جامعة بغداد

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قسم الرياضيات/ المرحلة الاولى

مادة التفاضل والتكامل

الفصل الرابع

الاشتقاق

Differentiation

اعضاء التدريس

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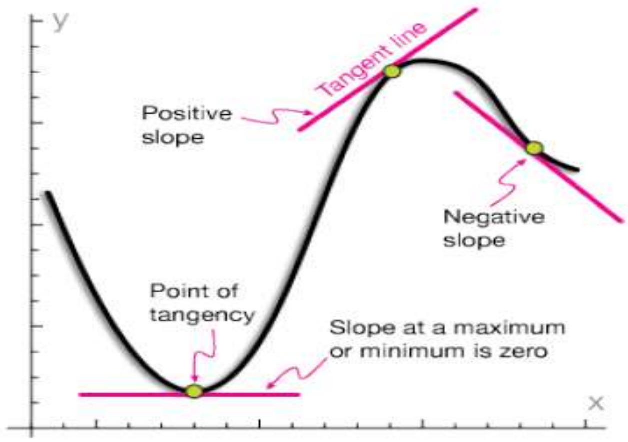
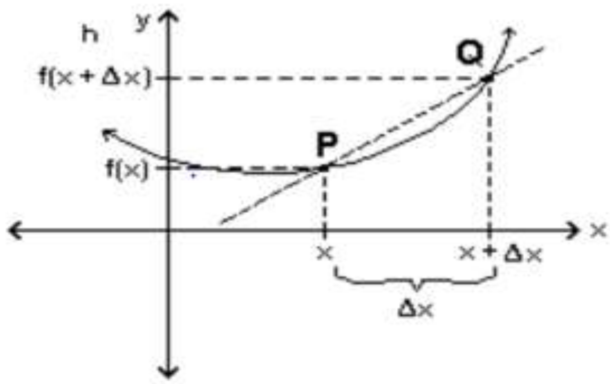
CHAPTER FOUR: Differentiation

For each point on the curve $y = f(x)$, there is a single straight tangent line at the point; the slop of straight tangent of the curve $y = f(x)$ at the point $(x, f(x))$ represents the derivative at that point



Let $P(x, f(x))$ be a fixed point on the curve; and $Q(x + \Delta x, f(x + \Delta x))$ be another point, so $\Delta y = f(x + \Delta x) - f(x)$.

$$\boldsymbol{m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$



Note that: At Δx, decreasing length (close to zero) the straight secant PQ more and more applicability begins on the straight tangent at the point $(x, f(x))$. When $(\Delta x \to 0)$, knowing that the slop straight tangent at the point $(x, f(x))$ represents a derived function at that point.

$$\mathbf{m}_{tan} = \lim_{\Delta x \to 0} m_{sec} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\boldsymbol{f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

Remark: When the value of the limit exist, the function is called differentiable function, and $f\,'$ is called the derivative of f at x.

Remark: The equation of the tangent line at a point (x_1, y_1) is given by the following form:

$$(\boldsymbol{y}-\boldsymbol{y_1})=\boldsymbol{m_{tan}}(\boldsymbol{x}-\boldsymbol{x_1})$$

Definition: The normal line of the curve is the line that is perpendicular to the tangent of the curve at a particular

$$\boldsymbol{m_{\perp}}=\frac{\boldsymbol{-1}}{\boldsymbol{m_{tan}}}$$

Remark: The equation of the normal line at a point (x_1, y_1) is given by the following form:

$$(\boldsymbol{y}-\boldsymbol{y_1})=\boldsymbol{m_{\perp}}(\boldsymbol{x}-\boldsymbol{x_1})$$

Note: $\boldsymbol{f\,'(x)} = \mathbf{y}' = \frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \frac{\boldsymbol{df(x)}}{\boldsymbol{dx}}$

Example 1: Let $f(x) = 4x - 2$, find $f\,'(x)$by using the definition?

Solution:-

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[4(x+\Delta x) - 2] - (4x-2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4x + 4\Delta x - 2 - 4x + 2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4\Delta x}{\Delta x} = 4$$

Example 2: Let $f(x) = \sqrt{x}$, find the equation of the tangent line and normal line at the point (4,2) by using the definition?

Solution:-

We need to find: $m_{tan}\Big|_{(4,2)} = f'(x)\Big|_{(4,2)}$

$$\Rightarrow f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow m_{tan} = \frac{1}{2\sqrt{x}} \Rightarrow m_{tan}\Big]_{(4,2)} = f'(x)\Big]_{(4,2)} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

Now, we need to find the equation of the tangent line at the point

$$(x_1, y_1) = (4,2)$$

$$(y - y_1) = m_{tan}(x - x_1)$$

$$\Rightarrow y - 2 = \frac{1}{4}(x - 4)$$

$$\Rightarrow y = \frac{1}{4}x + 1$$

Next, we need to find the equation of the normal line at the point

$$(x_1, y_1) = (4,2)$$

$$\because m_\perp = -\frac{1}{m_{tan}} \to m_\perp = -\frac{1}{m_{tan}} = -\frac{1}{\frac{1}{4}} = -4$$

$$(y - y_1) = m_\perp(x - x_1)$$

$$\Longrightarrow (y-2) = -4(x-4) \Longrightarrow y = -4x + 18$$

Problem 4.1:

1. find $f'(x)$by using the definition of the following functions:
   (a)x^2 (b) $4 - \sqrt{x+3}$
2. Let $f(x) = x^2$, find the equation of the tangent line and normal line at the point (3,9) by using the definition?
3. Let $f(\mathrm{x}) = \sqrt{\mathrm{x}+3}$, find the equation of the tangent line at x=3?

