Chapter 4

Inner Product Space

Definition 4.1.
Let L is a linear space over F. A mapping $\langle\ ,\rangle : L \times L \to F$ is called an **inner product on** L if the following axioms hold

(1) $\langle x, x\rangle \geq 0 \quad \forall x \in L$.

(2) $\langle x, x\rangle = 0 \iff x = 0_L$.

(3) $\overline{\langle x, y\rangle} = \langle y, x\rangle \quad \forall x, y \in L$, where $\overline{\langle x, y\rangle}$ =conjugate of $\langle x, y\rangle$.

(4) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle \quad \forall x, y, z \in L$ and $\alpha, \beta \in F$.

$(L, \langle\ ,\rangle)$ is called **inner product space** (briefly, I.P.S) or **Pre-Hilbert space**.

Remark 4.2.

(1) If $F = \mathbb{R}$ then axiom (3) becomes $\langle x, y\rangle = \langle y, x\rangle \quad \forall x, y \in L$.

(2) Every subspace of inner product space is an inner product space.

4.1 Examples of Inner Product Space

Example 4.3.

Let $L = \mathbb{R}^2$ and let $\langle \ , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to F$ is defined as $\langle x, y \rangle = x_1y_1 + x_2y_2 \quad \forall x, y \in \ \mathbb{R}^2$ where $x = (x_1, x_2), y = (y_1, y_2)$. Show that $\langle \ , \rangle$ is an inner product on $\mathbb{R}^2$.

<u>Solution:</u> (i) We check the I.P.S axioms

(1) $\langle x, x \rangle = x_1^2 + x_2^2 \geq 0 \quad \forall x = (x_1, x_2) \in \mathbb{R}^2$

(2) $\langle x, x \rangle = 0 \iff x_1^2 + x_2^2 = 0 \iff x_1 = x_2 = 0 \iff x = (0, 0)$

(3) $\langle x, y \rangle = x_1y_1 + x_2y_2 = \overline{\langle x, y \rangle}$ (since $F = \mathbb{R}$)

(4) Let $\alpha, \beta \in \mathbb{R}$ and let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$

$$
\begin{aligned}
\langle \alpha x + \beta y, z \rangle &= \langle (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2), (z_1, z_2) \rangle \\
&= (\alpha x_1 + \beta y_1)z_1 + (\alpha x_2 + \beta y_2)z_2 \\
&= (\alpha x_1z_1 + \alpha x_2z_2) + (\beta y_1z_1 + \beta y_2z_2) \\
&= \alpha(x_1z_1 + x_2z_2) + \beta(y_1z_1 + y_2z_2) \\
&= \alpha\langle x, z \rangle + \beta\langle y, z \rangle
\end{aligned}
$$

Thus, $\langle \ , \rangle$ is an inner product on $\mathbb{R}^2$.

<u>As an application to Example 4.3:</u>

Let $x = (2, 1), y = (0, -3), z = (3, 4)$. Find $\langle x, z \rangle, \langle x, x \rangle, \langle x + y, z \rangle$.

Remark 4.4.

As a generalization of Example 4.3, let $L = \mathbb{R}^n$ and $\langle \ , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to F$ is defined as $\langle x, y\rangle = x_1y_1 + x_2y_2 + ... + x_ny_n \quad \forall x, y \in \mathbb{R}^n$ where $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Then, $(\mathbb{R}^n, \langle \ , \rangle)$ is an inner product space (**check!**). The space $(\mathbb{R}^n, \langle \ , \rangle)$ is called **usual inner space**.

Example 4.5.

Let $L = \mathbb{R}^2$, which of the following is an inner product on L.

(i) $\langle x, y\rangle = 3x_1y_1 + x_2y_2$ (**H.W.**)

(ii) $\langle x, y\rangle = x_1^2y_1^2 + x_2^2y_2^2$

where $x = (x_1, x_2), y = (y_1, y_2)$

<u>**Solution:**</u> (i) We check the I.P.S axioms

(ii) The first three axioms of the definition of inner product hold but the forth condition does not satisfy.

If $\alpha = \beta = 1$ and let $x = (1, -1), y = (-1, 0), z = (-2, 2)$. Then

$\langle \alpha x + \beta y, z\rangle = \langle (0, -1), (-2, 2)\rangle = 0^2(-2)^2 + (-1)^2 2^2 = 4$

and $\alpha\langle x, z\rangle + \beta\langle y, z\rangle = \langle (1, -1), (-2, 2)\rangle + \beta\langle (-1, 0), (-2, 2)\rangle$

$$= 1^1.(-2)^2 + (-1)^2.2^2 + (-1)^2.2^2 + 0^2.2^{=}12$$

Thus, $\langle \alpha x + \beta y, z\rangle \neq \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

Example 4.6.

Let $L = F^n$ be a linear space and let $\langle\ ,\rangle : F^n \times F^n \to F$ defined as

$\langle x, y\rangle = \sum_{i=1}^{n} x_i\overline{y}_i = x_1\overline{y}_1 + ... + x_n\overline{y}_n \quad \forall x, y \in\ F^n$ where $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Show that $\langle\ ,\rangle$ is an inner product on F^n.

<u>Solution</u>:

(1) $\langle x, x\rangle = \sum_{i=1}^{n} x_i\overline{x}_i = \sum_{i=1}^{n} |x_i|^2 \geq 0$

(2) $\langle x, x\rangle = 0 \iff \sum_{i=1}^{n} |x_i|^2 = 0 \iff x_i = 0\ \ \forall i = 1, ..., n$

$$\iff x = (x_1, ..., x_n) = (0, ..., 0) = 0_{F^n}$$

(3) $\langle \overline{x, y}\rangle = \overline{\sum_{i=1}^{n} x_i\overline{y}_i} = \sum_{i=1}^{n} \overline{x}_i y_i = \sum_{i=1}^{n} y_i\overline{x}_i = \langle y, x\rangle$

(4) Let $\alpha, \beta \in F$ and let $x, y, z \in F^n$

$\alpha x + \beta y = (\alpha x_1 + \beta y_1, ..., \alpha x_n + \beta y_n)$

$\langle \alpha x + \beta y, z\rangle = \sum_{i=1}^{n}(\alpha x_i + \beta y_i)\overline{z}_i = \alpha \sum_{i=1}^{n} x_i\overline{z}_i + \beta \sum_{i=1}^{n} y_i\overline{z}_i = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

Thus, $\langle\ ,\rangle$ is an inner product on C^n.

