Chapter Two

1- Estimation Theory

Let $x_1, x_2, ..., x_n$ be a r.s. from a distribution having P.d.f $f(x, \theta)$, where $f(x, \theta)$ is of known from with unknown parameter θ, therefore it have to be estimator from the sample date. Two types of estimation can be done, namely the point estimation and the interval estimation.

Def:

The point estimation of θ is a rule (function) that assigns each element of the sample a value (estimate) of θ denoted as $\hat{\theta} = (x_1, x_2, ..., x_n)$.

Properties of Good Estimator

(1) Unbiasedness:

An estimator $\hat{\theta}$ is said to be unbiased estimator of θ if $\boxed{E(\hat{\theta}) = \theta}$. Otherwise, the estimator is said to be biased.

The value of biased $b(\theta)$ is defined as

$$\boxed{b(\theta) = E(\hat{\theta} - \theta) = \; = E(\hat{\theta}) - \theta}$$

Ex:

Let $x_1, x_2, ..., x_n$ be a r.s from $N(\mu, 1)$ show that $\hat{\theta} = \bar{x}$ is unbiased estimator of μ.

Solution:

We have to show that $E(\bar{X}) = \mu$

$$E(\bar{X}) = E(\frac{\sum x_i}{n}) = \frac{1}{n} E(\sum x_i) = \frac{1}{n} E(x_i)$$

since $x_i \sim N(\mu, 1)$, then $E(x_i) = \mu$

$$E(\bar{x}) = \frac{1}{n} \sum_{i=1}^{n} \mu = \frac{1}{n} n\mu = \mu$$

$\hat{\theta} = \bar{x}$ is unbiased estimator of μ

Ex:

Let $x_1, x_2, ..., x_n$ be a r.s from $N(\mu, \delta^2)$, show that $\frac{1}{n-1} \sum (x_i - \bar{x})^2$ is unbiased est. of δ^2

Solution:

Recalling that $S^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$

$$\sum(x_i - \bar{x})^2 = nS^2 \Rightarrow E(\frac{1}{n-1}\sum(x_i - \bar{x})^2) = \frac{1}{n-1}E(nS^2)$$

$$\frac{n}{n-1}E(S^2) = \frac{n}{n-1}\left[\frac{n-1}{n}\delta^2\right] = \delta^2$$

$\Rightarrow \frac{1}{n-1}\sum(x_i - \bar{x})^2$ is unbiased est. of δ^2.

(2) Mean Square Error متوسط مربعات الخطأ

The mean square error (MSE) of an est. $\hat{\theta}$ is defined as

$$\boxed{\text{MSE}\ (\hat{\theta}) = E\ (\hat{\theta} - \theta)^2 = \text{var}\ (\hat{\theta}) + b^2(\theta)}$$

if $\hat{\theta}$ is unbiased then $b(\theta) = 0$ and MSE ($\hat{\theta}$) = var ($\hat{\theta}$).

The good estimator has MSE as small as possible.

Ex.

Let $x_1, x_2, ..., x_n$ be a r.s from $f(x, \theta) = \theta^x (1-\theta)^{1-x}, x = 0, 1,$ use MSE to compare between the two statistics (estimators) $\bar{x}, x_i$.

Solution:

Since $x_i \sim$ Bernoulli $(1, \theta)$, then $E(x_i) = \theta$, $i = 1, 2, ..., n$, and hence $\varepsilon(x_i) = \theta$

Also, $E(\bar{x}) = E(\frac{\sum x_i}{n}) = (\frac{1}{n})E(\sum x_i) = \frac{1}{n}\sum E(x_i) = \frac{1}{n}\sum\theta$

$$= \frac{1}{n}n\theta = \theta$$

Each of $\bar{x}, x_i$ are unbiased est. of θ.

MSE (x_i) = var $(x_i) = \theta(1- \theta)$

$$\text{MSE}(\bar{x}) = \text{var}\ (\bar{x}) = \text{var}(\frac{\sum x_i}{n}) = \frac{1}{n^2}\ \text{var}\sum x_i = \frac{1}{n^2}\sum \text{var}(x_1)$$

$$= \frac{1}{n^2}\sum\theta(1-\theta) = \frac{1}{n^2}n\theta(1-\theta) = \frac{\theta(1-\theta)}{n}$$

$MSE(\bar{x}) < MSE(x_i)$. This means that $\bar{x}$ is better than x_i.

(3) Consistency الاتساق

$\hat{\theta}$ is consistent est. of θ if

1) $\hat{\theta}$ is unbiased 2) $\lim_{n\to\infty} var(\hat{\theta}) = 0$

Ex.

Let $x_1, x_2, ..., x_n$ be a r.s from $p(\theta)$. Show that $\hat{\theta} = \bar{x}$ is consistent est. of θ.

Solution:

Since $x \sim p(\theta)$, then $f(x,\theta) = \frac{e^{-\theta}\theta^x}{x!}$, $x = 0, 1, 2, ...$

$$E(\hat{\theta}) = E(\bar{x}) = E(\frac{\sum x_i}{n}) = \frac{1}{n}\sum E(x_i) = \frac{1}{n}\sum \theta = \frac{1}{n} n\theta$$

$= \theta \Rightarrow \hat{\theta} = \bar{x}$ is unbiased est. of θ.

$$var(\hat{\theta}) = var(\frac{\sum x_i}{n}) = \frac{1}{n^2}\sum var(x_i) = \frac{1}{n^2}\sum \theta = \frac{n\theta}{n^2} = \frac{\theta}{n}$$

$\lim_{n\to\infty} var(\hat{\theta}) = \lim_{n\to\infty} \frac{\theta}{n} = 0$, the two conditions consistency are satisfied $\Rightarrow \hat{\theta} = \bar{x}$ is consistent est. of θ

(4) Minimum Variance Unbiased Estimate

If a statistic $T = t(x_1, x_2, ..., x_n)$ is such that

1) T is unbiased statistic of θ.

2) It has smallest variance among all the unbiased statistics of θ,

then T is called a minimum variance unbiased estimate (MVUE) of θ.

Ex.

Let y_1 and y_2 be two stochastically independent unbiased statistics for θ. Say the variance of y_1 is twice the variance of y_2. Find the constants k_1 and k_2 so that $k_1y_1 + k_2y_2$ is an unbiased statistic with smallest possible variance for such a linear combination.

Solution:

Since each of y_1, y_2 and $k_1y_1 + k_2y_2$ are unbiased then $E(y_1) = \theta$, $E(y_2) = \theta$,
$E(k_1y_1 + k_2y_2) = \theta$
$k_1E(y_1) + k_2E(y_2) = \theta$ y_1 and y_2 be two stochastically independent
$k_1\theta + k_2\theta = \theta \Rightarrow (k_1 + k_2)\theta = \theta$
$k_1 + k_2 = 1 \Rightarrow$ $k_2 = 1 - k_1$……..1
let $var(y_2) = \delta^2 \Rightarrow var(y_1) = 2\delta^2$
Putting $Q = var(k_1y_1 + k_2y_2)$ then

$$Q = k_1^2\, var(y_1) + k_2^2\, var(y_2)$$

$$= 2k_1^2\delta^2 + (1 - k_1)^2\delta^2$$

$$\frac{\partial Q}{\partial k_1} = 4k_1\delta^2 - 2(1 - k_1)\delta^2 = 0$$

$$4k_1\, \delta^2 - 2\, \delta^2 + 2k_1\, \delta^2 = 0$$

$$\delta^2\, (4k_1 - 2 + 2k_1) = 0$$

$$4k_1 - 2 + 2k_1 = 0 \Rightarrow 6k_1 = 2 \Rightarrow k_1 = \frac{1}{3}$$

$$k_2 = 1 - k_1 = 1 - \frac{1}{3} = \frac{2}{3}$$

(5) Efficiency الكفاءة

Let T be unbiased est. for a parameter θ. Then T is called an efficient estimator of θ iff the variance of T attains the Rao-Cramer lower bound given by:

$$var(T) \geq \frac{1}{nE(\frac{\partial \ln f(x,\theta)}{\partial\theta})^2}$$

it can be shown that:

$$E(\frac{\partial \ln f(x,\theta)}{\partial\theta})^2 = -nE(\frac{\partial^2 \ln f(x,\theta)}{\partial\theta^2})$$

Ex.

Let $x_1, x_2, ..., x_n$ be ar,s from $p(\theta)$, show that $\bar{x}$ is an efficient statistic for θ.

Solution:

$$f(x,\theta) = \frac{e^{-\theta}\theta^x}{x!},\ x = 0, 1, 2, ...$$

$$\ln f(x,\theta) = \ln\frac{e^{-\theta}\theta^x}{x!} = \ln e^{-\theta}\theta^x - \ln x!$$

$$= \ln e^{-\theta} + \ln\theta^x - \ln x! = -\theta + x\ln\theta - \ln x!$$

$$\frac{\partial \ln f(x,\theta)}{\partial\theta} = -1 + \frac{x}{\theta} = \frac{x-\theta}{\theta}$$

$$E\left(\frac{\partial \ln f(x,\theta)}{\partial\theta}\right)^2 = E\left(\frac{x-\theta}{\theta}\right)^2 = \frac{1}{\theta^2}E(x-\theta)^2$$

$$= \frac{1}{\theta^2}E[(x-E(x)]^2 \text{ (because } x \sim p(\theta) \text{ and } E(x)=\theta)$$

$$= \frac{1}{\theta^2}\text{var}(x) = \frac{1}{\theta^2}\ \theta = \frac{1}{\theta}$$

$$R.C.L.B = \frac{1}{n\frac{1}{\theta}} = \frac{\theta}{n}$$

on the other hand we have

$$\text{var}(\bar{x}) = \text{var}\frac{\sum x_i}{n} = \frac{1}{n^2}\sum \text{var}\, x_1 = \frac{1}{n^2}\sum\theta = \frac{1}{n^2}n\theta = \frac{\theta}{n}$$

$$\text{var}(\bar{x}) = R.C.L.B$$

$\bar{x}$ is an efficient statistic for θ

Ex.

Let $x_1, x_2, ..., x_n$ be ar.s from $N(0, \theta)$, show that $\hat{\theta} = \frac{\sum x_i^2}{n}$ is:

1) efficient statistic for θ.
2) consistent statistic for θ.

Solution:

$$1)\ E(\hat{\theta}) = E\left(\frac{\sum x_i^2}{n}\right) = \frac{1}{n}\sum E(x_i^2)$$

$var\ (x) = E(x^2) - [E(x)]^2$

$$= \frac{1}{n}\sum[\text{var}(x_i) + (E(x_i))^2] = \frac{1}{n}\sum(\theta+0)$$

$$= \frac{1}{n}n\theta = \theta$$

$\hat{\theta} = \frac{\sum x_i^2}{n}$ is unbiased statistic for θ

$$f(x,\theta) = \frac{1}{\sqrt{2\pi\theta}}e^{-\frac{x^2}{2\theta}}$$

$$\ln f(x,\theta) = -\frac{1}{2}\ln 2\pi\theta + \ln e^{-\frac{x^2}{2\theta}}$$

$$= -\frac{1}{2}\ln 2\pi \ -\frac{1}{2}\ln\theta + -\frac{x^2}{2\theta}$$

$$\frac{\partial \ln f(x,\theta)}{\partial\theta} = \frac{-1}{2\theta} + \frac{x^2}{2\theta^2}$$

$$\frac{\partial^2 \ln f(x,\theta)}{\partial\theta^2} = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}$$

$$E[\frac{\partial^2 \ln f(x,\theta)}{\partial\theta^2}] = e[\frac{1}{2\theta^2} - \frac{x^2}{\theta^3}]$$

$$= \frac{1}{2\theta^2} - \frac{1}{\theta^3}E(x^2) = \frac{1}{2\theta^2} - \frac{1}{\theta^3}[var(x) + [E(x)]^2]$$

$$= \frac{1}{2\theta^2} - \frac{1}{\theta^3}(\theta + 0) = \frac{1}{2\theta^2} - \frac{1}{\theta^2} = \frac{-1}{2\theta^2}$$

$E(x^2) = var\ (x) + \ [E(x)]^2$

$$R.C.L.B = \frac{1}{-nE[\frac{\partial^2 \ln f(x,\theta)}{\partial\theta^2}]} = \frac{1}{-n(-\frac{1}{2\theta^2})} = \frac{2\theta^2}{n}$$

The derive $var(\hat{\theta}) = var(\sum x_i^2 / n)$ it is known that since $x_i \sim\sim N(0, \theta)$ then

$\frac{x_i}{\sqrt{\theta}} \sim N(0, 1)$ and $\frac{x_i^2}{\theta} \sim x^2(1)$

$$\Rightarrow \frac{\sum x_i^2}{\theta} \sim x^2(n)$$

$$E(\frac{\sum x_i^2}{\theta}) = n,\ var\ (\frac{\sum x_i^2}{\theta}) = 2n$$

$$\frac{1}{\theta^2} var\ (\sum x_i^2) = 2n \Rightarrow var \sum x_i^2 = 2n\theta^2$$

$$var(\hat{\theta}) = var(\frac{\sum x_i^2}{n}) = \frac{1}{n^2} var \sum x_i^2 = \frac{1}{n^2} 2n\theta^2$$

$$var(\hat{\theta}) = \frac{2n\theta^2}{n^2} = \frac{2\theta^2}{n}$$

$$var(\hat{\theta}) = R.C.L.B$$

$\hat{\theta} = \dfrac{\sum x_i^2}{n}$ is an eff. stat. for θ.