Differentiable VS. Continuous:

Theorem: If $f(x)$ is a differentiable function at x_0, then it is a continuous function at x_0.

Proof: To prove $f(x)$ is continuous function at x_0, we need to show:

$$\lim_{x \to x_0} f(x) = f(x_0) \text{ (i.e., } \lim_{x \to x_0} [f(x) - f(x_0)] = 0)$$

Suppose that:

$$\Delta x = x - x_0 \Longrightarrow x = x_0 + \Delta x \Longrightarrow f(x) = f(x_0 + \Delta x)$$

Hence, when $x \to x_0, \Delta x \to 0$

$$
\begin{aligned}
\lim_{\Delta x \to 0}[f(x) - f(x_0)] &= \lim_{\Delta x \to 0}[f(x_0 + \Delta x) - f(x_0)] \\
&= \lim_{\Delta x \to 0}\left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \cdot \Delta x\right] \\
&= \lim_{\Delta x \to 0}\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \cdot \lim_{\Delta x \to 0}\Delta x \\
&= f'(x_0) \cdot 0 = 0
\end{aligned}
$$

Note: The inverse of the above theorem is not true.

(i.e., If f(x) is a continuous at x_0, then it is not necessary to be differentiable at x_0)

Example: Let $f(x) = |x|$, and $x_0 = 0$.

From the above plot $f(x) = |x|$ is contimous at $x_0 = 0$.

However, $f(x) = |x|$ is not differentiable at $x_0 = 0$.

Proof:

$$
\because |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}
$$

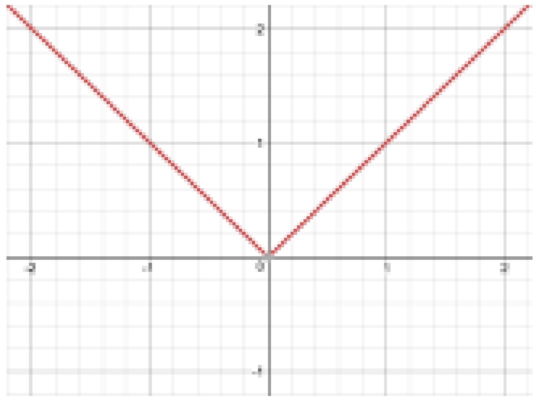
$$
\therefore |\Delta x| = \begin{cases} \Delta x & \Delta x \geq 0 \\ -\Delta x & \Delta x < 0 \end{cases}
$$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0}\frac{f(x + \Delta x) - f(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0}\frac{|x + \Delta x| - |x|}{\Delta x} \\
f'(0) &= \lim_{\Delta x \to 0}\frac{|0 + \Delta x| - |0|}{\Delta x} \\
&= \lim_{\Delta x \to 0}\frac{|\Delta x|}{\Delta x}
\end{aligned}
$$

Hence, $L^+ = \lim_{\Delta x \to 0^+} = 1 \& L^- = \lim_{\Delta x \to 0^-} = -1$

Since, $L^+ \neq L^- \Longrightarrow$ The limit does not exists.

$\therefore f(x)$ is not a differentiable function at $x_0 = 0$



<u>General Theorems of Differentiation</u>

Theorem(1): If $\boldsymbol{f(x) = \mathbf{c}}$, c be a constant, then $\boldsymbol{f'(x) = 0}$.

Theorem(2): If f is a differentiable function at x and let c be a constant, then (c f) is differentiable at x and $\boldsymbol{(c\, f)'(x) = \mathbf{c}\, f'(x)}$.

Theorem(3): If f *and* g are two differentiable functions at x then $(f + g)$ is differentiable at x and $\boldsymbol{(f + g)'(x) = f'(x) + g'(x)}$.

Remark: In general, If $f_1, f_2, \dots, f_n$ are differentiable functions at x then $(f_1 \pm f_2 \pm \dots \pm f_n)$ is differentiable at x and

$$\boldsymbol{(f_1 \pm f_2 \pm \dots \pm f_n)'(x) = (f_1)'(x) \pm (f_2)'(x) \pm \cdots \pm (f_n)'(x)}.$$

Theorem(4): If $\boldsymbol{f(x) = \mathbf{x}^n}$, where n>0, then $\boldsymbol{f'(x) = n x^{n-1}}$.

Theorem(5): If f *and* g are two differentiable functions at x then $(f.g)$ is differentiable at x and $\boldsymbol{(f.g)'(x) = \mathbf{f}(\mathbf{x}).g'(x) + f'(x).g(x)}$.

Remark: In general, If f, g *and* h are differentiable functions at x then $(f.g.h)$ is differentiable at x and $\boldsymbol{(f.g.h)'(x) = \mathbf{f}(\mathbf{x})(gh)'(x) + (f)'(x)(g.h)(x)}$

$$= \mathbf{f}(\mathbf{x})\big(\mathbf{g}(\mathbf{x}).\boldsymbol{h}'(\boldsymbol{x}) + \boldsymbol{g}'(\boldsymbol{x}).\boldsymbol{h}(\boldsymbol{x})\big) + (\boldsymbol{f})'(\boldsymbol{x}).\boldsymbol{g}(\boldsymbol{x}).\boldsymbol{h}(\boldsymbol{x})$$

$$= (\boldsymbol{f})'(\boldsymbol{x}).\boldsymbol{g}(\boldsymbol{x}).\boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{f}(\boldsymbol{x}).\boldsymbol{g}'(\boldsymbol{x}).\boldsymbol{h}(\boldsymbol{x}) + \mathbf{f}(\mathbf{x}).\mathbf{g}(\mathbf{x}).\boldsymbol{h}'(\boldsymbol{x}) .$$

Theorem(6): If f and g are two differentiable functions at x and $g(x) \neq 0$ then $(\frac{f}{g})$ is differentiable at x and $(\frac{\boldsymbol{f}}{\boldsymbol{g}})'(\boldsymbol{x}) = \frac{\boldsymbol{g}(\mathbf{x}).\boldsymbol{f}'(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}).\boldsymbol{g}'(\boldsymbol{x})}{(\boldsymbol{g}(\boldsymbol{x}))^{2}}$.

