=3 , n=3

$$g(y_i, y_j) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\,[F(y_i)]^{i-1}[F(y_j)F(y_i)]^{j-i-1}[1F(y_j)]^{n-j} f(y_i)f(y_j) \ldots \quad (3)$$

$$g(y_1,y_3) = \frac{3!}{0!\ \ 1!\ \ 0!}\,[F(y_1)]^0[F(y_3)\quad F(y_1)]^{3-1-1}[1-F(y_3)]^0 f(y_1)f(y_3)$$

$= 6[1\text{-}e^{-y_3} - (1-e^{-y_1})]e^{-y_1}e^{-y_3}$

$= 6[1\text{-}e^{-y_3} - 1 + e^{-y_1}]e^{-y_1}e^{-y_3}$

$= 6(\text{-}e^{-y_3} + e^{-y_1})e^{-y_1}e^{-y_3} = \ 6(e^{-y_3} + e^{-y_1})e^{-(y_1+y_3)} \qquad 0 < y_1 < y_3 < \infty$

(iii)the median is the observation of order $\frac{n+1}{2} = \frac{3+1}{2} = 2$ which is y_2

$g(y_2) = \frac{3!}{1!\ \ 1!}\quad [1-e^{-y_2}]^1\,[1-1+e^{-y_2}]^{3-2}\,f(y_2)$

$g(y_k) = \frac{n!}{(k-1)!(n-k)!}\,[F(y_k)]^{k-1}\,[1-F(y_k)]^{n-k}\,f(y_k)\,....(2)$

$= 6(1-e^{-y_2})\varrho^{-y_2}\varrho^{-y_2} = 6\varrho^{-2y_2}(1-e^{-y_2}) \qquad 0 < y_2 < \infty$

To find the value of y_2 we get $F(y_2) = \frac{1}{2} \Rightarrow 1-e^{-y_2} = \frac{1}{2} \Rightarrow \quad e^{-y_2} = \frac{1}{2} \Rightarrow y_2 = ln2$

The Range :- the range of the sample is the difference between the largest and smallest value of the sample that is $R = y_n - y_1$

Ex:- let x_1, x_2, x_3 be ar.s from B(2,1) and let $y_1 < y_2 < y_3$ be the order statistics of the sample

(a) find the probability distribution of R
(b) find the mean and variance of the distribution

solution :- $x \sim B(2,1)$, $x \sim B(\propto, B)$ $\propto = 2, B=1$

$f(x, \propto, B) = \frac{\Gamma_{\propto+B}}{\Gamma_{\propto}\Gamma_B} x^{\propto-1}(1-x)^{B-1} \qquad 0 < x < 1$

if $\propto = 2$, $B = 1$ then $f(x) = \frac{\Gamma_3}{\Gamma_1\Gamma_2} \; x(1-x)^0 = 2x \qquad 0 < x < 1$

$F(x) = \int_0^x f(u)du = \int_0^x 2udu = \frac{2u^2}{2} \; \Big|_0^x = x^2$

Applying formula (3) we get $g(y_1, y_3) = 6[F(y_3) - F(y_1)]f(y_1)f(y_3)$

$= 6\,[{y_3}^2 - {y_1}^2] \quad (2y_1)\ (2y_3)$

$g(y_1, y_3) \quad = 24 \quad y_1 y_3\, [{y_3}^2 - {y_1}^2] \qquad 0 < y_1 < y_3 < 1$

$R = y_3 - y_1$
$y_1 = y_3 - R$, $Z = y_3$

$y_1 = Z - R$

Let

$\left.\begin{matrix} R = y_3 - y_1 = u_1(y_1, y_3) \\ Z = y_3 = u_2(y_1, y_3) \end{matrix}\right\}$ (1-1) transformation from space of y_1, y_3 to space of R, Z