2) We proved the first condition of consistent that is $\hat{\theta}$ is unbiased.

$\lim\limits_{n\to\infty} var(\hat{\theta}) = \lim\limits_{n\to\infty} \dfrac{2\theta^2}{n} = 0$, thus the second condition of consistency is satisfied.

$\hat{\theta} = \dfrac{\sum x_i^2}{n}$ is consistent est. for θ.

(6) Sufficiency الكفاية

1) The Fisher Neyman Theorem

Let $x_1, x_2, ..., x_n$ denote ar.s from a dist. that has p.d.f f(x, θ).

Let $y = u(x_1, x_2, ..., x_n)$ be a statistic whose p.d.f is g(y, θ).

Define $L(x_1, x_2, ..., x_n, \theta) = f(x_1, \theta).f(x_2, \theta) ... f(x_n, \theta)$

$$= \prod_{i=1}^{n} f(x_i, \theta)$$

Then $y = u(x_1, x_2, ..., x_n)$ is a sufficient statistic for θ iff:

$L(x_1, x_2, ..., x_n, \theta)/ g(y, \theta). = H(x_1, x_2, ..., x_n)$ does not depend upon θ.

Ex. Let $x_1, x_2, ..., x_n$ denote ar.s from a distribution that has a p.d.f.

$f(x, \theta) = \begin{cases} \theta^x (1-\theta)^{1-x}, & x = 0, 1, \quad 0 < \theta < 1 \\ 0 & \text{o.w} \end{cases}$ show that $y = \sum_{i=1}^{n} x_i$ is a suff. stat. for θ.

Solution:

$L(x_1, x_2, ..., x_n, \theta) = \theta^{x_1} (1-\theta)^{1-x_1} \theta^{x_2} (1-\theta)^{1-x_2} ... \theta^{x_n} (1-\theta)^{1-x_n}$

$L = \theta^{\sum x_i} (1-\theta)^{n-\sum x_i}$

since x_i ~ Bernoulli (1, θ) ⇒ $y = \sum x_i$ ~ binomial (n, θ), since

$M_y(t) = E(e^{ty}) = E(e^{t\sum x_i}) = E(e^{tx_1} e^{tx_2} ... e^{tx_n})$

$= E(e^{tx_1}) E(e^{tx_2}) ... E(e^{tx_n}) = M_{x_1}(t) M_{x_2}(t) ... M_{x_n}(t)$

$$= [1-\theta+\theta e^t][1-\theta+\theta e^t]\ldots[1-\theta+\theta e^t] = [1-\theta+\theta e^t]^n$$

$$\therefore g(y,\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y},\ y = 0, 1, \ldots, n$$

$$\frac{L}{g} = \frac{\theta^{\sum x_i}(1-\theta)^{n-\sum x_i}}{\binom{n}{y}\theta^y(1-\theta)^{n-y}} = \frac{\theta^{\sum x_i}(1-\theta)^{n-\sum x_i}}{\binom{n}{\sum x_i}\theta^{\sum x_i}(1-\theta)^{n-\sum x_i}} = \frac{1}{\binom{n}{\sum x_i}}$$

$H(x_1, x_2, \ldots, x_n)$ does not contain θ.

$\therefore y = \sum x_i$ is a suff. stat. for θ.

Ex. Let $y_1 < y_2 < \ldots < y_n$ denote the order statistics of ar.s $x_1, x_2, \ldots, x_n$ from the dist. that has p.d.f

$$f(x,\theta) = e^{-(x-\theta)} \quad ,\theta < x < \infty \quad ,-\infty < \theta < \infty$$

show that y_1 is a suff. stat. for θ.

Solution:

$$L(x_1, x_2, \ldots, x_n, \theta) = e^{-(x_1-\theta)}e^{-(x_2-\theta)}\ldots e^{-(x_n-\theta)}$$

$$= e^{-\sum(x_i-\theta)} = e^{-\sum x_i + \sum\theta} = e^{-\sum x_i + n\theta}$$

$$g(y_k) = \frac{n!}{(k-1)!(n-k)!}[F(y_k)]^{k-1}[1-F(y_k)]^{n-k} f(y_k) \quad k=1$$

$$g(y,\theta) = \frac{n!}{(1-1)!(n-1)!}[F(y_1)]^{1-1}[1-F(y_1)]^{n-1} f(y_1)$$

$$F(x) = \int_\theta^x e^{-(u-\theta)}du = -e^{-(u-\theta)}\Big|_\theta^x = -e^{-(x-\theta)} + e^{-(\theta-\theta)}$$

$$= -e^{-(x-\theta)} + 1 = 1 - e^{-(x-\theta)}$$

$$F(y_1) = 1 - e^{-(y_1-\theta)}$$

$$g(y_1,\theta) = \frac{n(n-1)!}{(n-1)!}[1-(1-e^{-(y_1-\theta)}]^{n-1}e^{-(y_1-\theta)} = ne^{-n(y-\theta)}$$

$$g(y_1,\theta) = ne^{-n(y-\theta)} \qquad \theta < y_1 < \infty$$

[from formula 2 of order stat.] *let* y_1 be the smallest of these x_i y_2 the next x_i in order of magnitude , … and y_n the largest x_i $y_1 = \min x_i$

$$g(y_1,\theta) = ne^{-n(\min x_i)+n\theta}$$

$$\frac{L}{g} = \frac{e^{-\sum x_i} \cdot e^{n\theta}}{n\, e^{-n(\min xi)} e^{n\theta}} = \frac{e^{-\sum x_i}}{n\, e^{-n(\min xi)}} = H(x_1, x_2, ..., x_n)$$

(does not contain θ)

$y_1 = \min(x_1, x_2, ..., x_n)$ is a suff. stat. for θ.

2) The Factorization Theorem

Let $x_1, x_2, ..., x_n$ denote ar.s from a distribution that has p.d.f. $f(x, \theta)$. The statistic $T = t(x_1, x_2, ..., x_n)$ is a sufficient statistic for θ iff we can find two non negative functions k_1 and k_2 such that:

$$\boxed{L(x_1, x_2, ..., x_n, \theta) = k_1(T, \theta)\, k_2(x_1, x_2, ..., x_n)}$$

where $k_2(x_1, x_2, ..., x_n)$ does not depend on θ

يقال عن المقدار T أنه suff. للمعلمة θ إذا كان بالإمكان كتابة L على شكل حاصل ضرب دالتين إحداهما تحوي التقدير والمعلمة $k_1(T, \theta)$ والأخرى خالية من المعلمة θ تماماً $k_2(x_1, x_2, ..., x_n)$

Ex.

Let $x_1, x_2, ..., x_n$ denote ar.s from a dist. which is $N(\theta, \delta^2)$, $\infty - < x < \infty$, where the variance δ^2 is known. Show that $\bar{x} = \frac{\sum x_i}{n}$ is suff. stat. for θ.

Solution:

$x \sim N(\theta, \delta^2)$

$$\therefore f(x, \theta) = \frac{1}{\sqrt{2\pi\delta^2}} e^{\frac{-(x-\theta)^2}{2\delta^2}}, \qquad -\infty < x < \infty$$

$$L(x_1, x_2, ..., x_n, \theta) = \frac{1}{\sqrt{2\pi\delta^2}} e^{\frac{-(x_1-\theta)^2}{2\delta^2}} \frac{1}{\sqrt{2\pi\delta^2}} e^{\frac{-(x_2-\theta)^2}{2\delta^2}} \cdots \frac{1}{\sqrt{2\pi\delta^2}} e^{\frac{-(x_n-\theta)^2}{2\delta^2}}$$

$$L(x_1, x_2, ..., x_n, \theta) = (2\pi\delta^2)^{-n/2} e^{\frac{-\sum (x_i-\theta)^2}{2\delta^2}}$$

$$= (2\pi\delta^2)^{-n/2} e^{\frac{-\sum [(x_i-\bar{x})+(\bar{x}-\theta)]^2}{2\delta^2}}$$

$$= (2\pi\delta^2)^{-n/2} e^{\frac{-\sum [(x_i-\bar{x})^2+(\bar{x}-\theta)^2+2(x_i-\bar{x})(\bar{x}-\theta)]}{2\delta^2}}$$

$$= (2\pi\delta^2)^{-n/2} e^{\frac{-\sum(x_i-\bar{x})^2}{2\delta^2} + \frac{n(\bar{x}-\theta)^2}{2\delta^2}}$$

$$= (2\pi\delta^2)^{-n/2} e^{\frac{-n}{2\delta^2}(\bar{x}-\theta)^2} e^{\frac{-\sum(x_i-\bar{x})^2}{2\delta^2}}$$

$\therefore \bar{x}$ is a suff. stat. for θ.

Ex. Let $x_1, x_2, ..., x_n$ denote ar.s from a dist. $f(x, \theta) = \theta x^{\theta-1}$, $0 < x < 1$. Show that the product $T(x_1, x_2, ..., x_n) = x_1 . x_2 . ... x_n$ is a suff. stat. for θ.

Solution:

$$L(x_1, x_2, ..., x_n, \theta) = (\theta x_1^{\theta-1})(\theta x_2^{\theta-1})...(\theta x_n^{\theta-1}) =$$

$$= \theta^n (x_1 . x_2 ..... x_n)^{\theta-1}$$

$$= \theta^n (x_1 . x_2 ..... x_n)^{\theta} . (x_1 . x_2 ..... x_n)^{-1} .$$

$$= \theta^n (x_1 . x_2 ..... x_n)^{\theta} . \frac{1}{(x_1 . x_2 ..... x_n)} .$$

The product $x_1, x_2, ..., x_n$ is a suff. stat. for θ

Problems

(1) Let $x_1, x_2, ..., x_n$ be ar.s from

$$f(x, \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & 0 < x < \infty, 0 < \theta < \infty \\ 0 & \text{o.w} \end{cases}$$

Show that $\bar{x}$ is unbiased statistic for θ.

(2) Let $y_1 < y_2 < y_3$ be the order statistics of ar.s of size 3 from the uniform dist. having p.d.f. $f(x, \theta) = \frac{1}{\theta}$, $0 < x < \theta$, $0 < \theta < \infty$. Show that $4y_1$, $2y_2$ and $\frac{4}{3} y_3$ are all unbiased statistics for θ. Find the variance of each of these unbiased statistics.

(3) Let $x_1, x_2, ..., x_n$ be ar.s from P(θ). Show that $\sum x_i$ is a suff. stat. for θ.

(4) Show that the nth order statistic of ar.s of size n from the uniform dist. having p.d.f. $f(x, \theta) = \frac{1}{\theta}$, $0 < x < \theta$, $0 < \theta < \infty$. is a suff. statistic for θ.

