Chapter Three

Banach Space

In this chapter, we introduce the following

Banach Space

Definition (3.1):-

Let L be a normed space, we say that L is a complete space if every Cauchy sequence is convergent.

The complete normed space is called Banach space.

$i.e$, $(L, \|.\|)$ is Banach space if

(1) $(L, \|.\|)$ is normed space

(2) $(L, \|.\|)$ is complete space.

Examples of Banach Spaces

Example (3.2) :-

The space F^n with the norm $\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$, $x \in F^n$, is Banach space ?

Solution :- To prove that :

(1) $(F^n. \|.\|)$ is normed space (H.W)

(2) $(F^n. \|.\|)$ is complete space ?

Let $< x_m >$ be a Cauchy sequence in $F^n \Longrightarrow x_m \in F^n$, for each m =1,2,3,...

$\langle x_m \rangle = \langle x_1, x_2, \dots, x_m, \dots \rangle$

$= \langle (x_{11}, x_{12}, \dots, x_{1n}), (x_{21}, x_{22}, \dots, x_{2n}), \dots, (x_{m1}, x_{m2}, \dots, x_{mn}), \dots \rangle$

Then $\forall \epsilon > 0, \exists k \in Z_+$ such that

$\|x_m - x_j\| < \epsilon,\ \forall m, j > k \implies \|x_m - x_j\|^2 < \epsilon^2$ ... (1)

Since,

$x_m = (x_{m1}, x_{m2}, \dots, x_{mn}), x_{mi} \in F, i = 1, \dots, n$

$x_j = \left(x_{j1}, x_{j2}, \dots, x_{jn}\right),\quad x_{ji} \in F, i = 1, \dots, n$

$x_m - x_j = \left(x_{m1} - x_{j1}, x_{m2} - x_{j2}, \dots, x_{mn} - x_{jn}\right)$

So, $\|x_m - x_j\|^2 = \sum_{i=1}^{n} \left|x_{mi} - x_{ji}\right|^2$ ... (2)

From (1) & (2) $\implies \sum_{i=1}^{n} \left|x_{mi} - x_{ji}\right|^2 < \epsilon^2$

$$\implies \left|x_{mi} - x_{ji}\right|^2 < \epsilon^2, \forall m, j > k$$

$\implies \left|x_{mi} - x_{ji}\right| < \epsilon \implies < x_{mi} > \forall\, i = 1,2, \dots \dots, n$ is Cauchy in F

But F is complete space $\implies \forall i = 1,2, \dots \dots, n,\ \exists x_i \in F\ s.t$

$x_{mi} \to x_i \implies \forall \epsilon > 0, \exists k_i \in N$ such that

$$|x_{mi} - x_i| < \frac{\epsilon}{\sqrt{n}}, \forall m > k_i$$

Put, $x = (x_1, x_2, \dots, x_n)$. Let $k = \{k_1, k_2, \dots, k_n\}$

$\Longrightarrow \forall m > k, \quad |x_{mi} - x_i| < \frac{\epsilon}{\sqrt{n}} \Longrightarrow |x_{mi} - x_i|^2 < \frac{\epsilon^2}{n}$

$\Longrightarrow \sum_{i=1}^{n} |x_{mi} - x_i|^2 < n.\frac{\epsilon^2}{n} = \epsilon^2$

But $\|x_m - x\|^2 = \sum_{i=1}^{n} |x_{mi} - x_i|^2 < \epsilon^2$

$$\Longrightarrow \|x_m - x\|^2 < \epsilon^2 \;\Longrightarrow\; \|x_m - x\| < \epsilon$$

$\Longrightarrow < x_m >$ is convergent sequence $\Longrightarrow (F^n, \|.\|)$ is Banach space .

Example (3.3) :-

The space F^n with the norm $\|x\| = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$, $p \geq 1$, $x \in F^n$, is Banach space ?

(H.W.)

Example(3.4) :-

a .The space $\mathbb{R}^n (or\ \mathbb{C}^n)$ with the norm

$\|x\| = \{|x_1| , \dots , |x_n|\}$, $\forall x = (x_1, \dots , x_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$) is a Banach space .