<u>As an application to Example 4.6</u>:

Let $L = C^2$ and $\langle x, y\rangle = \sum_{i=1}^{2} x_i\overline{y}_i \quad \forall x, y \in\ C^2$ where $x = (x_1, x_2), y = (y_1, y_2)$. If $x = (2 + 3i, 1 + i), y = (1 + i, 1 - i), z = (2, 1 + i)$

Find $\langle x, x\rangle, \langle x + y, z\rangle, \langle x, y + z\rangle$

<u>Solution</u>: $\langle x, x\rangle = (2 + 3i)(\overline{2 + 3i}) + (1 + i)(\overline{1 + i})$

$$= (2 + 3i)(2 - 3i) + (1 + i)(1 - i)$$

$$= (4 + 9) + (1 + 1) = 15$$

$x + y = (3 + 4i, 2)$

$\langle x + y, z\rangle = (3 + 4i)2 + 2(\overline{1 + i}) = (6 + 8i) + 2(1 - i) = 8 + 6i$

$\langle x, y + z\rangle =$

Example 4.7.

Let $L = C[0,1]$ be a linear space over $\mathbb{R}$, and let $\langle\ ,\ \rangle : L \times L \to \mathbb{R}$ is defined by $\langle f, g\rangle = \displaystyle\int_0^1 f(x)g(x)\,dx$. Prove that $\langle\ ,\ \rangle$ is an inner product on L.

<u>**Solution**</u>**:** (1) $\langle f, f\rangle = \displaystyle\int_0^1 f(x)f(x)\,dx = \int_0^1 [f(x)]^2\,dx \geq 0$

(2) $\langle f, f\rangle = 0 \iff \displaystyle\int_0^1 [f(x)]^2\,dx = 0 \iff [f(x)]^2 = 0\ \forall x \in [0,1]$

$$\iff f(x) = 0\ \forall x \in [0,1] \iff f = \hat{0}$$

(3) Let $\alpha, \beta \in \mathbb{R}$ and $f, g, h \in L$

$$\langle \alpha f + \beta g, h\rangle = \int_0^1 (\alpha f + \beta g)(x)h(x)\,dx$$

$$= \int_0^1 (\alpha f(x) + \beta g(x)\ h(x)\,dx$$

$$= \alpha \int_0^1 f(x)\ h(x)\,dx + \beta \int_0^1 g(x)\ h(x)\,dx$$

$$= \alpha\langle f, h\rangle + \beta\langle g, h\rangle$$

(4) $\langle f, g\rangle = \displaystyle\int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx = \langle g, f\rangle$

As an application to Example 4.7:

Let $f(x) = x + 1, \;\; g(x) = x^2, \;\; h(x) = 3x + 2 \quad \forall x \in [0, 1]$

Find $\langle f, f\rangle, \langle f + g, h\rangle, \langle f, h\rangle, \langle 2f + 3g, h\rangle, \langle f - g, h - g\rangle$

Example 4.8.

Let $L = \mathbb{R}$ and $\langle\ ,\ \rangle : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\langle x, y\rangle = |xy| \quad \forall x, y \in \mathbb{R}$. Is $(L, \langle\ ,\ \rangle)$ I.P.S? (**H.W.**)

4.2 Some Properties of Inner Product Space

Theorem 4.9.

Let $(L, \langle\ ,\ \rangle)$ be an inner product space (I.P.S). Then, $\forall x, y, z \in L$ and $\alpha, \beta \in F$

(1) $\langle x, 0_L \rangle = \langle 0_L, x \rangle = 0$

(2) $\langle x, \alpha y + \beta z \rangle = \overline{\alpha}\langle x, y \rangle + \overline{\beta}\langle x, z \rangle$.

Proof. (1) $\langle 0_L, x \rangle = \langle 0_L + 0_L, x \rangle$

$$= \langle 0_L, x \rangle + \langle 0_L, x \rangle$$

Hence, $\langle 0_L, x \rangle + 0 = \langle 0_L, x \rangle + \langle 0_L, x \rangle$

Thus, $0 = \langle 0_L, x \rangle$ (I)

Now, $\langle \overline{0_L, x} \rangle = \langle x, 0_L \rangle$

$$\overline{0} = \langle x, 0_L \rangle$$

$$0 = \langle x, 0_L \rangle$$

(2) $\langle x, \alpha y + \beta z \rangle = \langle \overline{\alpha y + \beta z, x} \rangle$

$$= \overline{\alpha \langle y, x \rangle + \beta \langle z, x \rangle}$$

$$= \overline{\alpha}\ \overline{\langle y, x \rangle} + \overline{\beta}\ \overline{\langle z, x \rangle}$$

$$= \overline{\alpha}\ \langle x, y \rangle + \overline{\beta}\ \langle x, z \rangle$$

□

Corollary 4.10.

If $(L, \langle\ ,\ \rangle)$ is an I.P.S. Then

(i) $\left\langle \sum_{i=1}^{n} \alpha_i x_i, y \right\rangle = \sum_{i=1}^{n} \alpha_i \langle x_i, y \rangle$ where $x_1, ..., x_n, y \in L$

(ii) $\langle x, \sum_{i=1}^{n} \beta_i y_i \rangle = \sum_{i=1}^{n} \overline{\beta}_i \langle x, y_i \rangle$ where $x, y_1, ..., y_n \in L$

(iii) $\langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j \rangle = \sum_{i=1}^{n} \alpha_i \big(\sum_{j=1}^{m} \overline{\beta}_j \langle x_i, y_j \rangle \big)$

where $x_1, ..., x_n, y_1, ..., y_m \in L$

Proof. (i) We proof using induction.

If $n = 1$ then $\langle \alpha_1 x_1, y \rangle = \alpha_1 \langle x_1, y \rangle$ (by definition of norm)

If $n = 2$ then $\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$ (by definition of norm)

Suppose (i) hold when $n = k$

$\langle \sum_{i=1}^{k} \alpha_i x_i, y \rangle = \sum_{i=1}^{k} \alpha_i \langle x_i, y \rangle$ **(I)**

To prove (i) hold when $n = k + 1$

T.p. $\langle \sum_{i=1}^{k+1} \alpha_i x_i, y \rangle = \sum_{i=1}^{k+1} \alpha_i \langle x_i, y \rangle$

$$\begin{aligned}
\langle \sum_{i=1}^{k+1} \alpha_i x_i, y \rangle &= \langle \sum_{i=1}^{k} \alpha_i x_i + \alpha_{k+1} x_{k+1}, y \rangle \\
&= \langle \sum_{i=1}^{k} \alpha_i x_i, y \rangle + \langle \alpha_{k+1} x_{k+1}, y \rangle \\
&= \sum_{i=1}^{k} \alpha_i \langle x_i, y \rangle + \alpha_{k+1} \langle x_{k+1}, y \rangle \\
&= \sum_{i=1}^{k+1} \alpha_i \langle x_i, y \rangle
\end{aligned}$$

(ii) The proof is similar to the proof of (i).