$\left.\begin{matrix} y_1 = Z - R = {u_1}^{-1}(R, Z) \\ y_3 = Z = {u_2}^{-1}(R, Z) \end{matrix}\right\}$ (1-1) transformation from space of R, Z to space of y_1, y_3

$$J = \begin{vmatrix} \frac{dy_1}{dZ} & \frac{dy_1}{dR} \\ \frac{dy_3}{dZ} & \frac{dy_3}{dR} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

$g(y_1, y_3) \quad = 24 \quad y_1 y_3\, [{y_3}^2 - {y_1}^2]$

g(R , Z)=

$g[u_1^{-1}(R,Z), u_2^{-1}(R,Z)]\ |J|$

$=24[Z^2-(Z-R)^2](Z-R)Z] \qquad 0<R<Z<1$

$=24\ [Z^2-Z^2+2RZ-R^2](Z-R)Z$

$=24\ [2Z^2R-R^2Z]\ (Z-R)$

$=24\ [2Z^3R-R^2Z^2-2Z^2R^2+R^3Z]$

$=24\ R\ [2Z^3-RZ^2-2Z^2R+R^2Z]$

$=24R\ [2Z^3-3Z^2R+R^2Z] \qquad 0<R<Z<1$

h (R)= $\int_R^1 g(R,Z)dZ=\int_R^1 24R\ [2Z^3-\ 3Z^2R+R^2Z]dZ$

$=24R(\frac{2Z^4}{4}-\frac{3Z^3R}{3}+\frac{R^2Z^2}{2}\ |_R^1$

$=24R\ (\frac{1}{2}-R+\frac{R^2}{2}-\frac{1}{2}R^4+R^4-\frac{R^4}{2})$

$=\frac{24}{2}R[1-2R+R^2]=12R(R-1)^2$

$=12\ (1-R)^2R$

$R\sim B\ (2,3) \qquad f(x,\propto,B)=\frac{\Gamma_{\propto+B}}{\Gamma_{\propto}\Gamma_B}\ x^{\propto-1}(1-x)^{B-1} \qquad 0<x<1$

E (R)= $\frac{\propto}{\propto+B}=\frac{2}{5}$

Var (R)= $\frac{\propto B}{(\propto+B)^2(\propto+B+1)}=\frac{6}{25(6)}=\frac{1}{25}$

Ex:- let $y_1<y_2<\cdots<y_5$ denote the order statistics of ar.s. of size 5 from a distribution having p.d.f. $f(x)=e^{-x}, 0<x<\infty$ show that the statistics

$Z_1=y_2, Z_2=y_4-y_2$ are stochastically independent .

Solution :- $F(x)=1-e^{-x}, 0<x<\infty$ and according to formula (2) we get .
$g(y_2,y_4)=\frac{5!}{1!\ \ 1!\ \ !1}(1-e^{-y_2})[1-e^{-y_4}-(1-e^{-y_2})]$
[1-(1-e^{-y_4})]$e^{-y_2-y_4}$

$=5!\,(1-e^{-y_2})[e^{-y_2}-e^{-y_4}]e^{-y_4}\;e^{-y_2-y_4} \qquad 0<y_2<y_4<\infty$

Let the space of y_2, y_4 is $A=\{(\,y_2,y_4): 0<y_2<y_4<\infty\,\}$ the space of Z_1, Z_2 is $\qquad B=\{(Z_1,Z_2): 0<Z_1<\infty, 0<Z_2<\infty\}$

$\left.\begin{matrix} Z_1=y_2=u_1(y_1,y_2) \\ Z_2=y_4-y_2=u_2(y_1,y_2)\end{matrix}\right]$ (1-1) transformation maps A onto B

$\left.\begin{matrix} y_2=Z_1=u_1^{-1}(Z_1,Z_2) \\ y_4=Z_2+Z_1=u_2^{-1}(Z_1,Z_2)\end{matrix}\right]$ (1-1) transformation maps B onto A

$$J=\begin{vmatrix} \frac{\partial y_2}{\partial Z_1} & \frac{\partial y_2}{\partial Z_2} \\ \frac{\partial y_4}{\partial Z_1} & \frac{\partial y_4}{\partial Z_2}\end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1\end{vmatrix} = 1$$