Solution : Let $\langle x_m \rangle$ be a Cauchy sequence in F^n

$\langle x_m \rangle = \langle x_1, x_2, \dots , x_m , \dots \rangle$

$= \langle (x_{11}, x_{12}, \dots , x_{1n}), (x_{21}, x_{22}, \dots , x_{2n}), \dots , (x_{m1}, x_{m2}, \dots , x_{mn}), \dots \rangle$

Then $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|x_m - x_j\| < \epsilon \ \forall m, j > k$ (1)

Since $x_m, x_j \in F^n$, then

$x_m = (x_{m1}, x_{m2}, \dots, x_{mn}), x_{mi} \in F, i = 1, \dots, n$

$x_j = (x_{j1}, x_{j2}, \dots, x_{jn}), x_{ji}, \in F, i = 1 , \dots, n$

$x_m - x_j = (x_{m1} - x_{j1}, x_{m2} - x_{j2} , \dots, x_{mn} - x_{jn})$

Then,

$\|x_m - x_j\| = \{|x_{m1} - x_{j1}| , |x_{m2} - x_{j2}| , \dots, |x_{mn} - x_{jn}|\} < \epsilon \quad \forall m, j > k$

It follows that $|x_{mi} - x_{ji}| < \epsilon, \ \forall i = 1, \dots, n \ and \ \forall m, j > k$

Hence $\langle x_{mi} \rangle$ is a Cauchy sequence in $\mathbb{R}$(or $\mathbb{C}$)

So, it is convergent to $x_i \ in \ F$

Hence , for any $\in > 0, \exists k_i \in Z_+$ such that $|x_{mi} - x_i| < \in, \forall \ m_i > k_i$

$put \ l = \{ k_1, \dots, k_n\}$. then for each $\in > 0$

$|x_{mi} - x_i| < \in, \forall \ m > l, \forall i = 1, \dots, n$

For each $\epsilon > 0$,

$\|x_m - x\| = \{|x_{m1} - x_1|, |x_{m2} - x_2|, \dots, |x_{mn} - x_n|\} < \in, \forall \ \ m > l$

Thus $\langle x_m \rangle$ be a Cauchy sequence in $\mathbb{R}^n$(or $\mathbb{C}^n$)and $x_m \to x$. Thus, $\mathbb{R}^n$(or $\mathbb{C}^n$) is a Banach space

Example (3.4):-

b . Show that $(l^{\infty}, \|.\|)$ is Banach space where $\|x\| = sup\ |x_i|$, $\forall x = (x_1, x_2, \dots) \in l^{\infty}$?

Solution :- (1) To prove that $(l^{\infty}, \|.\|)$ is normed space **(H.W.)**

(2) To prove that $(l^{\infty}, \|.\|)$ is complete space

Let $< x_m >$ be a Cauchy sequence in $l^{\infty} \Longrightarrow x_m \in l^{\infty}$

$$x_m = (x_{m1}, x_{m2}, \dots, x_{mn}, \dots)$$

$$x_m - x_j = (x_{m1} - x_{j1}, \dots, x_{mn} - x_{jn}, \dots)$$

$$\|x_m - x_j\| = sup\ |x_{mi} - x_{ji}| < \epsilon\ ,\ \ \forall m, j > k$$

$$\Longrightarrow |x_{mi} - x_{ji}| < \epsilon\ , \forall m\ , j > k$$

$\Longrightarrow |x_{mi} - x_{ji}| < \epsilon\ , \forall m\ , j > k \Longrightarrow < x_m >$ is Cauchy in F , but F is complete $\Longrightarrow$ $\forall i,\ \exists x_i \in F$ such that

$$x_{mi} \to x_i \Longrightarrow \forall \epsilon > 0, \exists k_i \in Z_+ \text{ such that } |x_{mi} - x_i| < \epsilon$$

Let $k = \{k_1, k_2, \dots\} \Longrightarrow |x_{mi} - x_i| < \epsilon, \qquad \forall\ m > k \dots (1)$

Put $x = (x_1, x_2, \dots)$ to prove that $x \in l^{\infty}$

And $x_m \to x$. Now , since $x_m \in l^{\infty} \Longrightarrow \exists k_m \in \mathbb{R}^{+}$ such that

$$|x_{mi}| < k_m\ , \forall i\ , \text{but } \ x_i = (x_i - x_{mi}) + \ x_{mi}$$

$$|x_i| \leq |x_i - x_{mi}| + |x_{mi}| < \epsilon + k_m \Longrightarrow x \in l^{\infty}$$

By (1) we get, $\quad sup\ |x_{mi} - x_i| < \epsilon\ ,\ \forall m > k$

$\Longrightarrow \|x_m - x\| < \epsilon \ \Longrightarrow\ < x_m >$ is Cauchy sequence

$\Longrightarrow (l^{\infty}, \|.\|)$ is complete. So, $(l^{\infty}, \|.\|)$ is Banach space.