2- - Methods of Estimator

2-1 The maximum likelihood Method

Def. Let $x_1, x_2, ..., x_n$ be a r.s from a distribution having p.d.f. $f(x, \theta)$ then:

1) The joint function $f(x_1, \theta).\ f(x_2, \theta)...\ f(x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta)$ is called the likelihood function denoted as $L(x_1, x_2, ..., x_n, \theta)$.

2) Let $\hat{\theta}$ be the value of θ that maximize L. Thus, $\hat{\theta}$ is the root of the equation

$$\frac{\partial L}{\partial \theta} = 0$$

such that $\frac{\partial^2 L}{\partial^2 \theta} = 0$ and it is called the maximum likelihood estimate (MLE) for θ.

3) The value of θ that maximize L, maximize ln L also. Thus $\hat{\theta}$ may be regard as a solution of $\frac{\partial \ln L}{\partial \theta} = 0$, such that $\frac{\partial^2 \ln L}{\partial \theta^2} < 0$.

The following assumptions have to be done:

1) The first and second partial derivatives are continuous function of θ.

2) The range of the r.v x does not depend upon θ.

Properties of MLE

1) MLE are consistent estimators.

2) If MLE exist then it is the most efficient in the class of such estimators.

3) If $\hat{\theta}$ is MLE for θ and $g(\theta)$ is the single valued function of θ, then $g(\hat{\theta})$ is the MLE for $g(\theta)$. This is called the invariance property.

Ex.1

Let $x_1, x_2, ..., x_n$ be a r.s from the distribution having p.d.f

$$f(x, \theta) = \begin{cases} \theta x^{\theta - 1} & 0 < x < 1 \\ 0 & o.w \end{cases}$$

Fine the MLE for θ.

Solution:

$$L(x_1, x_2, ..., x_n, \theta) = (\theta x_1^{\theta-1})(\theta x_2^{\theta-1})...(\theta x_n^{\theta-1})$$

$$= \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

$$\ln L = \ln \theta^n \prod_{i=1}^{n} x_i^{\theta-1} = \ln \theta^n + \ln \prod_{i=1}^{n} x_i^{\theta-1}$$

$$= n \ln \theta + (\theta - 1) \ln \prod_{i=1}^{n} x_i$$ by pro. $\ln(x1.x2) = lnx1 + lnx2$

$$= n \ln \theta + (\theta - 1) \sum_{i=1}^{n} \ln x_i == n \ln \theta + \theta \sum_{i=1}^{n} \ln x_i - 1 \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \sum \ln x_i = 0 \Rightarrow \frac{n}{\theta} = -\sum \ln x_i$$

$\theta^{\wedge} = \dfrac{-n}{\sum \ln x_i}$ is the MLE for θ.

Ex.2

Let $x_1, x_2, ..., x_n$ be a r.s from $N(\mu, \delta^2)$, use the MLE method to estimate μ and δ^2.

Solution:

$$f(x, \mu, \delta^2) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(x-\mu)^2}{2\delta^2}}, -\infty < x < \infty$$

$$L(x_1, x_2, ..., x_n, \mu, \delta^2) = \prod_{i=1}^{n} f(x_i, \mu, \delta^2)$$

$$L = (2\pi\delta^2)^{-\frac{n}{2}} e^{\frac{-\sum_{i=1}^{n}(x_i-\mu)^2}{2\delta^2}}$$

$$\ln L = \frac{-n}{2} \ln(2\pi) - \frac{n}{2} \ln(\delta^2) - \frac{1}{2\delta^2} \sum (x_i - \mu)^2$$ -------------2

$$\frac{\partial \ln L}{\partial \mu} = \frac{1}{\delta^2} \sum (x_i - \mu) = 0 \Rightarrow$$

$$\sum (x_i - \mu) = 0$$

$\sum x_i - n\mu = 0 \Rightarrow \hat{\mu} = \dfrac{\sum x_i}{n} = \bar{x}$ is the MLE for μ.

$$\frac{\partial \ln L}{\partial \delta^2} = \frac{-n}{2\delta^2} + \frac{1}{2\delta^4} \sum (x_i - \hat{\mu})^2 = 0$$

$$\frac{n}{2\delta^2} = \frac{1}{2\delta^4}\sum(x_i - \hat{\mu})^2$$

$$\frac{2\delta^4}{2\delta^2} = \frac{\sum(x_i - \hat{\mu})^2}{n}$$

$$\delta^2 = \frac{\sum(x_i - \hat{\mu})^2}{n} = \frac{\sum(x_i - \bar{x})^2}{n}$$

$$\hat{\delta} = \sqrt{\frac{\sum(x_i - \hat{\mu})^2}{n}} = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}}$$

Ex.3

Let $x_1, x_2, ..., x_n$ be a r.s drawn from

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} & 0 < x \le \theta, 0 < \theta < \infty \\ 0 & \text{o.w} \end{cases}$$

Find the MLE for θ.

Solution:

$$L(x_1, x_2, ..., x_n, \theta) = (\frac{1}{\theta})(\frac{1}{\theta})...(\frac{1}{\theta}) = \frac{1}{\theta^n}$$

Ln1-n ln Θ , -n/Θ=0

We can't use the differentiation method because the range of x depend upon θ, but it is clear that L has maximum value at the smallest value of θ, which coincide with the maximum value of x. Hence, $\hat{\theta} = \max(x_i) =$ the largest order statistic of the sample.

Ex.4

Let $x_1, x_2, ..., x_n$ be a r.s from a distribution having p.d.f.

$$f(x,\alpha,\beta) = \begin{cases} \beta e^{-\beta(x-\alpha)} & \alpha \le x \le 0, \beta \ge 0 \\ 0 & ow \end{cases}$$

Find the MLE for α, β.

Solution:

$$L(x_1, x_2, ..., x_n, \alpha, \beta) = \beta^n e^{-\beta\sum_{i=1}^{n}(x_i - \alpha)}$$

The MLE for α can't be found by the method of differentiation since the range of x depend upon α.

It is clear that L has maximum value at the largest value of α which coincide with the smallest value of x. Hence, $\hat{\alpha} = \min(x_i) =$ the smallest order statistic of the sample.
To find $\hat{\beta}$ we can use the differentiation method as follows:

$$L(x_1, x_2, ..., x_n, \alpha, \beta) = \beta^n e^{-\beta\sum_{i=1}^{n}(x_i - \alpha)}$$

$$\ln L = \ln \beta^n e^{-\beta\sum_{i=1}^{n}(x_i - \hat{\alpha})}$$

$$= \ln \beta^n - \beta\sum(x_i - \hat{\alpha})$$

$$= n \ln \beta - \beta\sum(x_i - \hat{\alpha})$$

$$= n \ln \beta - \beta\sum(x_i - \min(x_i))$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} - \sum(x_i - \min(x_i)) = 0$$

$$\frac{n}{\beta} = \sum(x_i - \min(x_i))$$

$$\hat{\beta} = \frac{n}{\sum(x_i - \min(x_i))} = \frac{n}{\sum x_i - \sum \min(x_i)} = \frac{n}{n\bar{x} - n \min(x_i)}$$

$$= \frac{1}{\bar{x} - \min(x_i)}$$

Ex.5

Let $x_1, x_2, ..., x_n$ be a r.s from a dist. Having p.d.f.

$$f(x, \theta) = \begin{cases} \theta^x (1-\theta)^{1-x} & x = 0, 1 \\ 0 & \text{o.w} \end{cases}$$

Find the MLE for $w = \dfrac{\theta}{1-\theta}$

Solution:

At the first we find MLE for θ

$$L(x_1, x_2, ..., x_n, \theta) = f(x_1, \theta) f(x_2, \theta) ... f(x_n, \theta)$$

$$= \theta^{x_1}(1-\theta)^{1-x_1}\ \theta^{x_2}(1-\theta)^{1-x_2} \ ...\ \theta^{x_n}(1-\theta)^{1-x_n}$$

$$= \theta^{\sum x_i}(1-\theta)^{\sum(1-x_i)} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$\ln L = \ln(\theta^{\sum x_i}(1-\theta)^{n-\sum x_i}) = \ln \theta^{\sum x_i} + \ln(1-\theta)^{n-\sum x_i}$$

$$= \sum x_i \ln \theta + (n - \sum x_i) \ln (1-\theta)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{(1-\theta)} = 0$$

$$\frac{\sum x_i}{\theta} = \frac{n - \sum x_i}{(1-\theta)} \Rightarrow$$

$$\frac{1-\theta}{\theta} = \frac{n - \sum x_i}{\sum x_i}$$

$$\frac{1}{\theta} - 1 = \frac{n}{\sum x_i} - 1 \Rightarrow \hat{\theta} = \frac{\sum x_i}{n} = \overline{x}$$

$$\hat{w} = \frac{\hat{\theta}}{1-\hat{\theta}} = \frac{\overline{x}}{1-\overline{x}} \text{ is the MLE for w}$$

(according to the invariance property)

Ex.6

Eight trials are conducted of a given system with the following results (S, F, S, F, S, S, S, S) where S denote success and F denote failure. Find the MLE of P the probability of the successful events.

Solution:

Let the r.v. x denote the success event, then

$$x = \begin{cases} 1 & \text{if the event S occur} \\ 0 & \text{if the event S does not occur} \end{cases}$$

X ~ Ber (1, P) then $f(x) = P^x(1-P)^{1-x}$, x = 0, 1

$$L = f(x_1, p) \ldots f(x_n, p) = p^{x_1}(1-p)^{1-x_1} \ldots p^{x_n}(1-p)^{1-x_n}$$

$$= p^{\sum x_i}(1-p)^{n-\sum x_i} = p^6(1-p)^{8-6} = p^6(1-p)^2$$

$$\ln L = \ln (p^6(1-p)^2) = 6 \ln p + 2 \ln (1-p)$$

$$\frac{\partial \ln L}{\partial p} = \frac{6}{p} - \frac{2}{1-p} = 0$$

$$\frac{6}{p} = \frac{2}{1-p} \Rightarrow \frac{1-p}{p} = \frac{2}{6} \Rightarrow \frac{1}{p} - 1 = \frac{2}{6}$$

$$\frac{1}{p} = 1 + \frac{2}{6} = \frac{8}{6} \Rightarrow \hat{p} = \frac{6}{8} = \frac{3}{4} \quad \text{is the MLE of p.}$$

2- **The Moments Method**

Let $f(x,\ \theta_1,\ \theta_2,\ \ldots,\ \theta_n)$be the p.d.f of the population with k parameters θ_1, $\theta_2,\ \ldots,\ \theta_n$. By this method, we equate the population moments $M_r = E(x^r)$ with the sample moments $m_r = \frac{1}{n}\sum_1^n x^r \quad r = 1,2,\ldots,k$. Then solving for the unknown parameters

$m_1 = \frac{\sum_1^n x_i}{n} = \bar{X}$ r=1

Ex.

Let $x_1, x_2, \ldots, x_n$ be ar.s from p(θ). Find the moment estimator for θ.

Solution:

f(x) $=\frac{e^{-\theta}\ \theta^x}{x!}$ $E(x) = \theta$, var (x) = θ

We have the population moment $M_1 = E(x^1) = E(x) = \theta$

and the sample moment

M$=E(x) = \theta$, $m = \bar{X} = \frac{\sum_1^n x_i}{n}$

$M = m \Rightarrow \theta = \bar{X}$

$\hat{\theta} = \bar{X}$ is the moment est. for θ

Ex.

Let $x_1, x_2, \ldots, x_n$ be ar.s from $f(x,\theta) = \frac{1}{\theta} \quad 0 < x \le \theta$

Find the moment est. for θ.

Solution:

$M_1 = E(x) = \int_0^\theta x\, f(x)\, dx = \int_0^\theta x\, \frac{1}{\theta}\, dx = \frac{1}{\theta}\frac{x^2}{2}\Big]_0^\theta = \frac{1}{\theta}\left[\frac{\theta^2}{2} - 0\right] = \frac{1}{\theta}\frac{\theta^2}{2} = \frac{\theta}{2}$

$$m_1 = \frac{1}{n}\sum_1^n x_i = \bar{X}$$

$M = m$ then $\frac{\theta}{2} = \bar{X}$ then $\hat{\theta} = 2\,\bar{X}$

is the moment est. for θ

Ex.