Theorem(7): If g is a differentiable function at x , f is a differentiable function at g(x) and $h = f \circ g$ then $(f \circ g)$ is differentiable at x and

$$\boldsymbol{h}'(\boldsymbol{x}) = (\boldsymbol{f} \circ \boldsymbol{g})'(\boldsymbol{x}) = \boldsymbol{f}'(\boldsymbol{g}(\boldsymbol{x})).\boldsymbol{g}'(\boldsymbol{x}).$$

Theorem(8): If f is a differentiable function at x and $y = (f(x))^n$ $where\ n \in \mathbb{Z}$, then (y) is differentiable at x and

$$\boldsymbol{y}' = \frac{\mathbf{dy}}{\mathbf{dx}} = ((\boldsymbol{f}(\boldsymbol{x}))^{\boldsymbol{n}})' = \boldsymbol{n}\big(\boldsymbol{f}(\boldsymbol{x})\big)^{\boldsymbol{n}-\mathbf{1}}.(\boldsymbol{f})'(\boldsymbol{x}).$$

<u>Problem (4.2):</u>

1. Find the derivative of the following functions:

a) $y = \left(\frac{x^2+2}{x+1}\right)^4$

(f) $f(x) = \frac{(1+2x^2)(1+x^3)}{x^2}$

b) $y = (2\sqrt{x} - 1)^3$

(g) $f(x) = \sqrt{x + \sqrt{1 + \sqrt{x}}}$

(c) $y = \sqrt{3 - x^2}$

(h) $f(t) = t^3 - \frac{1}{t^2+1}$

(d) $f(w) = \sqrt{w} + \sqrt[3]{w} + \sqrt[4]{w}$

(i) $f(t) = \frac{\sqrt{t^2+1}}{(t+2)^4}$

(e) $f(x) = (x^3 + 2)^2(1 - x^2)^3$

(j) $f(z) = z^2(z^2 + 1)^{-\frac{1}{3}}$

2. Let $f(x) = x$ and $g(x) = x^2$, what is the value of x that makes the tangent line of two curves are parallel.

3. Let $f(x) = \frac{1}{\sqrt{x}}$, what is the value of x that make the tangent of the curve when it is parallel to the line $x + 8y = 10$

Chain Rule:

1. If $y = f(x)$ and $x = g(t)$, then $\boldsymbol{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$

2. If $y = f(x)$ and $t = g(x)$, then $\boldsymbol{\frac{dy}{dt} = \frac{\frac{dy}{dx}}{\frac{dt}{dx}}}$

Example1: Let $y = 3x - 1$ and $x = 2t$, find $\frac{dy}{dt}$?

Solution:

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} \\ &= (3) \cdot (2) = 6\end{aligned}$$

OR: $y = 3x - 1 = 3(2t) - 1 = 6t - 1 = 6$

Example2: Let $y = t^2 - 1$ and $x = 2t + 3$, find $\frac{dy}{dx}$?

Solution:-

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = \ \mathrm{t} = \frac{x-3}{2}$$

Problems (4.3): Find $\frac{dy}{dx}$ for the following functions:

1. $y = u^3 + 1,\ u = x^2 + 3$
2. $y = 3t^2 - 1,\ x = 6t - 1$
3. $y = \frac{t^2}{1+t},\ x = \frac{t}{2+t}$
4. $y = t^2,\ x = \frac{t}{1-t}$
5. $y = z^{\frac{2}{3}},\ z = x^2 + 1$
6. $y = w^2 - w^{-1},\ w = 3x$
7. $y = 2v^3 + \frac{2}{v^3},\ v = (2x + 2)^{\frac{2}{3}}$
8. $y = \frac{u^2}{u^2+1},\ u = \sqrt{2x + 1}$

Implicit Differentiation

Example1: Let $x^2 + xy + y^5 = 0$, find $\frac{dy}{dx}$ and $\frac{dx}{dy}$?

Solution:-
To find $\frac{dy}{dx}$, we derive implicitly for x by considering y is an implicit function of x.

$$\because x^2 + \mathrm{xy} + y^5 = 0$$

$$\rightarrow 2x\frac{dx}{dx} + \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 5y^4\frac{dy}{dx} = 0$$

$$\rightarrow 2x + xy' + \mathrm{y} + 5y^4y' = 0$$

$$\rightarrow xy' + 5y^4y' = -2\mathrm{x} - \mathrm{y}$$

$$\rightarrow (x + 5y^4)y' = -2\mathrm{x} - \mathrm{y} \rightarrow y' = \frac{dy}{dx} = \frac{-2\mathrm{x} - \mathrm{y}}{x + 5y^4}$$

To find $\frac{dx}{dy}$, we derive implicitly for y by considering x is an implicit function of y.

$$\because x^2 + \mathrm{xy} + y^5 = 0$$

$$\rightarrow 2x\frac{dx}{dy} + \left(x\frac{dy}{dy} + y\frac{dx}{dy}\right) + 5y^4\frac{dy}{dy} = 0$$

$$\rightarrow 2x\frac{dx}{dy} + x + \mathrm{y}\frac{dx}{dy} + 5y^4 = 0$$

$$\rightarrow 2x\frac{dx}{dy} + \mathrm{y}\frac{dx}{dy} = -\mathrm{x} - 5y^4$$

$$\rightarrow (2x + y)\frac{dx}{dy} = -\mathrm{x} - 5y^4 \rightarrow \frac{dx}{dy} = \frac{-\mathrm{x} - 5y^4}{2x + y}$$

Note that: $\frac{dx}{dy} = x' = \frac{1}{y'} = \frac{1}{\frac{dy}{dx}}$

Example 2: Find the equation of the tangent line and normal line of the curve $x^2 + y^2 = 2$ at (1,1).