Example (3.5) :-

The space $C[a,b]$ with the norm $\|f\| = \{|f(x)|: x \in [a,b]\}\ \ \forall f \in C[a,b]$ is a Banach space

Solution : Let $\langle f_n \rangle$ be a Cauchy sequence in $C[a,b]$

Then $\forall \in > 0, \exists k \in Z_+$ such that $\|f_m - f_n\| < \in, \forall m,n > k$

Hence , $\forall \in > 0, \exists k \in Z_+$ such that:

$\{|f_m(x) - f_n(x)|: x \in C[a,b]\} < \in\ \forall m,n > k$

It follows that $|f_m(x) - f_n(x)| < \in\ \forall x \in C[a,b],\ \ \forall m,n\ > k$

Hence, $\langle f_n(x) \rangle$ is a Cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is a Banach space , then $\langle f_n(x) \rangle$ is convergent to $f(x)$ in $\mathbb{R}$ thus ,

$\forall \in > 0, \exists k \in N$ such that $|f_m(x) - f(x)| < \in\ \ \forall m \geq k$

Thus , $\|f_m - f\| = \{|f_m(x) - f(x)|: x \in [a,b]\} < \in\ \ \ \forall m \geq k$

Hence , $f_m \to f\ as\ m\ \to \infty$ thus , $C[a,b]$ is a Banach space .

Example(3.6) :-

The space $(C[0,1], \|.\|)$ is not Banach space where $\|f\| = \int_0^1 |f(x)|\, dx$

Solution :- The space $(C[0,1], \|.\|)$ is normed space but not complete space , since there exist Cauchy sequence but not convergent , for example consider

$$f_n(x) = \{1 \qquad if \quad 0 \le x \le \frac{1}{2}$$

$$-nx + \frac{1}{2}n + 1 \qquad if \quad \frac{1}{2} < x \le \frac{1}{2} +$$

$$\frac{1}{n}\ 0 \qquad if \quad \frac{1}{2} + \frac{1}{n} < x \le 1$$

To prove $f_n(x)$ is Cauchy ? Let $m > n > 3$

$$\|f_m - f_n\| = \int_0^1 |(f_m - f_n)(x) dx|$$

$$= \int_0^1 |f_m(x) - f_n(x)|\, dx$$

$$= \int_0^{\frac{1}{2}} |f_m(x) - f_n(x)|\, dx + \int_{\frac{1}{2}}^1 |f_m(x) - f_n(x)|\, dx$$

$$= \int_0^{\frac{1}{2}} |1 - 1|\, dx + \int_{\frac{1}{2}}^1 |f_m(x) - f_n(x)|\, dx$$

$$\le \int_{\frac{1}{2}}^1 |f_m(x)|\, dx + \int_{\frac{1}{2}}^1 |f_n(x)|\, dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} \left|-mx + \frac{1}{2}m + 1\right| dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \left|-nx + \frac{1}{2}n + 1\right| dx$$

$$= \frac{1}{2m} + \frac{1}{2n} .$$

So , $\|f_m - f_n\| \leq \frac{1}{2m} + \frac{1}{2n}$, as $n, m \to \infty \Longrightarrow \|f_m - f_n\| \to 0$

$\Longrightarrow < f_m >$ is Cauchy sequence

But $f_m \to f$, where $f(x) = \begin{cases} 1 & if\ 0 \leq x \leq \frac{1}{2} \\ 0 & if\ \frac{1}{2} < x \leq 1 \end{cases}$

and f is not continuous function $\to f \notin C[a, b]$

$\Longrightarrow < f_m >$ not convergent $\Longrightarrow C[0,1]$ is not Banach space.

Some Important Theorems in Banach Space

Theorem (3.7): Let H be a subspace of Banach space L . then H is Banach space iff H is closed in L

Proof :- $\Longrightarrow$)

If H is Banach space $\Longrightarrow H$ is complete . To prove that H is closed ?