Let $x_1, x_2, \ldots, x_n$ be ar.s from U($\propto, \beta$)

Find the moment est. for $\propto$ and β.

Solution:

$$f(x,\propto,\beta) = \begin{cases} \frac{1}{\beta-\propto} & ,\propto < x < \beta \\ 0 & 0.w \end{cases}$$

$$E(x) = \frac{\propto+\beta}{2}$$

$$var\ (x) = \frac{(\beta-\propto)^2}{12}$$

$M_1 = E(x) = \frac{\propto+\beta}{2}$, $m1 = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X}$ $\qquad M_r = E(x^r)$,

$$m_r = \frac{1}{n}\sum_{1}^{n} x^r \quad r = 1,2,\dots, \quad k$$

$M1 = m\,1 \quad ===\Rightarrow \quad \frac{\propto+\beta}{2} = \bar{X} - - - - (1)$

$$var\ (x) = E(x^2) - [E(x)]^2$$

$M2 = E(x^2) = var\ (x) + [E(x)]^2 \qquad E(x^2) = var\ (x) + [E(x)]^2$

$$= \frac{(\beta-\propto)^2}{12} + \left(\frac{\propto+\beta}{2}\right)^2$$

$m_2 = \frac{\sum_{i=1}^{n} x^2{}_i}{n}$

$$\frac{\propto+\beta}{2} = \bar{X} \Rightarrow$$

$$\propto+\beta = 2\bar{X} \implies$$

$$\beta = 2\bar{X} - \propto$$

$M_2 = m_2$

$\frac{(\beta-\propto)^2}{12} + \left(\frac{\propto+\beta}{2}\right)^2 = \frac{\sum_{i=1}^{n} x^2{}_i}{n}$ --------(2)

By (1) $\Rightarrow \frac{(2\bar{X} - \propto - \propto)^2}{12} + \bar{X}^2 = \frac{1}{n}\sum_{i=1}^{n} x^2{}_i$

$\frac{(2\bar{X} - 2\propto)^2}{12} + \bar{X}^2 = \frac{1}{n}\sum_{i=1}^{n} x^2{}_i$

$\frac{4(\bar{X} - \propto)^2}{12} + \bar{X}^2 = \frac{1}{n}\sum_{i=1}^{n} x^2{}_i$

$\frac{(\bar{X} - \propto)^2}{3} + \bar{X}^2 = \frac{1}{n}\sum_{i=1}^{n} x^2{}_i$

$$\frac{\propto+\beta}{2} = \bar{X} \Rightarrow$$

$$\propto = 2\bar{X} - \beta$$

$(\bar{X} - \propto)^2 + 3\bar{X}^2 = \frac{3}{n}\sum_{i=1}^{n} x^2{}_i$

$(\bar{X} - \propto)^2 = \frac{3}{n}\sum_{i=1}^{n} x^2{}_i - 3\bar{X}^2$

$\bar{X} - \propto = \sqrt{\frac{3}{n}\sum_{i=1}^{n} x^2{}_i - 3\bar{X}^2}$

$\hat{\propto} = \bar{X} - \sqrt{\frac{3}{n}\sum_{i=1}^{n} x^2{}_i - 3\bar{X}^2}$

$\beta = 2\bar{X} - \propto$

$\hat{\beta} = 2\bar{X} - (\bar{X} - \sqrt{\frac{3}{n}\sum_{i=1}^{n} x^2{}_i - 3\bar{X}^2}\ )$u

$\hat{\beta}$ =$2\bar{X} - \bar{X} + \sqrt{\frac{3}{n}\sum_{i=1}^{n} x^2{}_i \quad - \quad 3\,\bar{X}^2)}$

$\hat{\beta}$ =$\bar{X} + \sqrt{\frac{3}{n}\sum_{i=1}^{n} x^2{}_i \quad - \quad 3\,\bar{X}^2}$

Exercises:

1) Let $x_1, x_2, ..., x_n$ be ar.s from P(θ). Find the MLE for Pr(x>0) .
2) Let $x_1, x_2, ..., x_n$ be ar.s from

$$f(x, \theta_1\ \theta_2) = \begin{cases} \frac{1}{\theta_2}\ e^{-\frac{(x-\theta_1)}{\theta_2}} & ,\theta_1 \le x \le \infty, \quad -\infty < \theta_1 < \infty\ , \\ & 0 < \ \theta_2 < \infty \\ 0 & 0.\text{w} \end{cases}$$

Find the MLE for θ_1 and θ_2 .
3) Let $x_1, x_2, ..., x_n$ be ar.s from $N(\ \mu, \delta^2\ )$. Find the moment est. for μ and δ^2.
4) Let $x_1, x_2, ..., x_n$ be ar.s from $G(\ \alpha, \beta\ )$, Find the moment est. for α and β.

3-- **The Method of Least Squares**

Suppose that we can write the observations in the form:

$y_i = g_i\ (\theta_1, \theta_2, ..., \theta_k) + \varepsilon_i$, i = 1, 2, ..., n

where g_i are known functions and the real numbers $\theta_1, \theta_2, ..., \theta_k$ are the unknown parameters of interest. Suppose that ε_i satisfy the conditions:

(*) $E(\varepsilon_i) = 0$, $var(\varepsilon_i) = \delta^2 > 0$, $cov(\varepsilon_i, \varepsilon_j) = 0$, i = 1, 2, .., n, j = 1, 2, ..., n

The method of least squares says that we should find the point

$\theta^n = (\theta_1^n, \theta_2^n, ..., \theta_k^n)$, which makes the expected value vector as close as possible to the observes value that is we should minimize.

$$\boxed{\sum_{i=1}^{n}[y_i - E(y_i)]^2}$$

Ex.

Let $y_i = \theta_1 + \varepsilon_i$, i = 1, 2, ..., n. Estimate θ_1 using LS method.

Solution:

We have $E(y_i) = E(\theta_1 + \varepsilon_i)$

$= E(\theta_1) + E(\varepsilon_i) = \theta_1$

Let $Q = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} [y_i - E(y_i)]^2 = \sum_{i=1}^{n} (y_i - \theta_i)^2$

$$\frac{\partial Q}{\partial \theta_1} = -2\sum (y_i - \theta_1) = 0 \Rightarrow \sum (y_i - \theta_1) = 0$$

$$\sum y_i - n\theta_1 = 0 \Rightarrow \sum y_i = n\theta_1 \Rightarrow \theta_1^n = \frac{\sum y_i}{n} = \bar{y}$$

Is the LS est. for θ_1

Ex.

Let x_1, x_2, x_3 be three random variables with the same variance δ^2. Let $E(x_1) = \theta_1$, $E(x_2) = \theta_1 + \theta_2$ and $E(x_3) = 2\theta_1 + \theta_2$. Find the LS estimators for θ_1 and θ_2, then find the mean and variance for each est.

Solution:

Let $Q = \sum_{i=1}^{3} [x_i - E(x_i)]^2$

$= [x_1 - E(x_1)]^2 + [x_2 - E(x_2)]^2 + [x_3 - E(x_3)]^2$

$= (x_1 - \theta_1)^2 + (x_2 - \theta_1 - \theta_2)^2 + (x_3 - 2\theta_1 - \theta_2)^2$

$$\frac{\partial Q}{\partial \theta_1} = 0 \Rightarrow -2(x_1 - \theta_1) - 2(x_2 - \theta_1 - \theta_2) - 4\,(x_3 - 2\theta_1 - \theta_2) = 0$$

$$-2x_1 + 2\theta_1 - 2x_2 + 2\theta_1 + 2\theta_2 - 4x_3 + 8\theta_1 + 4\theta_2 = 0$$

$$-x_1 + \theta_1 - x_2 + \theta_1 + \theta_2 - 2x_3 + 4\theta_1 + 2\theta_2 = 0$$

$$6\theta_1 + 3\theta_2 = x_1 + x_2 + 2x_3 \text{ .... (1)}$$

$$\frac{\partial Q}{\partial \theta_2} = 0 \Rightarrow -2(x_2 - \theta_1 - \theta_2) - 2\,(x_3 - 2\theta_1 - \theta_2) = 0$$

$$x_2 - \theta_1 - \theta_2 + x_3 - 2\theta_1 - \theta_2 = 0$$

$$-3\theta_1 - 2\theta_2 + x_2 + x_3 = 0$$

$$3\theta_1 + 2\theta_2 = x_2 + x_3 \text{ .... (2)} \Rightarrow \theta_1 = \frac{x_2 + x_3 - 2\theta_2}{3} \text{ .... (*)}$$

Solving eq. (1), eq. (2) for θ_1, θ_2, we get:

By eq. (1) we get

$$6\left(\frac{x_2 + x_3 - 2\theta_2}{3}\right) + 3\theta_2 = x_1 + x_2 + 2x_3$$

$$2x_2 + 2x_3 - 4\theta_2 + 3\theta_2 = x_1 + x_2 + 2x_3$$

$$2x_2 - x_1 - x_2 = \theta_2$$

$$\boxed{\hat{\theta}_2 = x_2 - x_1}$$

نعوضها في (*)

$$\theta_1 = \frac{2x_1 - x_2 + x_3}{3}$$

$$\boxed{\hat{\theta}_1 = \frac{2x_1 - x_2 + x_3}{3}}$$

$\therefore \hat{\theta}_1, \hat{\theta}_2$ are the LS est. for θ_1, θ_2

Ex.

Let $y_1 = \beta_0 + \beta_1 x_i + \varepsilon_i$, $i = 1, 2, ..., n$ (simple linear regression model).

Estimate β_0, β_1 using LS method.

Solution:

Let $Q = \sum G_1^2 = \sum [y_i - E(y_i)]^2$

$$= \sum (y_i - \beta_0 - \beta_0 x_i)^2$$

$$\frac{\partial Q}{\partial \beta_0} = 0 \Rightarrow -2\sum (y_i - \beta_0 - \beta_0 x_i) = 0$$

$$\sum (y_i - \beta_0 - \beta_0 x_i) = 0 \text{ .... (1)}$$

$$\frac{\partial Q}{\partial \beta_1} = 0 \Rightarrow -2\sum x_i (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\sum x_i (y_i - \beta_0 - \beta_1 x_i) = 0 \text{ .... (2)}$$

From eq. (1) we obtain

$$\sum y_i - n\beta_0 - \beta_1 \sum x_i \Rightarrow \sum y_i = n\beta_0 + \beta_1 \sum x_i \text{ .... (3)}$$

From eq. (2) we obtain

$$\sum x_i y_i - \beta_0 \sum x_i - \beta_1 \sum x_i^2 \Rightarrow \sum x_i y_i = \beta_0 \sum x_i + \beta_1 \sum x_i^2 \text{ .... (4)}$$

By (3) we get $\boxed{\beta_0 = \frac{\sum y_i - \beta_1 \sum x_i}{n}}$ .... (*)

تعوض في (4)

$$\sum x_i y_i = \sum x_i \left(\frac{\sum y_i - \beta_1 \sum x_i}{n}\right) + \beta_1 \sum x_i^2$$

$$= \frac{\sum x_i \sum y_i - \beta_1 (\sum x_i)^2}{n} + \beta_1 \sum x_i^2$$

$$n\sum x_i \sum y_i = \sum x_i \sum y_i - \beta_1 (\sum x_i)^2 + n\beta_1 \sum x_i^2$$

$$\beta_1 (n\sum x_i^2 - (\sum (x_i))^2) = n\sum x_i y_i - \sum y_i \sum x_i$$

$$\boxed{\hat{\beta}_1 = \frac{n\sum x_i y_i - \sum y_i \sum x_i}{n\sum x_i^2 - (\sum x_i)^2}}$$

تعوض في (*)

$$\beta_0 = \sum y_i - (\frac{n\sum x_i y_i - \sum y_i \sum x_i}{n\sum x_i^2 - (\sum x_i)^2})\sum x_i$$

$$n\beta_0 = \frac{n\sum y_i \sum x_i^2 - \sum y_i(\sum(x_i))^2 - n\sum x_i y_i \sum x_i + \sum y_i (\sum x_i)^2}{n\sum x_i^2 - (\sum x_i)^2}$$

$$\boxed{\hat{\beta}_0 = \frac{\sum y_i \sum x_i^2 - \sum x_i y_i \sum x_i}{n\sum x_i^2 - (\sum x_i)^2}}$$

$\therefore \hat{\beta}_0, \hat{\beta}_1$ are the LS estimators for β_0, β_1.