(iii) Let $z = \sum_{j=1}^{m} \beta_j y_j$

$$\begin{aligned}
\langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j \rangle &= \langle \sum_{i=1}^{n} \alpha_i x_i, z \rangle \\
&= \sum_{i=1}^{n} \alpha_i \langle x_i, z \rangle \qquad \text{(by part (i))}
\end{aligned}$$

$$= \sum_{i=1}^{n} \alpha_i \big\langle x_i, \sum_{j=1}^{m} \beta_j y_j \big\rangle$$

$$= \sum_{i=1}^{n} \alpha_i \big(\sum_{j=1}^{m} \overline{\beta}_j \langle x_i, y_j \rangle \big) \qquad \text{(by part (ii))} \qquad \square$$

Theorem 4.11.

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. such that $\langle v_1, w\rangle = \langle v_2, w\rangle \quad \forall w \in L$. Then $v_1 = v_2$. Also, if $\langle v_1, w\rangle = 0 \quad \forall w \in L$ then $v_1 = 0_L$.

Proof. By assumption, $\langle v_1 - v_2, w\rangle = \langle v_1, w\rangle - \langle v_2, w\rangle = 0, \quad \forall w \in L$.

Put $w = v_1 - v_2$, then $\langle v_1 - v_2, v_1 - v_2\rangle = 0 \implies v_1 - v_2 = 0 \implies v_1 = v_2$.

Now, $\langle v_1, w\rangle = 0, \quad \forall w \in L \implies \langle v_1, v_1\rangle = 0 \implies v_1 = 0_L$. $\square$

Theorem 4.12. ***General Cauchy Schwarz's Inequality***

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. and let $\|\ \| : L \to \mathbb{R}$ is defined by $\|x\| = \sqrt{\langle x, x\rangle} \quad \forall x \in L$. Then,

$$\big|\langle x, y\rangle\big| \leq \|x\|\,\|y\| \quad \forall x, y \in L.$$

Proof. If $x = 0$ or $y = 0$ then $\langle x, y\rangle = 0$, and hence $\langle x, y\rangle = 0 \leq \|x\|\,\|y\|$

If $y \neq 0$, put $z = \dfrac{y}{\|y\|} \qquad$ **(I)**

$$\|z\|^2 = \langle z, z\rangle = \big\langle \frac{y}{\|y\|}, \frac{y}{\|y\|} \big\rangle$$

$$= \frac{1}{\|y\|^2}\langle y, y\rangle = \frac{1}{\|y\|^2}\,\|y\|^2 = 1 \qquad \textbf{(II)}$$

Next, it is enough to show that $\big|\langle x, z\rangle\big| \leq \|x\|$

because if $\big|\langle x, z\rangle\big| \leq \|x\|$ then from **(I)**

$$\big|\langle x, z\rangle\big| = \left|\big\langle x, \frac{y}{\|y\|}\big\rangle\right| = \frac{1}{\|y\|}\,\big|\langle x, y\rangle\big| \leq \|x\|$$

$$\big|\langle x, y\rangle\big| \leq \|x\|\,\|y\|$$

Let $\alpha \in F$ then $\langle x-\alpha z, x-\alpha z\rangle \geq 0$

$$\langle x,x\rangle - \alpha\langle z,x\rangle - \overline{\alpha}\langle x,z\rangle + \alpha\overline{\alpha}\langle z,z\rangle \geq 0$$

$$\|x\|^2 - \overline{\alpha}\langle x,z\rangle - \alpha\langle z,x\rangle + \alpha\overline{\alpha}\underbrace{\|z\|^2}_{=1 \text{ from (I)}} \geq 0$$

$$\|x\|^2 - \langle x,z\rangle\overline{\langle x,z\rangle} + \langle x,z\rangle\overline{\langle x,z\rangle} - \overline{\alpha}\langle x,z\rangle - \alpha\langle z,x\rangle + \alpha\overline{\alpha} \geq 0$$

$$\|x\|^2 - \left|\langle x,z\rangle\right|^2 + \langle x,z\rangle\left(\overline{\langle x,z\rangle} - \overline{\alpha}\right) - \alpha\left(\langle z,x\rangle - \overline{\alpha}\right)) \geq 0$$

$$\|x\|^2 - \left|\langle x,z\rangle\right|^2 + \langle x,z\rangle\left(\overline{\langle x,z\rangle - \alpha}\right) - \alpha\left(\overline{\langle x,z\rangle - \alpha}\right)) \geq 0$$

$$\|x\|^2 - \left|\langle x,z\rangle\right|^2 + \left(\langle x,z\rangle - \alpha\right)\left(\overline{\langle x,z\rangle - \alpha}\right) \geq 0$$

$$\|x\|^2 - \left|\langle x,z\rangle\right|^2 + \left|\langle x,z\rangle - \alpha\right|^2 \geq 0 \quad \forall \alpha \in F \quad \textbf{(III)}$$

Put $\alpha = \langle x,z\rangle$, then (**III**) becomes

$$\|x\|^2 - \left|\langle x,z\rangle\right|^2 \geq 0 \implies \left|\langle x,z\rangle\right|^2 \leq \|x\|^2$$

$$\left|\langle x,z\rangle\right| \leq \|x\|$$

$$\left|\left\langle x, \frac{y}{\|y\|}\right\rangle\right| \leq \|x\| \quad \text{(using } \textbf{(I)}\text{)}$$

$$\left|\langle x,y\rangle\right| \frac{1}{\|y\|} \leq \|x\|$$

$$\left|\langle x,y\rangle\right| \leq \|x\|\,\|y\|. \qquad \square$$

As an application to Theorem 4.12:

If $L = \mathbb{R}^n$ and $\langle x,y\rangle = \sum_{i=1}^n x_i y_i$ for any $X = (x_1, ..., x_n), Y = (y_1, ..., y_n)$. Apply Cauchy Schwarz inequality.

Sloution: We have , $\|x\| = \left[\langle x,x\rangle\right]^{\frac{1}{2}} = \left[\sum_{i=1}^n x_i^2\right]^{\frac{1}{2}}$ and $\|y\| = \left[\langle y,y\rangle\right]^{\frac{1}{2}} = \left[\sum_{i=1}^n y_i^2\right]^{\frac{1}{2}}$

From Theorem 4.12, $\left|\langle x,y\rangle\right| \leq \|x\|\,\|y\|$; that is

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq [\sum_{i=1}^{n} x_i^2]^{\frac{1}{2}} [\sum_{i=1}^{n} y_i^2]^{\frac{1}{2}}$$

Theorem 4.13.