Solution:- $\because x^2 + y^2 = 2$

$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0 \Longrightarrow 2x + 2yy' = 0 \Longrightarrow y' = \frac{-x}{y}$$

Hence, $y'|_{(1,1)} = m_{\tan}\Big|_{(1,1)} = \frac{-1}{1} = -1$

The equation of the tangent line: $(y - y_1) = m_{\tan}(x - x_1)$

$$\Longrightarrow (y-1) = -1(x-1)$$
$$\Longrightarrow (y-1) = -x+1$$
$$\Longrightarrow y = -x+2$$

Since, $m_{\perp}|_{(1,1)} = \frac{-1}{m_{\tan}} \Longrightarrow m_{\perp}\Big|_{(1,1)} = \frac{-1}{-1} = 1$

The equation of the normal line: $(y - y_1) = m_{\perp}(x - x_1)$

$$\Longrightarrow (y-1) = 1(x-1) \Longrightarrow (y-1) = x-1 \Longrightarrow y = x$$

Problems (4.4):

1 Find the slop of the tangent line of the curve $x^2 + xy + y^2 = 7$ at the point $(1,2)$.

2 Find the slop of the tangent line of the circle equation $8x^2 + 8y^2 = 232$ at the point $(-5,2)$.

3 Find the equation of the tangent line and the normal line of the curve $xy^2 + yx^2 + y^2 = 0$ at the point $(1,1)$.

4 Find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ for the following functions:

a) $x^3y^2 + 2xy - x + 3y = 6$ b) $x^2 + x^3 = y + y^4$ c) $\frac{1}{x} + \frac{1}{y} = x + y$

d) $x^2 - \sqrt{xy} + y^2 = 6$ e) $x^3 + y^3 - 9xy = 0$ f) $xy^2 + yx^2 = 3y^3$

g) $2 - y^3 + yx^2 = 5$ h) $(1 + x^2y)^3 + x\sqrt{y} = 9$

High-Order Derivative

Let $y = f(x)$, then:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = y' = y^{(1)}$$ [First Derivative]

$$f''(x) = \lim_{\Delta x \to 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x} = \frac{d^2 y}{dx^2} = y'' = y^{(2)}$$ [Second Derivative]

$$f'''(x) = \lim_{\Delta x \to 0} \frac{f''(x + \Delta x) - f''(x)}{\Delta x} = \frac{d^3 y}{dx^3} = y''' = y^{(3)}$$ [Third Derivative]

$\vdots$

$$f^{(n)}(x) = \lim_{\Delta x \to 0} \frac{f^{(n-1)}(x + \Delta x) - f^{(n-1)}(x)}{\Delta x} = \frac{d^{(n)} y}{dx^{(n)}} = y^{(n)}, n \in N$$ [n^{th}Derivative]

Notes: $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right), \frac{d^3 y}{dx^3} = \frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right), \ldots, \frac{d^{\mathrm{n}} y}{dx^{\mathrm{n}}} = \frac{d}{dx}\left(\frac{d^{(\mathrm{n}-1)} y}{dx^{(\mathrm{n}-1)}}\right)$

Example: Let $y = 2x^3 + x^2 - 1$, Find $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$ and $y^{(5)}$?

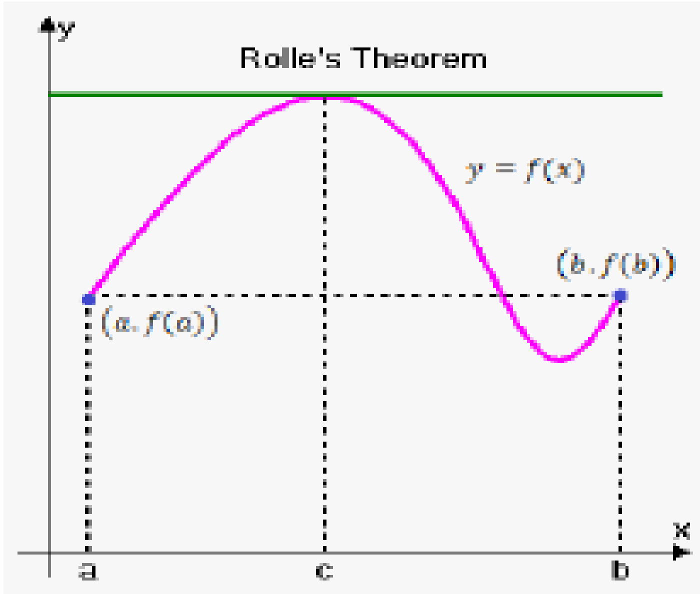
Solution:-

$$\begin{aligned} &\because y = 2x^3 + x^2 - 1 \\ \Longrightarrow\quad & y^{(1)} = 6x^2 + 2x \\ \Longrightarrow\quad & y^{(2)} = 12x + 2 \\ \Longrightarrow\quad & y^{(3)} = 12 \\ \Longrightarrow\quad & y^{(4)} = 0 \Longrightarrow y^{(5)} = 0 \end{aligned}$$

Problems (4.5): Find y', y'' and y''' for the following:

1. $y = x^7 - x^2 + 4x + 33$
2. $y = -4 + 2x^2 - 7x^3 + x^4$
3. $y = \frac{1}{2}x^2 - 100$
4. $y = x^3 - 9x - 5$
5. $y = -x^3 - 9x^2 - 23$
6. $y = -3x^2 - 4x^3 + x^4$

Rolle's Theorem: Let f(x) be a continuous function on [a,b], and f is differentiable on (a, b). If f(a)=f(b), then there exist $c \in (\mathrm{a,b})$ such that $f'(c) = 0$.