Ex.

For the simple linear regression model show that $\hat{\beta}_1$ can be written as:

$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2}$, then show that $\hat{\beta}_1$ is unbiased est.

for β.

Solution:

We have $\sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \bar{x}\sum y_i - \bar{y}\sum x_i + n\bar{x} - n\bar{x}\bar{y} - n\bar{y}\bar{x}$

$$= \sum x_i y_i - n\bar{x}\bar{y} = \sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}$$

$$\sum (x_i - \bar{x})^2 = \sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2 = \sum x_i^2 - n\bar{x}^2$$

$$= \sum x_i^2 - \frac{(\sum x_i)^2}{n}$$

$$\frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sum(x_i-\bar{x})^2}=\frac{\sum x_i y_i-\frac{(\sum x_i)(\sum y_i)}{n}}{\sum x_i^2-\frac{(\sum x_i)^2}{n}}$$

$$=\frac{n\sum x_i y_i-(\sum x_i)(\sum y_i)}{n\sum x_i^2-(\sum x_i)^2}=\hat{\beta}_1$$

Also we have $\sum(x_i-\bar{x})y_i=\sum x_i y_i-\bar{x}\sum y_i$

$$=\sum x_i y_i-\frac{(\sum x_i)(\sum y_i)}{n}$$

$$\hat{\beta}_1=\frac{\sum(x_i-\bar{x})y_i}{\sum(x_i-\bar{x})^2}$$

$$E(\hat{\beta}_1)=E\frac{\sum(x_i-\bar{x})y_i}{\sum(x_i-\bar{x})^2}=\frac{\sum(x_i-\bar{x})\,E(y_i)}{\sum(x_i-\bar{x})^2}$$

$$=\frac{\sum(x_i-\bar{x})\,(\beta_0+\beta_1 x_i)}{\sum(x_i-\bar{x})^2}=\beta_0\frac{\sum(x_i-\bar{x})}{\sum(x_i-\bar{x})^2}+\beta_1\frac{\sum x_i(x_i-\bar{x})}{\sum(x_i-\bar{x})^2}$$

$$=\beta_1\frac{\sum x_i^2-n\bar{x}^2}{\sum x_i^2-n\bar{x}^2}=\beta_1$$

2-2 Interval Estimation

Def: An (1-α) confidence interval (C. I.) estimator is an interval whose end points are functions of the sample statistics such that if we could generate indefinitely samples, the interval should contain the true parameters (1-α) % of the times.

Constructing C. I.: -

The following steps are necessary to construct the C.I.

step (1): obtain the probability distribution of the point estimator for the unknown parameter.

Step (2): Standardize the estimator such that we get a r.v with completely known distribution. Step (3): Construct C.I. for standardized r.v. then

1 - Solve for the unknown parameter.

2- 2-1 **C.I for Means of Normal Population**

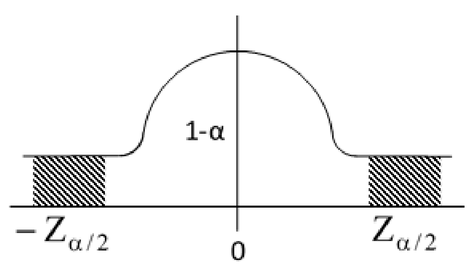
i- **if σ^2 is known:-**

Let $x_1, x_2, ..., x_n$ be a r.s from normal population with unknown mean μ and known Variance of δ^2 Applying the above steps:-

1- the sample mean $\bar{x}$ is a point estimate of μ with probability distribution N $(\mu, \frac{\sigma^2}{n})$.

2- standardizing $z = \frac{\bar{x}-\mu}{\delta/\sqrt{n}} \sim N(0,1)$

3- The values $-z\alpha/_2$, $z\alpha/_2$ place $\frac{1}{2}\alpha$ in each tail of normal distribution



Therefor

$p_r[-z\alpha/_2 < \frac{\bar{x}-\mu}{\delta/\sqrt{n}} < z\alpha/_2]$ =1-α

i.e

C.I

$pr(-c < \mu < c)= z\alpha/_2$

N(c) -N(-c)=N(c)- (1-N(c))

= N(c) -1+N(c) =2N(c)-1=N(c)=1-α/2

solving for μ we obtain

$p_r[\bar{x} - z\alpha/_2\, \sigma/\sqrt{n} < \mu < \bar{x} + z\alpha/_2\, \sigma/\sqrt{n}]$ =1-α

Where 0< α < 1 and selected often to be 0.1,0.01 or 0.5

Ex: Find 95% c.I for the mean of normal population N(μ,25) if it is known that normal $\bar{x} = 10 , n = 100.$

Solution: we have 1-α= 0.95 , α=0.05 ➔ $\frac{\alpha}{2} = \frac{0.05}{2} = 0.025.$

From tables of standard distribution, we get $z\alpha/_2$ = Z 0.025 = 1.96

$1-\alpha = p_r[-z\alpha/_2 < \frac{\bar{x}-\mu}{\delta/\sqrt{n}} < z\alpha/_2]$

$1-\alpha = p_r[-z\alpha/_2\, \delta/\sqrt{n} < \bar{x} - \mu < +z\alpha/_2\, \delta/\sqrt{n}]$

$1-\alpha = p_r[\bar{x} - z\alpha/_2\, \delta/\sqrt{n} < \mu < \bar{x} + z\alpha/_2\, \delta/\sqrt{n}]$

Pr [10- (1.96) $\frac{5}{\sqrt{100}} < \mu < x10 - (1.96) \frac{5}{\sqrt{100}}] = 0.95$

Pr [9.022< μ $< 10.98] = 0.95$

lower bound CL: 9.02

upper bound CU= 10.98

2) **if σ^2 is unknown**

$W=\frac{\bar{x}-\mu}{\delta/\sqrt{n}} \sim N(0,1)$
$V=\frac{ns^2}{\sigma^2} \sim x^2(n-1)$,
$r=n-1$
$T=\frac{w}{\sqrt{v/r}}$
$T=\frac{\frac{\bar{x}-\mu}{\delta/\sqrt{n}}}{\sqrt{\frac{\frac{ns^2}{\sigma^2}}{n-1}}}$ $T=\frac{w}{\sqrt{v/r}}$
$=\frac{\bar{x}-\mu}{s/\sqrt{n-1}}$

- In this case the r.v $\frac{\bar{x}-\mu}{\delta/\sqrt{n-1}} \sim T(n-1)$

Applying the steps stated earlier we get

$$p_r[\bar{x} - t\alpha_{/2}\, s/\sqrt{n-1} < \mu < \bar{x} + t\alpha_{/2}\, s/\sqrt{n-1}] = 1-\alpha$$

Ex: let $\bar{x}$=20, s^2 =29 denote the means and variance of ar.s of size 16 is from N(μ, σ^2). Find 95% C.I. form μ.

Solution: we have 1- α = 0.95 ➔ α= 0.05 , $\frac{\alpha}{2}$=0.025

From tables of T distribution. we get $t\alpha_{/2}$(n-1)= $t_{0.025}(14)$= 2.145 from table.

as another way to represent C.I we write C. I for

$=\bar{x}\mp t\alpha_{/2}\frac{s}{\sqrt{n-1}}$ = 20 ∓ (2.145) $\frac{\sqrt{9}}{\sqrt{15}}$ =20 ∓ (2.145) $\frac{3}{\sqrt{15}}$ μ

CL =18.338 , CU=21.661

(18.338 , 21.661)

b) **For Large Samples (n > 30)**

In this case and form statistical inference: theory the distribution of the r.v

.: $t=\frac{\sqrt{n}\ (\bar{x}-\mu)}{s}$ will converge to N(0,1).

which means that we can use the standard normal tables instead of t distribution table and hence.

c. I for μ = $\bar{x} \mp Z_{\alpha/2} \frac{\delta}{\sqrt{n}}$

Ex: let $\bar{x}$=20, s^2 =16 denote the means and variance of a r.s of size 100. Find 99% c.I. for μ.

Solution: we have 1- α = 0.99   α= 0.01 , $\frac{\alpha}{2}$ =0.005.

from tables of Normal we get $Z_{\alpha/2} = z_{0.005} = 2.58$

C.I= $\bar{x} \mp Z_{\alpha/2} \frac{s}{\sqrt{n}}$ $=20 \mp (2.58) \frac{\sqrt{16}}{\sqrt{100}}$

$= 20 \mp (2.58). \frac{4}{10}$

= (18.968 , 21.032).

2-2-2 C.I. for Difference Between Two Means

i) If δ_1^2, δ_2^2 are known

Let $\bar{x}_1, \bar{x}_2$ denote the means of two independent random samples of size n_1, n_2 from normal population with variance δ_1^2, δ_2^2 respectively. A $(1-\alpha)$ C.I. for $\mu_1 - \mu_2$ is

$$\text{C.I. for } \mu_1 - \mu_2 = (\bar{x}_1 - \bar{x}_2) \pm Z_{\alpha/2} \sqrt{\frac{\delta_1^2}{n_1} + \frac{\delta_2^2}{n_2}}$$

ii) If δ_1^2, δ_2^2 are unknown

a) For large sample ($n_1, n_2 > 30$)

A $(1-\alpha)\%$ C.I. for $\mu_1-\mu_2$ is given by

$$\text{C.I. for } \mu_1-\mu_2=(\bar{x}_1-\bar{x}_2)\pm Z_{\alpha/2}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$$

Where S_1^2, S_2^2 denote the variance of the two samples.

Ex.

Construct 96% C.I. for $\mu_1-\mu_2$ is it is known that $n_2 = 50$, $\bar{x}_2 = 76$, $S_2 = 6$, $n_1 = 75$, $\bar{x}_1 = 82$ and $S_1 = 8$.

Solution:

We have $(1-\alpha) = 0.96 \Rightarrow \alpha = 0.04$, $\frac{\alpha}{2} = 0.02$.

From tables of standard normal distribution $\alpha/2 = Z_{0.02} = 2.054$, then 96% C.I. for $\mu_1-\mu_2$

$$=(\bar{x}_1-\bar{x}_2)\pm Z_{\alpha/2}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$$

$$=(82-76)\pm(2.054)\sqrt{\frac{64}{75}+\frac{36}{50}}$$

$$=6\pm(2.054)\sqrt{\frac{64}{75}+\frac{36}{50}}$$

b) For small samples (n_1, n_2 < 30)

A $(1-\alpha)$ C.I. for $\mu_1-\mu_2$ is given by

$$\text{C.I. for } \mu_1-\mu_2=(\bar{x}_1-\bar{x}_2)\pm t_{\alpha/2}S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

Where S_p^2 is the pooled variance obtained from the sample variance S_1^2, S_2^2 as

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Ex.

Given that $n_1 = 12$, $\bar{x}_1 = 85$, $S_1 = 4$, $n_2 = 10$, $\bar{x}_2 = 81$, $S_2 = 5$. Find 90% C.I. for $\mu_1 - \mu_2$.

Solution:

We have $(1-\alpha) = 0.90 \Rightarrow \alpha = 0.10, \ \frac{\alpha}{2} = 0.05$

$t_{\alpha/2}(n_1 + n_2 + 2) = t_{0.05}(20) = 1.725$ from tables

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{(11)(16) + (9)\ 25}{12 + 10 - 2} = 20.05$$

$S_p = 4.478$

90% C.I. for $\mu_1 - \mu_2$ is $(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

$$(85 - 81) \pm (1.725)(4.478)\sqrt{\frac{1}{12} + \frac{1}{10}}$$

$= (0.69, 7.31)$

2- 2-3 C.I for Variance σ^2

i- **If mean μ is known**:- $\left[\frac{(n-1)\,s^2}{X^2{}_{\alpha/2}}, \frac{(n-1)\,s^2}{X^2{}_{1-\alpha/2}}\right]$

(1-α) % C.I for σ^2 is given by

$$p_r\left[\frac{(n-1)\,s^2}{X^2{}_{\alpha/2}} < \sigma^2 < \frac{(n-1)\,s^2}{X^2{}_{1-\alpha/2}}\right] = 1-\alpha$$

Let $X^2{}_{\alpha/2}$, $X^2{}_{1-\alpha/2}$ are the X^2 values obtained from X^2 distribution table with n degrees of freedom and level of significant $1-\alpha/2$, $\alpha/2$, respectively.