Every inner product space is a normed space and hence a metric space.

Proof. Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. and let the function $\|\ \| : L \to \mathbb{R}$ is defined by

$\|x\| = \sqrt{\langle x, x\rangle}\quad \forall x \in L.$ To prove $\|\ \|$ is a norm on L

(1) Since $\langle x, x\rangle \geq 0 \quad \forall x \in L \implies \|x\| = \sqrt{\langle x, x\rangle} \geq 0 \quad \forall x \in L$

(2) $\|x\| = 0 \iff \sqrt{\langle x, x\rangle} = 0 \iff \langle x, x\rangle = 0 \iff x = \mathbf{0_X}$

(3) Let $\alpha \in F$ and $x \in L$

$\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle = \alpha\overline{\alpha}\langle x, x\rangle = |\alpha|^2 \|x\|^2$

Thus, $\|\alpha x\| = |\alpha| \|x\|$

(4) T.P. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in L$

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \overline{\langle x, y\rangle} + \langle x, y\rangle + \|y\|^2 \\
&= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2\left|\langle x, y\rangle\right| + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{(by Cauchy Schwarz)}
\end{aligned}$$

$$= (\|x\| + \|y\|)^2$$

Thus, $\|x + y\| \leq \|x\| + \|y\|$. $\square$

Theorem 4.14.

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. and $x, y \in L$. Then

(1) $\|x+y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$ (Polarization Identity)

(2) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (Law of Parallelogram)

(3) $\langle x, y\rangle = \frac{1}{4}\left[\, \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \,\right]$

Proof. (1) $\|x+y\|^2 = \langle x+y, x+y\rangle$

$$= \langle x, x\rangle + \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle$$

$$= \|x\|^2 + \overline{\langle x, y\rangle} + \langle x, y\rangle + \|y\|^2$$

$$= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$$

(2) T.P. $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

By part (1), $\|x+y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$ (**I**)

$$\|x-y\|^2 = \langle x-y, x-y\rangle$$

$$= \langle x, x\rangle - \langle y, x\rangle - \langle x, y\rangle + \langle y, y\rangle$$

$$= \|x\|^2 - \overline{\langle x, y\rangle} - \langle x, y\rangle + \|y\|^2$$

$$= \|x\|^2 - 2Re\langle x, y\rangle + \|y\|^2 \qquad (\mathbf{II})$$

By summing up (**I**) and (**II**) we get $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(3) By parts (1) and (2), we have

$$\|x+y\|^2-\|x-y\|^2=\|x\|^2+2Re\langle x,y\rangle+\|y\|^2-\left(\|x\|^2-2Re\langle x,y\rangle+\|y\|^2\right)$$

$$=2Re\langle x,y\rangle+2Re\langle x,y\rangle$$

$$=\langle\overline{x,y}\rangle+\langle x,y\rangle+\langle\overline{x,y}\rangle+\langle x,y\rangle$$

$$=2\langle y,x\rangle+2\langle x,y\rangle \qquad (\mathbf{I})$$

$$\|x+iy\|^2=\langle x+iy,x+iy\rangle$$

$$=\langle x,x\rangle+i\langle y,x\rangle+\bar{i}\langle x,y\rangle+\langle y,y\rangle$$

$$=\|x\|^2+i\langle y,x\rangle-i\langle x,y\rangle+\|y\|^2$$

$$\|x-iy\|^2=\langle x-iy,x-iy\rangle$$

$$=\langle x,x\rangle-i\langle y,x\rangle-\bar{i}\langle x,y\rangle+\langle y,y\rangle$$

$$=\|x\|^2-i\langle y,x\rangle+i\langle x,y\rangle+\|y\|^2$$

Hence we get,

$$i\,\|x+iy\|^2-i\,\|x-iy\|^2=i\left[\,\|x\|^2+i\langle y,x\rangle-i\langle x,y\rangle+\|y\|^2\,\right]-i\left[\,\|x\|^2-i\langle y,x\rangle+i\langle x,y\rangle+\|y\|^2\,\right]$$

$$=i\,\|x\|^2-\langle y,x\rangle+\langle x,y\rangle+i\,\|y\|^2-i\,\|x\|^2-\langle y,x\rangle+\langle x,y\rangle-i\,\|y\|^2$$

$$=2\langle x,y\rangle-2\langle y,x\rangle \qquad (\mathbf{II})$$

By (**I**) and (**II**), we have

$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 2\langle y,x\rangle + 2\langle x,y\rangle + 2\langle x,y\rangle - 2\langle y,x\rangle$

$$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 4\langle x,y\rangle$$

$$\frac{1}{4}\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = \langle x,y\rangle \qquad \square$$

Remark 4.15.

Any normed linear space generated from inner product space must satisfies the three laws of Theorem 4.14.

Example 4.16.

Let $L = C[a,b]$ and let $\|f\| = \max\{|f(x)| : x \in [a,b]\}$. Then the converse of Theorem 4.13. i.e.,

(1) Show that $(L, \|\ \|)$ is a normed linear space (**H.W.**)

(2) Show that L is not generated by I.P.S (i.e, L is not I.P.S)

Solution: (2) To show that L is not I.P.S, we shall show that parallelogram law does not hold. i.e., $\|f+g\|^2 + \|f-g\|^2 \neq 2\|f\|^2 + 2\|g\|^2$ for some $f, g \in C[a,b]$.

Let $f(x) = 1$ and $g(x) = \dfrac{x-a}{b-a} \quad \forall x \in [a,b]$

Note that f, g are continuous on $[a,b]$. Thus, $f, g \in C[a,b]$.

$\|f\| = 1$ and $\|g\| = 1$

$$\|f+g\| = \left\|1 + \frac{x-a}{b-a}\right\| = \max\left\{\left|1 + \frac{x-a}{b-a}\right| : x \in [a,b]\right\} = 2$$

$$\|f-g\| = \left\|1 - \frac{x-a}{b-a}\right\| = \max\left\{\left|1 - \frac{x-a}{b-a}\right| : x \in [a,b]\right\} = 1$$

$\|f+g\|^2 + \|f-g\|^2 = 4 + 1 = 5 \qquad$ (**I**)

$2\|f\|^2 + 2\|g\|^2 = 2.1^2 + 2.1^2 = 4 \qquad$ (**II**)

By (**I**) and (**II**), we get $\|f+g\|^2 + \|f-g\|^2 \neq 2\|f\|^2 + 2\|g\|^2$

i.e., $5 \neq 4$

Example 4.17.