Example 1: Let $f(x) = \mathrm{x}^2 - 3x + 2$. Show that $f(x)$ satisfy Rolle's theorem on [1,2].

Solution:-

$f(x)$ is continuous on [1,2]. (because $f(x)$ is a polynomial function)

$f(x)$ is differentiable on (1,2). (because $f(x)$ is a polynomial function)

$$a = 1 \text{ and } b = 2$$
$$f(a) = f(1) = 1^2 - 3(1) + 2 = 0$$
$$f(b) = f(2) = 2^2 - 3(2) + 2 = 0$$
$$\Rightarrow f(a) = f(b)$$

From above Rolle's theorem is satisfied, and hence $\exists c \in (1,2)$ s.t.

$$f'(c) = 0$$
$$\because f'(x) = 2x - 3$$
$$\Longrightarrow f'(c) = 2c - 3 = 0$$
$$\Rightarrow 2c - 3 = 0 \Longrightarrow c = \frac{3}{2} \in (1,2)$$

Example 2: Let $f(x) = 1 - |x|$. Show that $f(x)$ does not satisfy Rolle's theorem on $[-1,1]$.

Solution:-

$f(x)$ is continuous on $[-1,1]$. (because $f(x)$ is a polynomial function) But, $f(x)$ is not differentiable at $= 0$?

$$f'(x) = \lim_{\Delta x \to 0} \frac{1 - |x + \Delta x| - 1 + |x|}{\Delta x} = \lim_{\Delta x \to 0} \frac{-|x + \Delta x| + |x|}{\Delta x}$$

$$f'(0) = \lim_{\Delta x \to 0} \frac{-|0 + \Delta x| + |0|}{\Delta x} = \lim_{\Delta x \to 0} \frac{-|\Delta x|}{\Delta x}$$

$$\mathrm{L}^{+} = \lim_{\Delta x \to 0^{+}} \frac{-\Delta x}{\Delta x} = -1 \ , \mathrm{L}^{-} = \lim_{\Delta x \to 0^{-}} \frac{-(-\Delta x)}{\Delta x} = 1$$

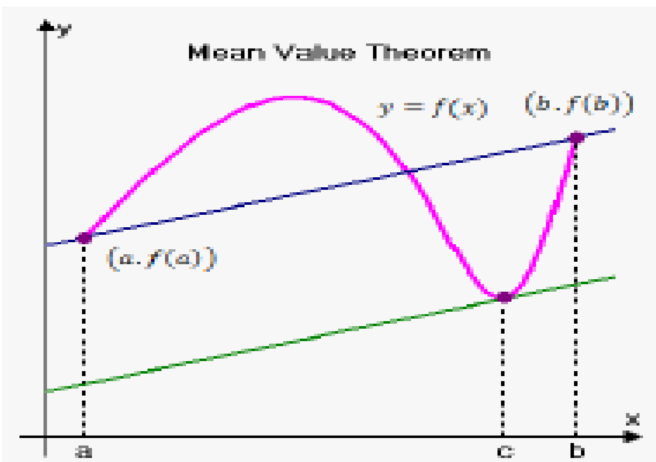
$\because \mathrm{L}^{+} \neq \mathrm{L}^{-} \rightarrow$ the limit does not exist at 0.

Hence, $f'(0)$ dose not exist.

Therefore, f(x) dose not satisfy Rolle's theorem on [-1,1].

The Mean Value Theorem: Let f(x) be a continuous function on [a, b] and f is differentiable on (a, b) then there exist at least one point $c \in (a, b)$ such that:

$$\boldsymbol{f'(\mathbf{c}) = \frac{f(b) - f(a)}{b - a}}$$



Note: Rolle's theorem is a special case from the Mean Value Theorem.

Example1: Find the value of c that satisfies the Mean Value Theorem, where $f(x) = \mathrm{x}^2,\ \ x \in [0, 1]$.

Solution:-
$f(x)$ is continuous on [0,2]. (because $f(x)$ is a polynomial function)

$f(x)$ is differentiable on (0,2).

From above Mean Value theorem is satisfied, and hence $\exists c \in (0,2)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\because \mathrm{a} = 0 \Longrightarrow \mathrm{f(a)} = \mathrm{f}(0) = 0^2 = 0$$

$$\because \mathrm{b} = 2 \Longrightarrow \mathrm{f(b)} = \mathrm{f}(2) = 2^2 = 4$$

$$\Longrightarrow f'(\mathrm{x}) = 2\mathrm{x} \Longrightarrow f'(\mathrm{c}) = 2\mathrm{c}$$

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{4 - 0}{2 - 0} = 2$$

$$\Longrightarrow 2\mathrm{c} = 4 \Longrightarrow \mathrm{c} = 1 \in (0, 1).$$

Example 2: Let $\mathrm{f}(x) = \mathrm{x}^3 - 3x\ and\ \ f: [a, 0] \to R$ where f satisfies the Mean Value Theorem at c=-1, find the value of a