Ex : let S^2 =9 denoted the variance of ar.s of size 25 from N(10 , σ^2) , find 95% c.I. for σ^2

Solution:

$$1-\alpha = p_r[\frac{(n)\,s^2}{X^2{}_{\alpha/2}} < \sigma^2 < \frac{(n)\,s^2}{X^2{}_{\alpha/2}}]$$

we have 1- α = 0.95 , α= 0.05 , $\frac{\alpha}{2}$ =0.025 , $1-\frac{\alpha}{2} = 1 - 0.025 = 0.975$

from X^2 table we get

$X^2{}_{\alpha/2}$ (n) = $X^2{}_{0.025}$ (25) =40.6465

$X^2{}_{1-\alpha/2}$ (n) = $X^2{}_{0.075}$ (25) = 13.1197

$$p_r[\frac{(n)\,s^2}{X^2{}_{\alpha/2}} < \sigma^2 < \frac{(n)\,s^2}{X^2{}_{1-\alpha/2}}] = 1-\alpha$$

$$p_r[\frac{24(9)}{40.6465} < \sigma^2 < \frac{24(9)}{13.1197}] = 0.95$$

$$p_r[\frac{216}{40.6465} < \sigma^2 < \frac{216}{13.1197}] = 0.95$$

$$p_r[5.5355 < \sigma^2 < 17.1498] = 0.95$$

CL= 5.5355 , CU =17.1498

ii- **<u>If mean μ is unknown</u>**:-

(1-α) % C.I for σ^2 is given by

$$p_r[\frac{(n-1)\,s^2}{X^2{}_{1-\alpha/2}} < \sigma^2 < \frac{(n-1)\,s^2}{X^2{}_{\alpha/2}}] = 1-\alpha$$

Let $X^2{}_{1-\alpha/2}$, $X^2{}_{\alpha/2}$ are the X^2 values obtained from X^2 distribution table with $(n-1)$ degrees of freedom and level of significant $\alpha/2$, $1-\alpha/2$ respectively.

Let $x_1, x_2, \ldots, x_{10}$ be a r.s from normal population N (μ,σ^2).where EX: both are unknown, suppose $\sum x_i = 159$, and $\sum x^2{}_i = 2531$. Compute 95% c.I. for σ^2 . if it $x^2{}_{0.025}(9) = 2.70$ and $x^2{}_{0.975}(9) = 19$

$$p_r[\frac{(n-1)\,s^2}{X^2{}_{1-\alpha/2}} < \sigma^2 < \frac{(n-1)\,s^2}{X^2{}_{\alpha/2}}] = 1-\alpha$$

$$p_r[\frac{(n-1)\,s^2}{X^2{}_{1-\alpha/2}} < \sigma^2 < \frac{(n-1)\,s^2}{X^2{}_{\alpha/2}}] = 1-\alpha$$

$$S^2 = \frac{1}{(n-1)} \sum (x_i - \bar{x})^2$$

$$(n-1)S^2 = \sum(x_i - \bar{x})^2 = \sum x^2{}_i - n\,\bar{x}^2$$

$$\sum(x^2{}_i - 2x_i\,\bar{x} + \bar{x}^2)$$

$$2\sum x_i\bar{x} + \sum \bar{x}^2 = \sum x^2{}_i -$$

$$= \sum x^2{}_i - 2\frac{n}{n}\sum x_i\bar{x} + n\bar{x}^2$$

$$= \sum x^2{}_i - 2n\frac{\sum x_i}{n}\bar{x} + n\bar{x}^2$$

$$2n\,\bar{x}^2 + n\bar{x}^2 \sum x^2{}_i -$$

$$(n-1)S^2 = 2531 - 10(\frac{159}{10})^2 = 2531 - 10(15.9)^2 = 2.90$$

we have $1-\alpha = 0.95$ $\alpha = 0.05$, $\frac{\alpha}{2} = 0.025$ $1-\frac{\alpha}{2} = 1 - 0.025 = 0.975$

95% C.I. for σ^2 is

$$p_r[\frac{(n-1)\,s^2}{X^2{}_{1-\alpha/2}} < \sigma^2 < \frac{(n-1)\,s^2}{X^2{}_{\alpha/2}}] = 1-\alpha$$

$$p_r[\frac{2.90}{19} < \sigma^2 < \frac{2.90}{2.70}] = 0.95$$

$$p_r[0.15 < \sigma^2 < 1.07] = 0.95$$

ii) C.I. for the Ration of Two Variances

Let S_1^2, S_2^2 be the variance of two independent random samples of size n_1, n_2 respectively.

Let $v_1 = n_1 - 1$, $v_2 = n_2 - 1$ be the degrees of freedom then $(1-\alpha)\%$ C.I. for the ratio $\frac{\delta_1^2}{\delta_2^2}$ is given by:

$Z1 = \frac{nS_1^2}{\sigma_1^2} \sim x^2(\text{n1-1})$,

$Z2 = \frac{nS_2^2}{\sigma_2^2} \sim x^2(\text{n2-1})$,

$F = \frac{Z_{1/n-1}}{Z_2/n2-1}$

$$\Pr[\frac{S_1^2}{S_2^2}\frac{1}{f_{\alpha/2}(v_1, v_2)} < \frac{\delta_1^2}{\delta_2^2} < \frac{S_1^2}{S_2^2} f_{\alpha/2}(v_2, v_1)] = 1-\alpha$$

The values of $f_{\alpha/2}(v_1, v_2)$ and $f_{\alpha/2}(v_2, v_1)$ obtained from the F distribution table.

Ex.

Find 98% C.I. for δ_1^2/δ_2^2 if it is known that $n_1 = 25$, $n_2 = 16$, $S_1 = 8$, $S_2 = 7$.

Solution:

We have $(1-\alpha) = 0.98 \Rightarrow \alpha = 0.02,\ \frac{\alpha}{2} = 0.01$

$$f_{\alpha/2}(v_1, v_2) = f_{0.01}(24, 15) = 3.29$$

$$f_{1-\alpha/2}(v_2, v_1) = f_{0.01}(15, 24) = 2.89$$

$$\Pr[\frac{S_1^2}{S_2^2}\frac{1}{f_{\alpha/2}(v_1, v_2)} < \frac{\delta_1^2}{\delta_2^2} < \frac{S_1^2}{S_2^2} f_{\alpha/2}(v_2, v_1)] = 1-\alpha$$

$$\Pr[\frac{64}{49}\frac{1}{3.29} < \frac{\delta_1^2}{\delta_2^2} < \frac{64}{49}(2.89)] = 0.98$$

C.I. = (0.397, 3.775)

Exercises

1) If it is known that n = 17 is the size of r.s. from $N(\mu, \delta^2)$ with $\bar{x}$ = 5.3, S^2 = 6.2. Find 95% C.I. for both μ and δ^2. The tabulated values are:

$$t_{0.025}(16) = 2.120,\ X^2_{0.025}(16) = 6.91,\ X^2_{0.975}(16) = 28.8$$

2) Given $\bar{x}$ = 18, is the mean of ar.s. of size 20 from N(μ, 25). Find 99% C.I. for μ if it is known that $Z_{0.005}$ = 2.58.

3) A r.s. of size 10 is drawn from $N(\mu, \delta^2)$. The values of individuals are 10.7, 12.6, 9.3, 9.5, 11.3, 12.2, 11.5, 11.1, 10.4 and 10.2. Find 95% C.I. for μ and δ^2, $t_{0.025}(9) = 2.262$.

4) Two random samples each of size 10 from $N(\mu_1, \delta^2)$, $N(\mu_2, \delta^2)$ yield $\bar{x}_1$ = 4.8, S_1^2 = 8.64, $\bar{x}_2$ = 5.6, S_2^2 = 7.88. Find 95% C.I. for $\mu_1 - \mu_2$ if it known that $t_{0.025}(18) = 2.101$.

5) Let $x_1, x_2, ..., x_n$ be ar.s. from $N(\mu, \delta^2)$. Let 0 < a < b. Show that the mathematical expectation of the length of random interval

$$[\frac{\sum(x_i-\mu)^2}{b}, \frac{\sum(x_i-\mu)^2}{a}] \text{ is } (b-a)\,(\frac{n\delta^2}{ab})$$

Problems

(1) Let $x_1, x_2, ..., x_n$ be a r.s from

$$f(x,\theta) = \begin{cases} \frac{1}{\theta}e^{-x/\theta} & 0 < x < \infty, 0 < \theta < \infty \\ 0 & \text{o.w} \end{cases}$$

Show that $\bar{x}$ is an unbiased statistic for θ.

Solution:

Since x ~ G(1, θ), α = 1, β = θ

E(x) = αβ = θ

Now $E(\bar{x}) = E(\frac{\sum x_i}{n}) = \frac{1}{n}E(\sum x_i) = \frac{1}{n}\sum E(x_i) = \frac{1}{n}\sum\theta$

$= \frac{1}{n}n\theta = \theta$

$\therefore \bar{x}$ is unbiased estimator of θ.

(2) Let $y_1 < y_2 < y_3$ be two order statistics of a r.s of size 3 from the uniform dist. Having p.d.f. $f(x, \theta) = \frac{1}{\theta}, 0 < x < \theta, 0 < \theta < \infty$.

Show that $4y_1$, $2y_2$ and $\frac{3}{4}y_3$ are all unbiased statistics for θ. Find the variance of each of these unbiased statistics.

Solution:

$$g(y_k) = \frac{n!}{(k-1)!(n-k)!}[F(y_k)]^{k-1}[1-F(y_k)]^{n-k} f(y_k)$$

$$g(y_1) = \frac{3!}{(1-1)!(2-k)!}[F(y_1)]^0[1-F(y_1)]^{3-1} f(y_k)$$

$$F(x) = \int_0^x \frac{1}{\theta}du = \frac{x}{\theta} \Rightarrow F(y_1) = \frac{y_1}{\theta}$$

$$g(y_1) = \frac{3!}{2!}[1-\frac{y_1}{\theta}]^2\frac{1}{\theta} = 3[1-\frac{y_1}{\theta}]^2\frac{1}{\theta}$$

$$E(4y_1) = 4E(y_1) = 4\int_0^\theta \frac{3}{\theta}[1-\frac{y_1}{\theta}]^2 y_1 dy_1$$

$$= \frac{12}{\theta}\int_0^\theta [1-\frac{2y_1}{\theta}+\frac{y_1^2}{\theta^2}]y_1 dy_1$$

$$=\frac{12}{\theta}\int_0^\theta [y_1-\frac{2y_1^2}{\theta}+\frac{y_1^3}{\theta^2}]dy_1$$

$$=\frac{12}{\theta}\int_0^\theta [\frac{2y_1^2}{\theta}y_1-\frac{2y_1^2}{\theta}+\frac{y_1^3}{\theta^2}]dy_1$$

$$=\frac{12}{\theta}(\frac{y_1^2}{2}-\frac{2y_1^3}{3\theta}+\frac{y_1^4}{4\theta^2})\Big|_0^\theta$$

$$=\frac{12}{\theta}[\frac{\theta^2}{2}-\frac{2}{3}\theta^2+\frac{\theta^2}{4}]=\frac{12}{\theta}\,\frac{6\theta^2-8\theta^2+3\theta^2}{12}=\frac{\theta^2}{\theta}=\theta$$

$\therefore 4y_1$ unbiased statistics for θ.

$$g(y_2)=\frac{3!}{(2-1)!\,(3-2)!}[F(y_2)]^{2-1}[1-F(y_2)]f(y_2)$$

$$F(y_2)=\frac{y_2}{\theta}$$

$$g(y_2)=6[\frac{y_2}{\theta}][1-\frac{y_2}{\theta}]\frac{1}{\theta}=6\frac{y_2}{\theta}[\frac{1}{\theta}-\frac{y_2}{\theta^2}]$$

$$=6[\frac{y_2}{\theta^2}-\frac{y_2^2}{\theta^3}]$$

$$E(2y_2)=2E(y_2)=2\int_0^\theta 6y_2(\frac{y_2}{\theta^2}-\frac{y_2^2}{\theta^3})dy_2$$

$$=12\int_0^\theta(\frac{y_2^2}{\theta^2}-\frac{y_2^3}{\theta^3})dy_2=12(\frac{y_2^3}{3\theta^2}-\frac{y_2^4}{4\theta^3})\Big|_0^\theta$$

$$=12(\frac{\theta^3}{3\theta^2}-\frac{\theta^4}{4\theta^3})=12(\frac{\theta}{3}-\frac{\theta}{4})=12\frac{4\theta-3\theta}{12}=\theta$$

$\therefore 2y_2$ unbiased statistics for θ.

$$g(y_3)=\frac{3!}{(3-1)!\,(3-3)!}[F(y_3)]^{3-1}[1-F(y_3)]^{3-3}f(y_3)$$

$$= 3[F(y_3)]^2 f(y_3) = 3[\frac{y_3}{\theta}]^2 \frac{1}{\theta} = \frac{3y_3^2}{\theta^3}$$

$$E(\frac{4}{3}y_4) = \frac{4}{3}E(y_3) = \frac{4}{3}\int_0^\theta y_3 \frac{3y_3^2}{\theta^3}dy_3 = 4\int_0^\theta \frac{y_3^3}{\theta^3}dy_3$$

$$= \frac{4}{\theta^3}\left.\frac{y_3^3}{4}\right|_0^\theta = \frac{\theta^4}{\theta^3} = \theta$$

$\therefore \frac{4}{3}y_3$ unbiased statistics for θ.