$g(Z_1,Z_2)=g(u_1^{-1}(Z_1,Z_2), u_2^{-1}((Z_1,Z_2)\;|J|$

$=120\,(1\text{-}e^{-Z_1})\left(e^{-Z_1}-e^{-(Z_2+Z_1)}\right)e^{-(Z_2+Z_1)}e^{-Z_1-Z_2-Z_1}$

$=120\,(1\text{-}e^{-Z_1})\;e^{-Z_1}(1-e^{-Z_2})\;e^{-Z_2}e^{-Z_1}e^{-2Z_1}e^{-Z_2}$

$=120\;e^{-4Z_1}(1-e^{-Z_1})e^{-2Z_2}(1-e^{-Z_2}) \qquad ,0<Z_1<\infty \quad , \; 0<Z_2<\infty$

The marginal distribution for each of Z_1, Z_2 are as follows :-

$h(Z_1)=\int_0^\infty g(Z_1,Z_2)d\,Z_2=\int_0^\infty 120\;e^{-4Z_1}(1-e^{-Z_1})\;e^{-2Z_2}(1-e^{-Z_2})dZ_2$

$=120\;e^{-4Z_1}(1-e^{-Z_1})\;\int_0^\infty e^{-2Z_2}(1-e^{-Z_2})dZ_2$

$=120\;e^{-4Z_1}\;(1-e^{-Z_1})\int_0^\infty(e^{-2Z_2}-e^{-3Z_2})\;dZ_2$

$=120\;\;e^{-4Z_1}\;(1-e^{-Z_1})\;(\frac{-1}{2}\;e^{-2Z_2}+\frac{1}{3}\;e^{-3Z_2}\Big|_0^\infty$

$=120\;\;e^{-4Z_1}\;(1-e^{-Z_1})\;(\frac{1}{2}-\;\frac{1}{3})$

$=\;\;120\;\;e^{-4Z_1}\;(1-e^{-Z_1})\;(\frac{3-2}{6})$

$h(Z_1)=20\;\;e^{-4Z_1}\;(1-e^{-Z_1}) \quad 1$

$h(Z_2)=\int_0^\infty g(\,Z_1,Z_2)dZ_1$

$=120\;e^{-2Z_2}\;(1-e^{-Z_2})\int_0^\infty e^{-4Z_1}\;(\,1-e^{-Z_1})dZ_1$

$= 120\ e^{-2Z_2}(1 - e^{-Z_2}) \int_0^\infty e^{-4Z_1} \ - e^{-sZ_1} dZ_1$

$= 120\ e^{-2Z_2}(1 - e^{-Z_2})\ (\frac{-1}{4} e^{-4Z_1} + \frac{1}{5} e^{-5Z_1}\Big|_0^\infty$

$= 120\ e^{-2Z_2}(1 - e^{-Z_2})\ (\frac{1}{4} - \frac{1}{5})$

$= 120\ e^{-2Z_2}(1 - e^{-Z_2})\ \frac{5-4}{20}$

$= 6e^{-2Z_2}(1 - e^{-Z_2}) \qquad 2$

Since $g(Z_1, Z_2) = h(Z_1)h(Z_2)$ then Z_1, Z_2 are stocuastically independent .

Problems :-

(1) let $y_1 < y_2 < y_3 < y_4$ be the orde. Statistics of r.s of size 4 from the dist having p.d.f f(x)=e^{-x}, $0 < x < \infty$, final $p_r(s \leq y_4)$

(2) let x_1, x_2, x_3 be ar.s from f(x)=2x , $0 < x < 1$, compute the probability that the smallest of these xi excecds the median of the distribution .