Solution:- $\qquad f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\Longrightarrow f'(\mathrm{x}) = 3\mathrm{x}^2 - 3 \Longrightarrow f'(\mathrm{c}) = f'(-1) = 3(-1)^2 - 3 = 0$$

$$\because \mathrm{a} = ?\,\text{and b} = 0 \Longrightarrow$$

Hence, $\qquad 0 = \frac{f(0)-f(a)}{0-a} = \frac{3(0)^2-3-3(\mathrm{a})^2+3}{0-a} = \frac{\mathrm{a}^2-3\mathrm{a}}{a}$

$$\Longrightarrow \mathrm{a}^2 - 3\mathrm{a} = 0 \Longrightarrow \mathrm{a}^2 = 3 \Longrightarrow \mathrm{a} = \pm\sqrt{3} \Longrightarrow \mathrm{a} = -\sqrt{3} = -1.7.$$

Problems (4.6):

1. Check whether the following functions satisfy the Rolle's theorem or not?
   a) $\mathrm{f}(x) = (2 - \mathrm{x})^2$ on [0,4] $\qquad$ b) $\mathrm{f}(x) = 9\mathrm{x} + 3\mathrm{x}^2 - x^3$ on [-1, 1].

2. Find the value of c that satisfies the Mean Value Theorem, where $\mathrm{f}(x) = \mathrm{x}^2 - 6x + 4,\ \ x \in$ [-1, 7].

3. Let $f(x) = x^2 - 4x,\ and\ f: [0, b] \to R$ where f satisfies the Mean value theorem at c=2, find the value of b?

L'Hopitals Rule:

Let f and g be differentiable functions at x_0, $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$ $, where \lim_{x \to x_0} g'(x) \neq 0$.
(or $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$)

Then $\boldsymbol{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$.

Example1: Find $\lim_{x \to 1} \frac{x^2-3x+2}{x^2-1}$.

Solution:-

$$\because \lim_{x \to 1} x^2 - 3x + 2 = 0 \text{ and } \lim_{x \to 1} x^2 - 1 = 0$$

$$\therefore \lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 1} = \lim_{x \to 1} \frac{2x - 3}{2x} = \frac{1}{2}$$

Example2: Find $\lim_{x \to 0} \frac{2-\sqrt{x+4}}{x}$.

Solution:-

$$\because \lim_{x \to 0} 2 - \sqrt{x + 4} = 0 \text{ and } \lim_{x \to 0} x = 0$$

$$\lim_{x \to 0} \frac{2 - \sqrt{x + 4}}{x}$$

$$= \quad \lim_{x \to 0} \frac{0 - \frac{1}{2}(x + 4)^{\frac{1}{2}}}{1}$$

$$= \quad -\frac{1}{2} \cdot \lim_{x \to 0} \frac{1}{\sqrt{x + 4}} = -\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4}$$

Another Method: by multiplying by the conjugate:

$$\lim_{x\to 0}\frac{2-\sqrt{x+4}}{x}$$

$$= \lim_{x\to 0}\frac{2-\sqrt{x+4}}{x}\cdot\frac{2+\sqrt{x+4}}{2+\sqrt{x+4}}$$

$$= \lim_{x\to 0}\frac{4-(x-4)}{x(2-\sqrt{2+4})}$$

$$= \lim_{x\to 0}\frac{-x}{(2+\sqrt{x+4})} = \frac{-1}{2+\sqrt{0+4}} = \frac{-1}{2+2} = \frac{-1}{4}$$

<u>Problems (4.7):</u> Find the following limits if it exists:

1) $\lim_{x\to 2}\dfrac{x^2+2x-8}{x^2-9x+14}$

2) $\lim_{x\to 0}\dfrac{\sqrt{x+9}-3}{x}$

3) $\lim_{x\to 1}\dfrac{x^2+5x+4}{x^2-4\text{x}-5}$

4) $\lim_{x\to 0}\dfrac{4x^3+3x^2-8x+1}{x^3-2x^2+3\text{x}-6}$

5) $\lim_{x\to 1}\dfrac{x^2-3\text{x}+2}{x^2-1}$

6) $\lim_{x\to 0}\dfrac{x^3+4x^2-5x}{x^3-2\text{x}}$

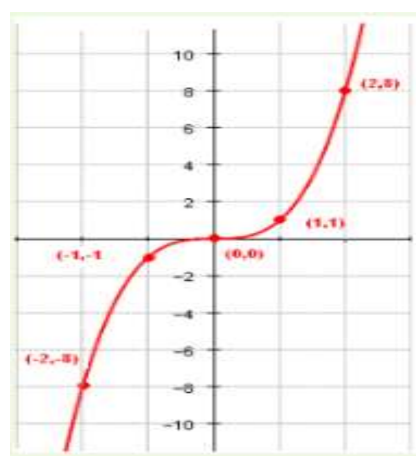
<u>Increasing and Decreasing Functions:</u>

<u>Definition:</u> A function f is defined on an interval $[a,b]$ is said to be **increasing** on $[a,b]$ if $\forall x_1, x_2 \ni a \le x_1 < x_2 \le b \Longrightarrow f(x_1) < f(x_2)$.

<u>Example:</u> Let $f(x) = x^3$ on $[-2,2]$.