Let $L = \mathbb{R}^2$ and let $\|x\| = |x_1| + |x_2| \quad \forall x = (x_1, x_2) \in \mathbb{R}^2$. Then the converse of Theorem 4.13. i.e.,

(1) Show that $(\mathbb{R}^2, \|\ \|)$ is a normed linear space (**H.W.**)

(2) Show that $\mathbb{R}^2$ is not generated by I.P.S (i.e, $\mathbb{R}^2$ is not I.P.S)

Solution: (2) To show that L is not I.P.S, we shall show that parallelogram law does not hold. i.e., $\|x+y\|^2 + \|x-y\|^2 \neq 2\|x\|^2 + 2\|y\|^2$ for some $x, y \in \mathbb{R}^2$

Let $x = (2, 3)$ and $y = (-6, 1)$

$\|x\| = |2| + |3| = 5 \implies 2\|x\|^2 = 50$

$\|y\| = |-6| + |1| = 7 \implies 2\|y\|^2 = 98$

$\|x+y\| = \|(-4, 4)\| = |-4| + |4| = 8$

$\|x+y\|^2 = 64$

$\|x-y\| = \|(8, 2)\| = |8| + |2| = 10$

$\|x-y\|^2 = 100$

Thus, $\|x+y\|^2 + \|x-y\|^2 = 64 + 100 = 164$

and $2\|x\|^2 + 2\|y\|^2 = 50 + 98 = 148$

Hence, $\|x+y\|^2 + \|x-y\|^2 \neq 2\|x\|^2 + 2\|y\|^2$

i.e., $\|\ \|$ does not satisfy paralleogram law.

Example 4.18.

Let $L = \mathbb{R}^2$ and let $\|x\| = \max\{|x_1|, |x_2|\} \quad \forall (x_1, x_2) \in \mathbb{R}^2$. Then

(1) Show that $(\mathbb{R}^2, \| \, \|)$ is a normed linear space (**H.W.**)

(2) Is $\mathbb{R}^2$ generated by I.P.S? (**H.W.**)

Theorem 4.19.

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. Then

(1) If $(x_n) \to x$ and $(y_n) \to y$ then $\langle x_n, y_n \rangle \to \langle x, y \rangle$

(2) If (x_n) and (y_n) are Cauchy sequences in L then $\langle x_n, y_n \rangle$ is a Cauchy sequence in F.

Proof. (1) $\langle x_n, y_n \rangle = \langle x + (x_n - x), y + (y_n - y) \rangle$

$$= \langle x, y \rangle + \langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle$$

$$\langle x_n, y_n \rangle - \langle x, y \rangle = \langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle$$

$$\left|\langle x_n, y_n \rangle - \langle x, y \rangle\right| = \left|\langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle\right|$$

$$\leq \left|\langle x, y_n - y \rangle\right| + \left|\langle x_n - x, y \rangle\right| + \left|\langle x_n - x, y_n - y \rangle\right|$$

$$\leq \|x\|\,\|y_n - y\| + \|x_n - x\|\,\|y\| + \|x_n - x\|\,\|y_n - y\|$$ (By Cauchy Schwarz)

But $(x_n) \to x$ and $(y_n) \to y$ then $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$

Hence, $\left|\langle x_n, y_n \rangle - \langle x, y \rangle\right| \to 0$, and hence, $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

(2) for any $n, m \in Z_+$

$$\langle x_n, y_n \rangle = \langle (x_n - x_m) + x_m, (y_n - y_m) + y_m \rangle$$

$$= \langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_n - x_m, y_m \rangle$$

$$\langle x_n, y_n \rangle - \langle x_m, y_m \rangle = \langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle$$

$$\left|\langle x_n, y_n \rangle - \langle x_m, y_m \rangle\right| = \left|\langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle\right|$$

$$\leq \left|\langle x_n - x_m, y_n - y_m \rangle\right| + \left|\langle x_m, y_n - y_m \rangle\right| + \left|\langle x_m, y_n - y_m \rangle\right|$$

$\leq \|x_n - x_m\| \, \|y_n - y_m\| + \|x_m\| \, \|y_n - y_m\| + \|x_n - x_m\| \, \|y_m\|$ (By Cauchy Schwarz)

But (x_n) and (y_n) are Cauchy sequences, then $\|x_n - x_m\| \to 0$ and $\|y_n - y_m\| \to 0$ as $n \to \infty$. Also, (x_n) and (y_n) are bounded sequences, then as $n \to \infty$

$$\left|\langle x_n, y_n \rangle - \langle x_m, y_m \rangle\right| \to 0$$ □

Corollary 4.20.

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. Then

(1) If $(x_n) \to x$ then $\|x_n\| \to \|x\|$

(2) If (x_n) is a Cauchy sequences in L then $\langle\, \|x_n\| \,\rangle$ is a convergent sequence in $\mathbb{R}$.

Proof. (1) Since $(x_n) \to x$ then $\langle x_n, x_n \rangle \to \langle x, x \rangle$ (By Theorem 4.19)

Hence, $\|x_n\|^2 \to \|x\|^2$. i.e., $\|x_n\| \to \|x\|$

(2) Since (x_n) is a Cauchy sequences in L, then by Theorem 4.19(2), $\langle x_n, x_n \rangle$ is a Cauchy sequence in F. Since $F = \mathbb{R}$ or C then F is complete. Thus, $\langle \|x_n\|^2 \rangle$ is a convergent sequence in F. Thus, $\langle\, \|x_n\| \,\rangle$ is a convergent sequence in F □

4.3 Hilbert Space

Definition 4.21.
Hilbert space is an I.P.S. $(L, \langle\ ,\ \rangle)$ which is a Banach space with respect to $\|x\| = \sqrt{\langle x, x\rangle}$.

Example 4.22.

Consider the I.P.S. $(\mathbb{R}^n, \langle\ ,\ \rangle)$ (or $(C^n, \langle\ ,\ \rangle)$ such that $\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y}_i$ where $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$ (or C^n). (see Example 4.6)

Show that $(\mathbb{R}^n, \langle\ ,\ \rangle)$ (or $(C^n, \langle\ ,\ \rangle)$ is Hilbert space.

Solution: Since $\sqrt{\langle x, x\rangle} = \left[\sum_{i=1}^{n} x_i \overline{x}_i\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} = \|x\|$

From Example 3.2, $\mathbb{R}^n$ (or C^n) is a Banach space w.r.t. $\|x\| = \sqrt{\langle x, x\rangle}$, and thus, $(\mathbb{R}^n, \langle\ ,\ \rangle)$ (or $(C^n, \langle\ ,\ \rangle)$ is a Hilbert space.

Example 4.23.