Let $x \in \overline{H} \Longrightarrow$ there exist a sequence $< x_n >$ in H s. t $x_n \to x$.

So , $< x_n >$ is Cauchy sequence

Since H is complete $\Longrightarrow \exists y \in H \;\; s.t \;\; x_n \to y$, But the limit point is unique

$So, x = y \;\Longrightarrow\; x \in H \;\;\Longrightarrow H = \bar{H} \;\;\Longrightarrow H$ is closed

$\Longleftarrow$) Suppose that H is closed set in L. To prove that H is a Banach space ?

It is clear that H is normed space (because every subspace of normed space is normed space). Now , let $< x_n >$ be a Cauchy sequence in $H \subseteq L$

$\Longrightarrow$ the sequence $< x_n >$ is Cauchy sequence in L , but L is complete.

$\Longrightarrow < x_n >$ is convergent sequence in L . i.e, $\exists x \in L \;\; s.t \;\; x_n \to x$

Since $x_n \in H \Longrightarrow x \in \bar{H}$ (by theorem) , but H is closed $. i.e, \;\; H = \bar{H}$

$So, x \in H \;\;\Longrightarrow\;\; < x_n >$ convergent in $H \;\;\Longrightarrow\;\; H$ is complete

Theorem (3.8):- Every finite dimensional normed space is complete space

Proof :- Let $dim \;\; L = n > 0$ and $\{x_1, x_2, \dots, x_n\}$ basis for L. T.p L is complete

Let $< x_m >$ be a Cauchy sequence in $L \Longrightarrow \|x_m - x_j\| < \in, \quad \forall m, j > k$,

i.e., $\|x_m - x_j\| \to 0$, $\quad \forall m, j > k$ ….. (1)

Since, $x_m, xj \in L$. By previous lemma $\Longrightarrow x_m = \sum_{i=1}^{n} \alpha_{mi} x_i \;\;, \alpha_{mi} \in F$

$x_j = \sum_{i=1}^{n} \alpha_{ji} x_i \;\;, \alpha_{ji} \in F \; and \;\; x_m - x_j = \sum_{i=1}^{n} (\alpha_{mi} - \alpha_{ji}) x_i$

Since $\{x_1, x_2, \dots, x_n\}$ is linearly independent $\Longrightarrow \exists c > 0 \;\; s.t$

$\|x_m - x_j\| = \| \sum_{i=1}^{n}(\alpha_{mi} - \alpha_{ji}) \| \geq c \sum_{i=1}^{n}|\alpha_{mi} - \alpha_{ji}|$ ……… (2)

From (1) & (2) we get $\sum_{i=1}^{n}\left|\alpha_{mi} - \alpha_{ji}\right| \to 0 \;\; as\; m, j \to \infty$.

$\Longrightarrow \left|\alpha_{mi} - \alpha_{ji}\right| \to 0$ as $m, j \;\to \infty$, $\forall i$.

$\Longrightarrow < \alpha_{mi} >$ is Cauchy in F & F is complete

$\Longrightarrow \alpha_{mi} \to \alpha_i,\;\; \forall i = 1,2, \dots\dots, n$

$i.e, x_m \to x$, where $x = \sum_{i=1}^{n} \alpha_i x_i$

$\Longrightarrow L$ is complete space

Corollary(3.9):- Every finite dimensional subspace of a Banach space is closed set.

Proof :- Let M is finite dimensional $\Longrightarrow M$ complete $\Longrightarrow M$ is closed (by above theorem)

Dentition (3.10) Quotient Space

Let X be a linear space over F. Let H be a subspace of a linear space L. Let $x + H = \{z;\; z = x + y, x \in L\,, y \in H\}$.

Define addition and scalar multiplication by

$$(x_1 + y) + (x_2 + y) = (x_1 + x_2) + y, \quad \forall \;\; x_1 + y\,,\;\; x_2 + y \in {}^{L}/_{H}$$

$$\alpha.(x + y) = \alpha.x + y, \forall\, x + y \in \frac{L}{H}\,,\;\; \alpha \in F$$

Then the space $(\frac{L}{H},+,.)$ is called quotient space (or factor space)

Proposition (3.11) :-

Prove that $(\frac{L}{H},+,\ .)$ is a linear space over F. **(H.W.)**

Theorem(3.12) :- Let L be a normed space and H be a closed subset of L, then L/H is normed space with $\|.\|_1$ where