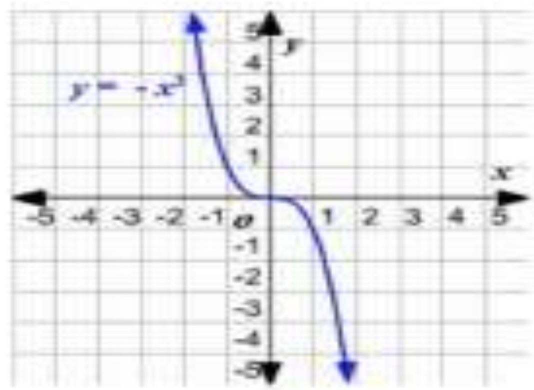
$$-2 < -1 \Longrightarrow f(-2) = -8 < -1 = f(-1)$$
$$-1 < 1 \Longrightarrow f(-1) = -1 < 1 = f(1)$$
$$1 < 2 \Longrightarrow f(1) = 1 < 8 = f(2)$$
$$\vdots$$
$$\because \forall a,b \in [-2,2] \Longrightarrow f(a) < f(b)$$

$\therefore f(x)$ is an increasing function on $[-2,2]$



Definition: A function f is defined on an interval $[a,b]$ is said to be **decreasing** on $[a,b]$ if $\forall x_1, x_2 \ni a \le x_1 < x_2 \le b \Longrightarrow f(x_1) > f(x_2)$.

Example: Let $f(x) = -x^3$on $[-2,2]$.



$$-2 < -1 \Longrightarrow f(-2) = 8 > 1 = f(-1)$$
$$-1 < 1 \Longrightarrow f(-1) = 1 > -1 = f(1)$$
$$1 < 2 \Longrightarrow f(1) = -1 > -8 = f(2)$$

$\because \forall a, b \in [-2,2] \Longrightarrow f(a) > f(b)$

$\therefore f(x)$ is a decreasing function on $[-2,2]$.

Definition: Let f be defined and continuous function on $[a,b]$, and let $x_0 \in [a,b]$, then $\left(x_0, f(x_0)\right)$ is said to be a Critical Point of $f \Leftrightarrow f'(x_0) = 0$ or $f'(x)$ is not defined.

Example 1: Let $f(x) = x^2$ be defined and continuous on $[-1,1]$. Find the critical points (if exists)?

Solution: $f'(x) = 2x$ When $f'(x) = 0 \Longrightarrow 2x = 0 \Longrightarrow x = 0$

Hence, $\left(x_0, f(x_0)\right) = (0,0)$ is a critical point.

Example 2: Let $f(x) = \frac{x^4}{3} - \frac{x^2}{2}$ be defined and continuous on all the real numbers. Find the critical points(if exists)?

Solution:

When $f'(\text{x}) = 0 \Longrightarrow x^2 - x = 0 \Longrightarrow \text{x}(\text{x} - 1) = 0 \Longrightarrow \text{x} = 0$ or $\text{x} = 1$

Hence, (0, f(0)) =(0,0) and (1, f(1))= $\left(1, -\frac{1}{6}\right)$ are the critical points.

Example 3: Let $f(x) = |\text{x}|$ be defined and continuous on [-1, 1]. Find the critical points(if exists)?

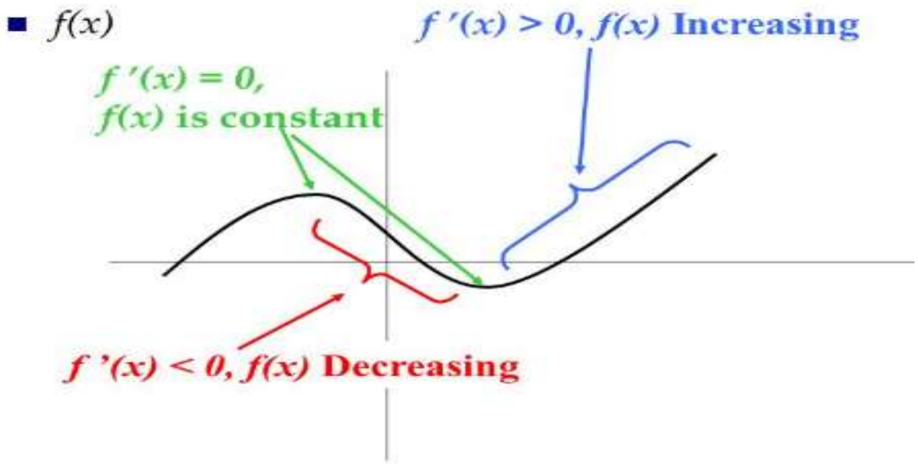
Solution:

$0 \in [-1, 1], but\ f'(0)\ does\ not\ exists$

Hence, (0, f(0)) =(0,0) is critical points.

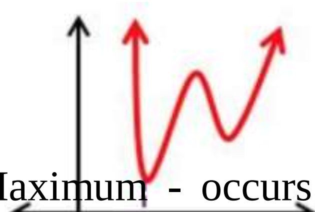
Theorem: Let f be a function that is continuous 0n [a, b] and differentiable on (a, b), then:

1. If $f'(\mathrm{x}) > 0\ \ \forall x \in (a, b), then\ \ f\ is\ increasing\ on\ [a, b]$.
2. If $f'(\mathrm{x}) < 0\ \ \forall x \in (a, b), then\ \ f\ is\ decreasing\ on\ [a, b]$.
3. If $f'(\mathrm{x}) = 0\ \ \forall x \in (a, b), then\ \ f\ is\ constant\ on\ [a, b]$.

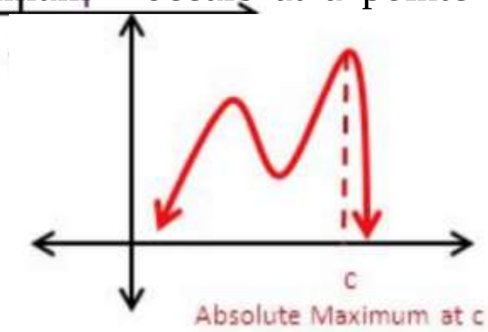


Definitions:

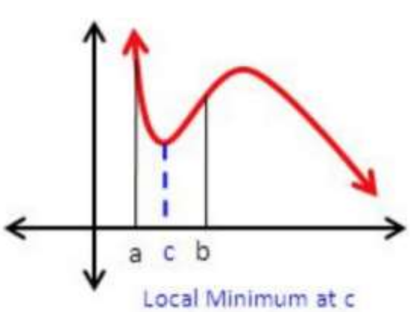
Absolute Minimum-occurs at a point c if $f(c) \leq f(x)$ for x all values in the domain.