The space $C[-1, 1]$ with the inner product defined by $\langle f, g\rangle = \displaystyle\int_{-1}^{1} f(x)\ g(x)\ dx$ is not a Hilbert space.

Solution: Let

$$f_n(x) = \begin{cases} 0 & \text{if} \quad -1 \le x \le 0 \\ nx & \text{if} \quad 0 < x < \frac{1}{n} \\ 1 & \text{if} \quad \frac{1}{n} \le x \le 1 \end{cases}$$

$$\|f_n - f_m\|^2 = \langle f_n - f_m, f_n - f_m\rangle$$

Suppose $n > m$, then $\dfrac{1}{n} < \dfrac{1}{m}$. We must find $f_n(x) - f_m(x)$

$$f_n(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ nx & \text{if} \quad 0 < x < \frac{1}{n} \\ 1 & \text{if} \quad \frac{1}{n} \leq x \leq 1 \end{cases}$$

and

$$f_m(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ mx & \text{if} \quad 0 < x < \frac{1}{m} \\ 1 & \text{if} \quad \frac{1}{m} \leq x \leq 1 \end{cases}$$

Then

$$f_n(x) - f_m(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ (n-m)x & \text{if} \quad 0 < x < \frac{1}{n} \\ 1 - mx & \text{if} \quad \frac{1}{n} \leq x \leq \frac{1}{m} \\ 0 & \text{if} \quad \frac{1}{m} \leq x \leq 1 \end{cases}$$

$$\|f_n - f_m\|^2 = \int_{-1}^{1} (f_n(x) - f_m(x))^2 \ dx = \int_0^{\frac{1}{n}} (n-m)^2 x^2 \ dx + \int_{\frac{1}{n}}^{\frac{1}{m}} (1 - mx)^2 \, dx$$

$$= \frac{(n-m)^2 x^3}{3}\Big]_0^{\frac{1}{n}} + (\frac{-1}{m}) \frac{(1-mx)^3}{3}\Big]_{\frac{1}{n}}^{\frac{1}{m}}$$

$$= \frac{(n-m)^2}{3} \frac{1}{n^3} - \frac{1}{m}\Big[0 - \frac{1}{3}(1 - \frac{m}{n})^3\Big]$$

$$= \frac{(n-m)^2}{3n^3} + \frac{1}{3m}\left(\frac{n-m}{n}\right)^3$$

$$= \frac{(n-m)^2}{3n^2 m}$$

Thus, $\|f_n - f_m\|^2 = \dfrac{(n-m)^2}{3n^2 m}$

Since $n > m$, then $n = m + t$

$\|f_n - f_m\|^2 = \dfrac{t^2}{3(m+t)^2 m} \to 0 \quad$ as $m \to \infty$

Hence, $\|f_n - f_m\| \to 0$. Thus, $\langle f_n \rangle$ is a Cauchy sequence.

But $f_n \to f$ where

$$f(x) = \begin{cases} 0 & \text{if} \quad -1 \le x \le 0 \\ 1 & \text{if} \quad 0 < x \le 1 \end{cases}$$

Thus, $f \notin C[-1,1]$. Then, $\langle f_n \rangle$ is not convergent in $C[-1,1]$. i.e., The space is not Hilbert space.

Remark 4.24.

Every Hilbert space is a Banach space but the converse is not true. For example, the space $C[a,b]$ with $\|f\| = \max\{|f(x)| : x \in [a,b]\}$ is a Banach space (see Example 3.5). However, $C[a,b]$ is not a Hilbert space since it does not satisfy parallelogram law; that is $\|\ \|$ can not be obtained from inner product (see Example 4.16).

4.4 Orthogonality and Orthonormality in Inner Product Space

Definition 4.25. ***orthogonal Elements***
Let $(L, \langle\ ,\rangle)$ be an I.P.S and $x, y \in L$. Then x is said to be **orthogonal** on y (denoted by $x \perp y$) if and only if $\langle x, y\rangle = 0$.

Example 4.26.

Let $L = \mathbb{R}^2$ is I.P.S such that $\langle x, y\rangle = x_1y_1 + x_2y_2$ is usual inner product $\forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Let $x = (-6, 3), y = (2, -1), z = (1, 2)$. Show that $x \perp z, y \perp z$ and $y \not\perp x$.

Solution: $\langle x, z\rangle = \langle(-6, 3), (1, 2)\rangle = -6 + 6 = 0$. Hence, $x \perp z$.

$\langle y, z\rangle =$

$\langle y, x\rangle =$

Proposition 4.27.

Let $(L, \langle\ ,\rangle)$ be an I.P.S and $x, y \in L$. Then

(i) If $x \perp y$ then $y \perp x$.

(ii) $\mathbf{0}_L \perp x \ \ \forall x \in L$. (**H.W.**)

(iii) if $x \perp x$ then $x = \mathbf{0}_L$. (**H.W.**)

Proof. (1) Let $x \perp y$ then $\langle x, y\rangle = 0$. From Definition 4.1(3), we have $\langle y, x\rangle = \overline{\langle x, y\rangle} = \overline{0}$. i.e., $y \perp x$. □

Proposition 4.28.

Let $(L, \langle\ ,\rangle)$ be an I.P.S and $x, x_1, ..., x_n \in L$ such that x is orthogonal on $x_1, ..., x_n$. Prove that x is orthogonal on any linear combination of $x_1, ..., x_n$.

Proof. Let w be a linear combination of $x_1, ..., x_n$. i.e., there exists $\alpha_i \in F$ such that $w = \sum_{i=1}^n \alpha_i x_i$. We must show $\langle x, w\rangle = 0$.

$$\begin{aligned}\langle x, w\rangle = \langle x, \textstyle\sum_{i=1}^n \alpha_i x_i\rangle &= \textstyle\sum_{i=1}^n \overline{\alpha_i}\langle x, x_i\rangle \quad \text{(by Corollary 4.9(ii))}\\ &= \textstyle\sum_{i=1}^n \overline{\alpha_i}.0 \quad \text{(From the assumption)}\\ &= 0.\end{aligned}$$

□

Example 4.29.

(1) Find the value of a that makes the vectors $X = (a, 2, -1), Y = (3, -5, 2)$ orthogonal vectors in $\mathbb{R}^3$ with usual inner product. (**H.W.**)

(2) Let $(L, \langle\ ,\rangle)$ be an I.P.S over $\mathbb{R}$ and let $x, y \in L$ such that $\|x\| = \|y\| = 1$ (i.e., x and y are normal elements). Prove that $x + y \perp x - y$.