Now, to find the variance of $4y_1$

$$var(4y_1) = 16\,var(y_1) = 16[E(y_1^2) - E(y_1)^2]$$

$$E(y_1^2) = \int_0^\theta y_1^2 \frac{1}{\theta} 3[1-\frac{y_1}{\theta}]^2 dy_1 = \frac{3}{\theta}\int_0^\theta y_1^2[1 - \frac{2y_1}{\theta} + \frac{y_1^2}{\theta^2}]dy_1$$

$$= \frac{3}{\theta}\int_0^\theta (y_1^2 - \frac{2y_1^3}{\theta} + \frac{y_1^4}{\theta^2})dy_1 = \frac{3}{\theta}\left.(\frac{y_1^3}{3} - \frac{2y_1^4}{4\theta} + \frac{y_1^5}{5\theta^2})\right|_0^\theta$$

$$= \frac{3}{\theta}[\frac{\theta^3}{3} - \frac{\theta^4}{2\theta} + \frac{\theta^5}{5\theta^2}] = \theta^2 - \frac{3}{2}\theta^2 + \frac{3}{5}\theta^2 = \frac{10\theta^2 - 15\theta^2 + 6\theta^2}{10}$$

$$= \frac{\theta^2}{10}$$

$$E(y_1) = \int_0^\theta y_1 \frac{1}{\theta} 3[1-\frac{y_1}{\theta}]^2 dy_1$$

$$= \frac{3}{\theta}\int_0^\theta [1 - \frac{2y_1}{\theta} + \frac{y_1^2}{\theta^2}]y_1 dy_1$$

$$= \frac{3}{\theta}\int_0^\theta [y_1 - \frac{2y_1^2}{\theta} + \frac{y_1^3}{\theta^2}]dy_1 = \frac{3}{\theta}\left.(\frac{y_1^2}{2} - \frac{2y_1^3}{3\theta} + \frac{y_1^4}{4\theta^2}\right|_0^\theta$$

$$= \frac{3}{\theta}(\frac{\theta^2}{2} - \frac{2}{3}\frac{\theta^3}{\theta} + \frac{\theta^4}{4\theta^2}) = \frac{3}{\theta}[\frac{\theta^2}{2} - \frac{2}{3}\theta^2 + \frac{\theta^2}{4}]$$

$$=\frac{3}{\theta}\frac{6\theta^2-8\theta^2+3\theta^2}{12}=\frac{\theta^2}{4\theta}=\frac{\theta}{4}$$

$$\mathrm{var}(4y_1)=16[\frac{\theta^2}{10}-\frac{\theta^2}{16}]=(\frac{8\theta^2-5\theta^2}{80})16=\frac{3\theta^2}{80}16=\frac{3}{5}\theta^2$$

To find the variance of $2y_2$

$$\mathrm{var}(2y_2)=4\,\mathrm{var}(y_2)=4[E(y_2^2)-E(y_2)]^2$$

$$E(y_2^2)=\int_0^\theta y_2^2 6(\frac{y_2}{\theta^2}-\frac{y_2^2}{\theta^3})dy_2$$

$$=6\int_0^\theta(\frac{y_2^3}{\theta^2}-\frac{y_2^4}{\theta^3})dy_2=6(\frac{y_2^4}{4\theta^2}-\frac{y_2^5}{5\theta^3})\Big|_0^\theta$$

$$=6(\frac{\theta^4}{4\theta^2}-\frac{\theta^5}{5\theta^3})=6(\frac{\theta^2}{4}-\frac{\theta^2}{5})=6\frac{\theta^2}{20}=\frac{3}{10}\theta^2$$

$$\mathrm{var}(2y_2)=4(\frac{3}{10}\theta^2-\frac{\theta^2}{4})=4\frac{6\theta^2-5\theta^2}{20}=\frac{\theta^2}{5}$$

To find the variance of $\frac{4}{3}y_3$

$$\mathrm{var}(\frac{4}{3}y_3)=\frac{16}{9}\mathrm{var}(y_3)=\frac{16}{9}[E(y_3^2)-E(y_3)]^2$$

$$E(y_3^2)=\int_0^\theta y_3^2\frac{3y_3^2}{\theta^3}dy_3=\frac{3}{\theta^3}\int_0^\theta y_3^4dy_3=\frac{3}{\theta^3}\frac{y_3^5}{3}\Big|_0^\theta$$

$$=\frac{3}{\theta^3}\frac{\theta^5}{3}=\frac{3}{5}\theta^2$$

$$\mathrm{var}(\frac{4}{3}y_3)=\frac{16}{9}[\frac{3}{5}\theta^2-\frac{9}{16}\theta^2]=\frac{16}{9}\frac{(48-45)\theta^2}{80}=\frac{\theta^2}{15}$$

(3) Let $x_1, x_2, ..., x_n$ be a r.s from $p(\theta)$. Show that $\sum x_i$ is a suff. Stat. for θ.

Solution:

Since $x_1, x_2, ..., x_n \sim p(\theta)$

$$f(x) = \frac{e^{-\theta}\theta^{x}}{x!}, x = 0, 1, ...$$

$$L = l(x_1, x_2, x_3, ..., x_n, \theta) = f(x_1, \theta)\ f(x_2, \theta)...f(x_n, \theta)$$

$$= \frac{e^{-\theta}\theta^{x_1}}{x_1!}\ \frac{e^{-\theta}\theta^{x_2}}{x_2!} \cdots \frac{e^{-\theta}\theta^{x_n}}{x_n!}$$

$$= \frac{e^{-n\theta}\theta^{\sum x_i}}{\pi x_i!} = e^{-n\theta}\theta^{\sum x_i}\ \frac{1}{\pi x_i!}$$

$\therefore \sum x_i$ is a suff. Stat. for θ.

(4) Show that the n^{th} order statistic of a r.s of size n from the uniform dif. Having p.d.f. $f(x, \theta) = \frac{1}{\theta}$, $0 < x < \theta$, $0 < \theta < \infty$ is a suff. statistic for θ.

Solution:

$$F(x) = \int_0^x \frac{1}{\theta} du = \frac{1}{\theta} u \Big|_0^x = \frac{x}{\theta}$$

$$F(y_n) = \frac{y_n}{\theta}$$

$$g(y_n) = \frac{n!}{(n-1)!(n-n)!}[F(y_n)]^{n-1}[1 - F(y_n)]^{n-n} f(y_n)$$

$$= \frac{n(n-1)!}{(n-1)!}[\frac{y_n}{\theta}]^{n-1}\frac{1}{\theta} = \frac{n}{\theta^n} y_n^{n-1}$$

$$L(x_1, x_2, ..., x_n, \theta) = f(x_1, \theta)\ f(x_2, \theta)...f(x_n, \theta)$$

$$= \frac{1}{\theta}.\frac{1}{\theta}.....\frac{1}{\theta} = \frac{1}{\theta^n}$$

Nyman Fisher تستخدم طريقة

$$\frac{L}{g} = \frac{\frac{1}{\theta^n}}{\frac{n}{\theta^n} y_n^{n-1}} = \frac{1}{n y_n^{n-1}} \text{ does not involve } \theta$$

$\Rightarrow y_n$ is suff. Stat. for θ.

Exercise

(1) Let $x_1, x_2, ..., x_n$ be a r.s from $p(\theta)$. Find the MLE for $pr(x > 0)$.

Solution:

$$f(x, \theta) = \frac{e^{-\theta}\theta^x}{x!}, \ x = 0, 1, ...$$

$L(x_1, x_2, ..., x_n, \theta) = f(x_1, \theta).f(x_2, \theta)...f(x_n, \theta)$

$$= \frac{e^{-\theta}\theta^{x_1}}{x_1!} \frac{e^{-\theta}\theta^{x_2}}{x_2!} ... \frac{e^{-\theta}\theta^{x_n}}{x_n!}$$

$$= \frac{e^{-n\theta}\theta^{\sum x_i}}{\pi x_i!}$$

$$\ln L(x_1, x_2, ..., x_n, \theta) = \ln \frac{e^{-n\theta}\theta^{\sum x_i}}{\pi x_i!} = \ln e^{-n\theta} + \ln \theta^{\sum x_i} - \ln \pi x_i!$$

$$= -n\theta + \sum x_i \ln\theta - \ln \pi x_i!$$

$$\frac{\partial \ln L}{\partial \theta} = -n + \frac{\sum x_i}{\theta} = 0 \Rightarrow n = \frac{\sum x_i}{\theta} \Rightarrow \overset{n}{\theta} = \frac{\sum x_i}{n} = \bar{x}$$

$$pr(x > 0) = 1 - pr(x = 0) = 1 - \frac{e^{-\theta}\theta^0}{0!} = 1 - e^{-\theta}$$

لإيجاد MLE إلى $pr(x > 0)$ نستبدل θ بمقدرها وهو $\bar{x}$ ، أي أن

MLE for $pr(x > 0) = 1 - e^{-\bar{x}}$

حسب خاصية الثبات (invariant property)

(2) Let $x_1, x_2, ..., x_n$ be a r.s from

$$f(x_1, \theta_1, \theta_2) = \begin{cases} \frac{1}{\theta^2} e^{-\frac{(x-\theta_1)}{\theta_2}} & \theta_1 \le x < \infty \\ & -\infty < \theta_1 < \infty, 0 < \theta_2 < \infty \\ 0 & \text{o.w} \end{cases}$$

Find the MLE for θ_1 and θ_2.

Solution:

To find MLE of θ_1

$L(x_1, x_2, ..., x_n, \theta_1, \theta_2) = f(x_1, \theta_1, \theta_2).f(x_2, \theta_1, \theta_2)...f(x_n, \theta_1, \theta_2)$

$$= \frac{1}{\theta_2} e^{\frac{-(x_1-\theta_1)}{\theta_2}} \frac{1}{\theta_2} e^{\frac{-(x_2-\theta_1)}{\theta_2}} ... \frac{1}{\theta_2} e^{\frac{-(x_n-\theta_1)}{\theta_2}}$$

$$= \frac{1}{\theta_2^n} e^{\frac{-\sum(x_i-\theta_1)}{\theta_2}}$$

We can't use the differentiation method because the range of x depend upon θ_1, but it is clear that L has maximum value at the largest value of θ_1 which coincide with the smallest value of x. Hence, $\hat{\theta}_1 = \min(x_i)$ = the smallest order statistic of the sample.