$\|x+H\|_1 = inf\ inf\ \{\|x+y\|: y \in H\}$

Proof (1) T.P $\|x+H\|_1 \geq 0$

For any $x+H \in L/H$

$\|x+y\| \geq 0,\ \forall y \in H$

$\{\|x+y\|: y \in H\} \geq 0$

$\|x+H\|_1 = inf\ \{\|x+y\|: y \in H\} \geq 0$

(2) T.P $\|x+H\|_1 = 0 \Leftrightarrow x+H = H = 0_{L/H}$

$(\Longrightarrow) If\ \|x+H\|_1 = 0 \Longrightarrow inf\ \{\|x+y\|: y \in H\} = 0$

Hence , $\exists \langle y_n \rangle \in H$ such that $\|x+y_n\| \to 0$ as $n \to 0$

Hence , $x+y_n \to 0$ as $n \to 0$.Thus , $y_n \to -x$.

Therefore , $\exists\langle y_n\rangle \in H$ such that $y_n \to -x$ thus $-x \in \overline{H}$ (by theorem).

Now, since H is closed , then $-x \in \overline{H} = H$, i.e., $-x \in H$

Since H is a subspace then $x \in H$ and $x + H = H$, that is , $x + H = 0_{L/H}$

$(\Leftarrow) If\ x + H = H = 0_{x/y}\ then\ x \in H. i.e., x + H \in H,\ \forall y \in H$

Hence , $\|x + H\|_1 = inf\ \{\|x + y\|: y \in H\} = inf\ \{\|z\|: z \in H\}$

Since $0 \in Y$ and $\|0\| = 0,$ so $inf\ \{\|z\|: z \in H\} = 0.$ Thus , $\|x + H\|_1 = 0$

(3) T.P $\|\alpha.(x + H)\|_1 = |\alpha|\|x + H\|_1,\ \alpha \in F$

If $\alpha = 0$ then (3) holds

If $\alpha \neq 0$ then

$\|\alpha.(x + H)\|_1 = inf\ \{\|\alpha(x + y)\|: y \in H\}$

$= inf\ \{|\alpha|\|x + y\|: y \in H\}$

$= |\alpha|\ inf\ \{\|x + y\|: y \in H\}$

By using the proposition (If A is bounded below and $\alpha \geq 0$, then $inf\ (\alpha A) = \alpha\ inf\ (A)$)

$= |\alpha|\|x + H\|_1$

(4) Let $x_1 + H$, $x_2 + H \in L/H$

$\|(x_1 + H) + (x_2 + H)\|_1 = \|(x_1 + x_2) + H\|_1$

$inf \ \{\|x_1 + x_2 + y\|: y \in H\}$

$inf \ \{\|x_1 + x_2 + z_1 + z_2\|: z_1, z_2 \in H\}$

$\leq inf \ \{\|x_1 + z_1\| + \|x_2 + z_2\|: z_1, z_2 \in H\}$

$= inf \ \{\|x_1 + z_1\|: z_1 \ \in H\} + inf \ \{\|x_2 + z_2\|: z_2 \in H\}$

$= \|x_1 \ + H\|_1 + \|x_2 + H\|_1$

Thus L/H is a normed space

Proposition (3.13) :- If L is a Banach space and H is a closed subspace of L .Then L/H is a Banach space.

Proof : $L/H = \{x + H: x \in L\}. Let$ $\langle x_n \rangle$ be a Cauchy sequence in L / H then , $x_n = x_n + H \ where \ x_n \in L, \forall n \in Z_+$

$\forall \in > 0, \exists k \in Z_+$ such that $\|x_m - x_n\| < \in \forall n, m > k$

So , $\forall \in > 0, \exists k \in Z_+$ such that $\|x_m - x_n + H\| < \in \forall n, m > k$

Then , $\forall \in > 0, \exists k \in Z_+$ such that:

$inf \ \{\|x_m - x_n + H\|: y \in H\} < \in \forall n, m > k$ .This implies $\forall y \in H,$ $\langle x_n + y \rangle$ is a Cauchy in L

Since L is a Banach space , then $\exists z \in L$ such that $x_n + y \rightarrow z = (z - y) + y$

$= w + y, \ \forall y \in H$

Thus , $x_n + H \rightarrow w + H$. Thus L / H is a Banach space .