Answer: $\langle x + y, x - y\rangle = \langle x, x\rangle - \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle = \|x\|^2 - \langle x, y\rangle + \langle x, y\rangle - \|y\|^2 = 0$. Hence, $x + y \perp x - y$.

(3) Let $(L, \langle\ ,\rangle)$ be an I.P.S and let $x, y \in L$ such that $x \perp y$. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2 = \|x - y\|^2$.

Answer: $$\begin{aligned}\|x + y\|^2 = \langle x + y, x + y\rangle &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle\\ &= \|x\|^2 + 0 + 0 + \|y\|^2 = \|x\|^2 + \|y\|^2\end{aligned}$$

Similarly, $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.

(4) Let $(L, \langle\ ,\rangle)$ be an I.P.S and let $x, y \in L$ such that $x \perp y$. Prove that $\|x + \lambda y\| = \|x - \lambda y\|$.

Answer: (H.W.)

(5) Let $(L, \langle\ ,\rangle)$ be an I.P.S and let $x_1, x_2, ..., x_n \in X$ such that $x_i \perp x_j\ \ \forall i \neq j$. Prove that $\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$.

Answer: We prove using induction. If $n = 1$, the statement is true.

If $n = 2$. Since $x_1 \perp x_2$ then $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$ (by part (3)).

Suppose the statement is true for $n = k$. i.e., $\left\|\sum_{i=1}^{k} x_i\right\|^2 = \sum_{i=1}^{k} \|x_i\|^2$

To prove the statement is true when $n = k + 1$. i.e.,

T.P. $\left\|\sum_{i=1}^{k+1} x_i\right\|^2 = \sum_{i=1}^{k+1} \|x_i\|^2$

$$\left\|\sum_{i=1}^{k+1} x_i\right\|^2 = \left\|\sum_{i=1}^{k} x_i + x_{k+1}\right\|^2 = \left\|\sum_{i=1}^{k} x_i\right\|^2 + \|x_{k+1}\|^2$$

$$= \sum_{i=1}^{k} \|x_i\|^2 + \|x_{k+1}\|^2 \text{ (by induction } n = k)$$

$$= \sum_{i=1}^{k+1} \|x_i\|^2.$$

Definition 4.30. ***Orthogonal to Set***

Let $(L, \langle\ ,\rangle)$ be an I.P.S, $x \in L$, and $A \subseteq X$. Then, x is said to be orthogonal on A ($x \perp A$) if $x \perp a\ \ \forall a \in A$.

Example 4.31.

Consider the space $\mathbb{R}^2$ with usual product space and $A = \{(0, a) : a \in \mathbb{R}\}$.

Then $(2, 0) \perp A$ because $\langle(2, 0), (0, a)\rangle = 2.0 + 0.a = 0$.

Definition 4.32. ***Orthogonal Sets***
Let $(L, \langle\ ,\rangle)$ be an I.P.S, and $A, B \subseteq L$. Then, A is said to be orthogonal to B ($A \perp B$) if $a \perp b$, $\forall a \in A, \forall b \in B$.

Example 4.33.

Consider the space $\mathbb{R}^2$ with usual inner product and $A = \{(0, a) : a \in \mathbb{R}\}$ and $B = \{(b, 0) : b \in \mathbb{R}\}$. Show that $A \perp B$.

Answer: for each $(0, a) \in A$ and for each $(b, 0) \in B$, then $\langle (a, 0), (0, b)\rangle = a.0 + 0.b = 0$. Thus, $A \perp B$.

Proposition 4.34.

Let $(L, \langle\ ,\rangle)$ be an I.P.S, and $A, B \subseteq L$ such that $A \perp B$ then $A \cap B = \{\mathbf{0}\}$.

Proof. Let $x \in A \cap B \Rightarrow x \in A$ and $x \in B$ (**I**)

Since $A \perp B \Rightarrow \langle a, b\rangle = 0, \quad \forall a \in A, \forall b \in B$.

From (**I**), $\langle a, b\rangle = \langle x, x\rangle = 0$.

Using Definition 4.1(2), $x = \mathbf{0}$, then $A \cap B = \{\mathbf{0}\}$. □

Definition 4.35.
Let $(L, \langle\ ,\rangle)$ be an I.P.S. and $\phi \neq A \subseteq L$. Then, the set

$$A^{\perp} = \{x \in L : x \perp a, \quad \forall a \in A\}$$

is called the orthogonal complement on A.

Proposition 4.36.

Let $(L, \langle\ ,\rangle)$ be an I.P.S. and $\phi \neq A, B \subseteq L$. Then,

(1) $L^{\perp} = \{\mathbf{0}\}$.

(2) $\{\mathbf{0}\}^{\perp} = L$. (**H.W.**)

(3) $A \cap A^{\perp} = \{\mathbf{0}\}$.

(4) $A \subseteq A^{\perp\perp}$.

(5) If $A \subseteq B$ then $B^{\perp} \subseteq A^{\perp}$. (**H.W.**)

(6) If $A \subseteq B^{\perp}$ then $B \subseteq A^{\perp}$.

Proof. (1) $L^{\perp} = \{x \in L : x \perp L\} = \{x \in L : \langle x, l\rangle = 0, \forall l \in L\} = \{\mathbf{0}\}$.

(3) Let $x \in A \cap A^{\perp} \Rightarrow x \in A$ and $x \in A^{\perp}$ **(I)**

Since $x \in A^{\perp}$ then $x \perp A$ **(II)**

From **(I)** and **(II)**, $x \perp x$. i.e., $\langle x, x\rangle = 0$, thus $x = \mathbf{0}$.

Then, $A \cap A^{\perp} = \{\mathbf{0}\}$.

(4) To prove $A \subseteq A^{\perp\perp}$. Let $x \in A$.

For any $y \in A^{\perp} \Rightarrow y \perp A$. In particular, $y \perp x$ $(x \in A)$

From Proposition 4.27(1), $x \perp y$, $\forall y \in A^{\perp}$. Thus, $x \in A^{\perp\perp}$.

(6) Let $A \subseteq B^{\perp}$, then from part (5), $B^{\perp\perp} \subseteq A^{\perp}$.

Now, from part (4), $B \subseteq B^{\perp\perp} \subseteq A^{\perp}$. Then, $B \subseteq A^{\perp}$. $\square$

Theorem 4.37.

Let $(L, \langle\ ,\rangle)$ be an I.P.S. and $\phi \neq A \subseteq L$. Then, $A^{\perp}$ is a closed subspace of L.

Proof. (1) To prove $A^{\perp}$ is a subspace of L.