Now, to find the MLE of θ_2

$$L(x_1, x_2, ..., x_n, \theta_1, \theta_2) = \frac{1}{\theta_2^n} e^{\frac{-\sum(x_i-\theta_1)}{\theta_2}}$$

$$\ln L = \ln(\frac{1}{\theta_2^n} e^{\frac{-\sum(x_i-\theta_1)}{\theta_2}})$$

$$= \ln \theta_2^{-n} - \frac{-\sum(x_i-\theta_1)}{\theta_2}$$

$$= -n \ln \theta_2 - \frac{\sum(x_i-\theta_1)}{\theta_2}$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{-n}{\theta_2} + \frac{\sum(x_i - \theta_1)}{\theta_2^2} = 0$$

$$\frac{n}{\theta_2} = \frac{\sum(x_i - \theta_1)}{\theta_2^2} \Rightarrow \theta_2^n = \frac{\sum(x_i - \theta_1)}{n} = \frac{\sum x_i - \theta n_1}{n}$$

$$\therefore \theta_2^n = \frac{\sum(x_i) - n\min(x_i)}{n} = \frac{\frac{n\sum x_i}{n} - n\min(x_i)}{n}$$

$$\theta_2^n = \bar{x} - \min x_i$$

(3) Let $x_1, x_2, ..., x_n$ be a r.s from $N(\mu, \delta^2)$. Find the moment est. for μ and δ^2.

Solution:

$$m_1 = \frac{1}{n}\sum x_i = \bar{x}$$

$$\mu_1 = E(x) = \mu$$

$$\mu_1 = m_1 \Rightarrow \overset{n}{\mu} = \bar{x} \text{ the moment of } \mu$$

$$m_2 = \frac{1}{n}\sum x_i^2$$

$$\mu_2 = E(x^2) = \text{var}(x) + [E(x)]^2 = \hat{\mu}^2 + \delta_x^2 = (\bar{x})^2 + \delta^2$$

$$\frac{1}{n}\sum x_i^2 = \delta_x^2 + \bar{x}^2$$

$$\hat{\delta}_x^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

$$\hat{\delta}_x^2 = \frac{1}{n}\sum(x_i - \bar{x})^2 \quad \text{the moment of } \delta^2$$

(4) Let $x_1, x_2, ..., x_n$ be a r.s from $G(\alpha, \beta)$. Find the moment est. for α and β.

Solution:

$$\mu_1 = E(x_1) = \alpha\beta$$

$$m_1 = \frac{1}{n}\sum x_i = \bar{x}$$

$$\mu_1 = \alpha\beta = \bar{x} = m_1 \Rightarrow \overset{n}{\beta} = \frac{\bar{x}}{\alpha}$$

$$m_2 = E(x^2) = \mathrm{var}(x) + [E(\alpha)]^2 = \alpha\beta^2 + \alpha^2\beta^2 = \alpha\beta^2(1+\alpha)$$

$$\mu_2 = \frac{1}{n}\sum x_i^2$$

$$m_2 = \mu_2 = \frac{1}{n}\sum x_i^2 = \beta^2\alpha(1+\alpha)$$

$$\frac{1}{n}\sum x_i^2 = \frac{\bar{x}^2}{\alpha^2}\alpha(1+\alpha)$$

$$\frac{1}{n}\sum x_i^2 = \frac{\bar{x}^2}{\alpha}(1+\alpha)$$

$$\frac{1}{n}\sum x_i^2 = \frac{\bar{x}^2}{\alpha} + \bar{x}^2 \Rightarrow \frac{1}{n}\sum x_i^2 - \bar{x}^2 = \frac{\bar{x}^2}{\alpha}$$

$$\alpha = \frac{\bar{x}^2}{\frac{1}{n}\sum x_i^2 - \bar{x}^2}$$

$$\beta^n = \frac{\bar{x}}{\frac{\bar{x}^2}{\frac{1}{n}\sum x_i^2 - \bar{x}^2}} = \frac{\bar{x}}{\bar{x}^2}.\left(\frac{1}{n}\sum x_i^2 - \bar{x}^2\right) = \frac{\frac{1}{n}\sum x_i^2 - \bar{x}^2}{\bar{x}}$$

$$\beta^n = \frac{s^2}{\bar{x}}$$

$$\alpha^n = \frac{\bar{x}}{\frac{s^2}{\bar{x}}} = \frac{\bar{x}.\bar{x}}{s^2} = \frac{\bar{x}^2}{s^2}$$

Exercises

(1) If it is known that n = 17 is the size of r.s from N(μ, δ^2) with $\bar{x}$ = 5.3, s^2 = 6.2. Find 95% C.I for both μ and δ^2. The tabulated values are $t_{0.025}(16) = 2.120$, $x^2_{0.025}(16) = 6.91$, $x^2_{0.975}(16) = 28.8$.

Solution:

n < 30, δ^2 is unknown

$$\Pr[\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}] = 1 - \alpha$$

We have $1 - \alpha = 0.95 \Rightarrow \alpha = 0.05, \frac{\alpha}{2} = 0.025$

From tables of t distribution we get

$$t_{\alpha/2}(n-1) = t_{0.025}(16) = 2.120$$

$$\text{C.I. for } \mu = \bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}} = 5.3 \pm (2.120)\frac{2.49}{\sqrt{17}}$$

$$= 5.3 \pm 1.28$$

CL = 4.0197

CU = 6.58

Now, we find C.I. for δ^2, when μ is unknown

$$p_r[\frac{(n-1)\,S^2}{X^2\,\alpha/_2} < \sigma^2 < \frac{(n-1)\,S^2}{X^2\,1-\alpha/_2}] = 1-\alpha$$

$$1 - \frac{\alpha}{2} = 1 - 0.025 = 0.975$$

From x^2 table, we get

$$X^2{}_{1-\alpha/_2}\,(n-1) = X^2{}_{0.025}\,(16) = 28.8$$

$X^2{}_{1-\alpha/2}\ (n-1) = X^2{}_{0975}\ (16) = 6.91$

$$p_r[\frac{(n-1)\,s^2}{X^2{}_{\alpha/2}} < \sigma^2 < \frac{(n-1)\,s^2}{X^2{}_{1-\alpha/2}}] = 1\text{-}\alpha$$

$$\Pr[\frac{(16)(6.2)}{28.8} < \delta^2 < \frac{(16)(6.2)}{6.91}] = 0.95$$

$$\Pr[3.44 < \delta^2 < 14.356] = 0.95$$

CL = 3.44, CU = 14.356

(2) Given $\overline{x}$ = 18 is the mean of a r.s. of size 20 from N(μ, 25). Find 99% C.I. for μ if it is known that $Z_{0.005} = 2.58$.

Solution:

We have 1 - α = 0.99 ⇒ α = 0.01

$$\frac{\alpha}{2} = \frac{0.01}{2} = 0.005$$

From tables of standard normal distribution we get

$Z_{\alpha/2} = Z_{0.005} = 2.58$

$$\Pr[\overline{x} - Z_{\alpha/2}\frac{\delta}{\sqrt{n}} < \mu < \overline{x} + Z_{\alpha/2}\frac{\delta}{\sqrt{n}}] = 1 - \alpha$$

$$\Pr[18 - (2.58)\frac{5}{\sqrt{20}} < \mu < 18 + (2.58)\frac{5}{\sqrt{20}}] = 0.99$$

$$\Pr[15.1155 < \mu < 20.881] = 0.99$$

CL = 15.1155

CU = 20.881

(3) A r.s of size 10 is drawn from N(μ, δ^2), the value of individuals 10.7, 12.6, 9.3, 9.5, 11.3, 12.2, 11.5, 11.1, 10.4 and 10.2. Find 95% C.I. for μ and δ^2, $t_{0.02}(9) = 2.262$, $x^2_{0.025}(9) = 2.70$, $x^2_{0.975}(9) = 19$.

Solution:

Since n < 30

$$\Pr[\bar{x} - t_{\alpha/2}\frac{\delta}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{\delta}{\sqrt{n}}] = 1 - \alpha$$

$$1 - \alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$$

From tables of t distribution we get:

$$t_{\alpha/2}(n-1) = t_{0.025}(9) = 2.262$$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{10.7 + 12.6 + 9.3 + 9.5 + 11.3 + 12.2 + 11.5 + 11.1 + 10.4 + 10.2}{10}$$

$$= \frac{108.8}{10} = 10.88$$

$$S^2 = \frac{1}{n}\sum(x_i - \bar{x})^2 = \frac{1}{n}\sum x_i^2 - \frac{1}{n}n\bar{x}^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

$$S^2 = \frac{1}{10}1194.18 - 118.3744 = 1.0436$$

$$S = 1.0216$$

$$\Pr[10.88 - (2.262)\frac{1.0216}{\sqrt{10}} < \mu < 10.88 + (2.262)\frac{1.0216}{\sqrt{10}}] = 0.95$$

$$\Pr[10.1493 < \mu < 11.611] = 0.95$$

CL = 10.1493

CU = 11.611

Now, to find C.I. for δ^2 when μ is unknown

$$p_r[\frac{(n-1)\,s^2}{X^2{}_{\alpha/2}} < \sigma^2 < \frac{(n-1)\,s^2}{X^2{}_{1-\alpha/2}}] = 1-\alpha$$

$$\alpha = 0.05, \frac{\alpha}{2} = 0.025, 1 - \frac{\alpha}{2} = 1 - 0.025 = 0.975$$

From x^2 tables we get

$$X^2{}_{1-\alpha/2}\,(n-1) = X^2{}_{0.025}\,(9) = 19$$

$$X^2{}_{1-\alpha/2}\,(n-1) = X^2{}_{0.975}\,(9) = 2.70$$

$$p_r[\frac{(n-1)\,s^2}{X^2{}_{\alpha/2}} < \sigma^2 < \frac{(n-1)\,s^2}{X^2{}_{1-\alpha/2}}] = 1-\alpha$$

$$\Pr[\frac{(9)(1.0436)}{19} < \delta^2 < \frac{(9)(1.0436)}{2.70}] = 0.95$$

(4) Two random samples each of size 10 from $N(\mu_1, \delta^2)$, $N(\mu_2, \delta^2)$ yield $\bar{x}_1 = 4.8$, $s_1^2 = 8.64$, $\bar{x}_2 = 5.6$, $s_2^2 = 7.88$. Find 95% C.I. for $\mu_1 - \mu_2$ if it is known that $t_{0.025}(18) = 2.101$

Solution:

Since $n_1, n_2 < 30$

We have $1-\alpha=0.95 \Rightarrow \alpha=0.05 \Rightarrow \frac{\alpha}{2}=0.025$

$t_{\alpha/2}(n_1+n_2-2)=t_{0.025}(18)=2.101$

C.I. for $\mu_1-\mu_2=((\bar{x}_1-\bar{x}_2)\mp t_{\alpha/2}sp\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$

$$s^2p=\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}$$

$$=\frac{9(8.64)+9(7.88)}{18}=8.26$$

Sp = 2.874

C.I. for $\mu_1-\mu_2=((4.8-5.6)\mp(2.101)(2.874)\sqrt{\frac{1}{10}+\frac{1}{10}})$

$=(-0.8\mp 2.7004)$

$=(-3.5004, 1.9004)$

(5) Let $x_1, x_2, ..., x_n$ be a r.s from $N(\mu, \delta^2)$. Let $0 < a < b$. Show that the mathematical expectation of the length of random interval

$[\frac{\sum(x_i-\mu)^2}{b}, \frac{\sum(x_i-\mu)^2}{a}]$ is $(b-a)\frac{n\delta^2}{ab}$.

Solution:

$$L=\frac{\sum(x_i-\mu)^2}{a}-\frac{\sum(x_i-\mu)^2}{b}=\frac{b\sum(x_i-\mu)^2-a\sum(x_i-\mu)}{ab}$$

$$L=\frac{\sum(x_i-\mu)^2(b-a)}{ab}$$

$$E(L)=E(\frac{\sum(x_i-\mu)^2(b-a)}{ab})=\frac{b-a}{ab}E(\sum(x_i-\mu)^2)$$

$$=\frac{b-a}{ab}\sum\delta^2=\frac{b-a}{ab}n\delta^2=(b-a)\frac{n\delta^2}{ab}$$