Let $x, y \in A^{\perp}$ and $\alpha, \beta \in F$. T.P. $\alpha x + \beta y \in A^{\perp}$

$$\text{T.P. } \langle \alpha x + \beta y, a\rangle = 0, \ \ \forall a \in A.$$

Since $x, y \in A^{\perp} \Rightarrow \langle x, a\rangle = \langle y, b\rangle = 0 \qquad$ (**I**)

$\langle \alpha x + \beta y, a\rangle = \alpha\langle x, a\rangle + \overline{\beta}\langle y, a\rangle = \alpha.0 + \overline{\beta}.0 = 0 \qquad$ [from (I)]

Thus, $A^{\perp}$ is a subspace of L.

(2) T.P. $A^{\perp}$ is a closed set (i.e., $A^{\perp} \subseteq \overline{A^{\perp}}$ and $\overline{A^{\perp}} \subseteq A^{\perp}$)

It is clear that $A^{\perp} \subseteq \overline{A^{\perp}} \qquad$ (**I**)

T.P. $\overline{A^{\perp}} \subseteq A^{\perp}$. Let $x \in \overline{A^{\perp}}$ then $\exists (x_n) \in A^{\perp}$ such that $(x_n) \to x$.

Since $(x_n) \in A^{\perp}, \quad \forall n \in N \Rightarrow x_n \perp A \Rightarrow x_n \perp a, \ \ \forall a \in A$

$$\Rightarrow \langle x_n, a\rangle = 0, \ \ \forall a \in A.$$

But $(x_n) \to x$ and $a \to a$. Thus, from Theorem 4.19(1), $\underbrace{\langle x_n, a\rangle}_{=0} \to \langle x, a\rangle$.

$\langle x, a\rangle = 0 \ \ \forall a \in A$. Then, $x \in A^{\perp}$. Thus, $\overline{A^{\perp}} \subseteq A^{\perp} \qquad$ (**II**).

From (**I**) and (**II**), $A^{\perp}$ is a closed set.

□

Definition 4.38. ***Orthonormal Set***

Let $(L, \langle\ ,\rangle)$ be an I.P.S. and $A \subseteq L$. Then, A is called **orthonormal set** if

(1) A is said to be **orthogonal** if $x \perp y \ \ \forall x, y \in A, \ \ x \neq y$.

(2) Each element $x \in A$ is a normal element. i.e., $\langle x, x\rangle^{\frac{1}{2}} = \|x\| = 1 \ \ \forall x \in A$.

Remark 4.39.

Orthonarmal set has no zero element ($\mathbf{0} \notin A$) because $\|\mathbf{0}\| \neq 1$ ($\mathbf{0}$ is not normal element).

Example 4.40.

Let $L = \mathbb{R}^3$ with usual inner product and $A = \{(1,2,2),(2,1,-2),(2,-2,1)\} \subset L$. Show that A is orthogonal but not orthonormal.

<u>**Solution:**</u> T.P. A is orthogonal set **(H.W.)**.

To show not every vector in A is normal. i.e.,

$\|(1,2,2)\|^2 = \langle(1,2,2),(1,2,2)\rangle = 1+4+4 = 9 \neq 1 \Rightarrow \|(1,2,2)\| \neq 1$.

Thus, A is not orthonormal.

Theorem 4.41.

Let L be an I.P.S. and $x_1, ..., x_n$ be orthonormal vectors in L. Then

$$\sum_{i=1}^{n} |\langle x, x_i\rangle|^2 \leq \|x\|^2 \quad \forall x \in L$$

.

Example 4.42.

Let $L = \mathbb{R}^3$ and $x_1 = \frac{1}{3}(1,2,2), x_2 = \frac{1}{3}(2,1,-2), x_3 = \frac{1}{3}(2,-2,1)$.

Let $x = (2,1,3)$. Then

$|\langle x, x_1\rangle|^2 = [\frac{1}{3}(2+2+6)]^2 = \dfrac{100}{9}$

$|\langle x, x_2\rangle|^2 = [\frac{1}{3}(4+1-6)]^2 = \dfrac{1}{9}$

$|\langle x, x_3\rangle|^2 = [\frac{1}{3}(4-2+3)]^2 = \dfrac{25}{9}$

$\displaystyle\sum_{i=1}^{3} |\langle x, x_i\rangle|^2 = \frac{100}{9} + \frac{1}{9} + \frac{25}{9} = 14.$

on the other hand, $\|x\|^2 = \langle x, x\rangle = 4 + 1 + 9 = 14$.

As in Theorem 4.41, $\displaystyle\sum_{i=1}^{3} |\langle x, x_i\rangle|^2 = \|x\|^2$

Take $x = (1, 1, 1)$ and apply Theorem 4.41. **(H.W.)**

Theorem 4.43.

Let $(L, \langle\ ,\rangle)$ be an I.P.S. Let (x_n) be an orthonormal sequence in L and (λ_n) be a sequence in F such that $\sum_{i=1}^{+\infty} |\lambda_i|^2 < +\infty$. Let $y_n = \sum_{i=1}^{n} \lambda_i x_i$. Then, (y_n) is a Cauchy sequence.

Proof. Let $y_n = \sum_{i=1}^{n} \lambda_i x_i$, $\quad y_m = \sum_{i=1}^{m} \lambda_i x_i$. Assume that $n < m$ then $m = n + k$ for some $k \in N$. We must prove $\|y_m - y_n\| \to 0$.

$y_m - y_n = \sum_{i=1}^{m} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n+k} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=n+1}^{n+k} \lambda_i x_i$.

$$\begin{aligned}
\|y_m - y_n\|^2 = \left\|\sum_{i=n+1}^{n+k} \lambda_i x_i\right\|^2 &= \langle \sum_{i=n+1}^{n+k} \lambda_i x_i,\ \sum_{i=n+1}^{n+k} \lambda_i x_i\rangle \\
&= \sum_{i=n+1}^{n+k} \lambda_i \sum_{i=n+1}^{n+k} \overline{\lambda_i}\langle x_i, x_i\rangle \\
&= \sum_{i=n+1}^{n+k} \lambda_i \overline{\lambda_i}\langle x_i, x_i\rangle \\
&= \sum_{i=n+1}^{n+k} |\lambda_i|^2 \|x_i\|^2 \\
&= \sum_{i=n+1}^{n+k} |\lambda_i|^2 \qquad (\|x_i\|^2 = 1 \ \ \forall i)
\end{aligned}$$

As $n \to +\infty$, $\sum_{i=n+1}^{n+k} |\lambda_i|^2 \to 0 \qquad (\sum_{i=1}^{+\infty} |\lambda_i|^2$ convergent)

Thus, $\|y_m - y_n\|^2 \to 0$ which means $\|y_m - y_n\| \to 0$. Hence, (y_n) is a Cauchy sequence. □