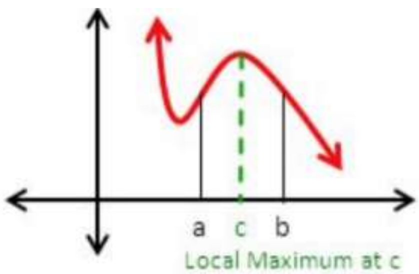
Absolute Maximum - occurs at a pointc if $f(c) \geq f(x)$ for all x values in the domain.



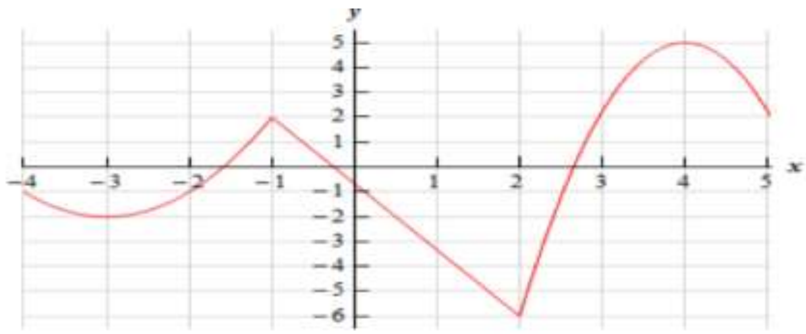
Local Minimum-occurs at a point c in an open interval, (a, b), in the domain if $f(c) \leq f(x)$ for all x values in the open interval.



Local Maximum-occurs at a point c in an open interval, (a, b), in the domain if $f(c) \geq f(x)$ for all x values in the open interval.



<u>Example:</u> Let $f(x)$ be define on $[-4,5]$ as given in the following plot. Find the absolute maximum, absolute minimum, local maximum and local minimum points.



Solution:

Absolute Maximum: (4, 5)

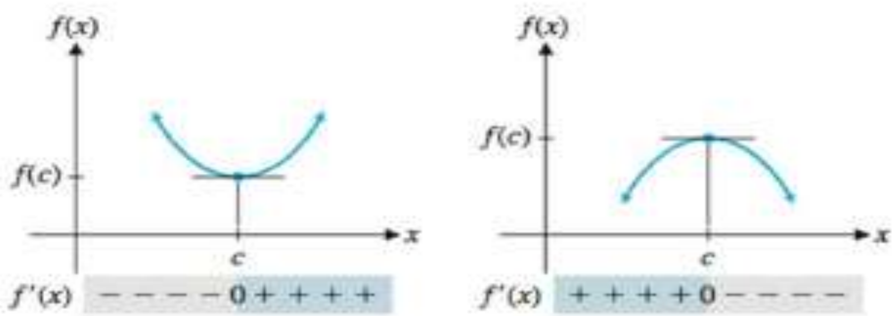
Absolute minimum: (2, -6)

Local maximum: (-1, 2) and (4, 5)

Local maximum: (-3, -2) and (2, -6)

First Derivative Test

1) If the sign changes from "+" to "-" at c, then c is called a **local maximum point**.
2) If the sign changes from "-" to "+" at c, then c is called a **local minimum point**.



(A) $f(c)$ is a local minimum

(B) $f(c)$ is a local maximum

Example 1: Let $f(x) = x^3 - 6x^2 + 1$. Using the First Derivative Test, find the local maximum and minimum points.

Solution:
First, we need to find the critical points ($f'(x) = 0$):

$$\because f(x) = x^3 - 6x^2 + 1 \Longrightarrow f'(x) = 3x^2 - 12x$$
$$f'(x) = 0 \Longrightarrow 3x^2 - 12x = 0 \Longrightarrow 3x(x-4) = 0$$

Hence, $f(x)$ has critical points at $x = 0,4$.

Increasing Intervals: $(-\infty, 0)$ and $(4, \infty)$
Decressing Interval: $(0,4)$

$f(x)$ has local maximum at $x = 0$, and $(0,1)$ is a local maximu point. $f(x)$ has local minimum at $x = 4$, and $(4, -31)$ is a local minimu point.

Example 2: Let $f(x) = x^3 - 6x^2 + 9x - 8$ on $(0,5)$. Using the Firs Derivative Test, find the local maximum and minimum points.

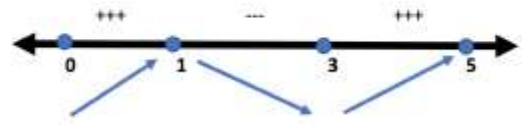
Solution:
First, we need to find the critical points ($f'(x) = 0$):

$$\because f(x) = x^3 - 6x^2 + 9x - 8 \Longrightarrow f'(x) = 3x^2 - 12x + 9$$

$$f'(x) = 0 \Longrightarrow 3x^2 - 12x + 9 = 0$$
$$\Longrightarrow \quad 3(x^2 - 4x + 3) = 0 \Longrightarrow (x-1)(x-3) = 0$$

Hence, $f(x)$ has critical points at $x = 1,3$.

Increasing Intervals: $(0,1)$ and $(3,5)$
Decressing Interval: $(1,3)$



$f(x)$ has local maximum at $x = 1$, and $(1, -4)$ is a local maximum point. $f(x)$ has local minimum at $x = 3$, and $(3, -8)$ is a local minimum point.

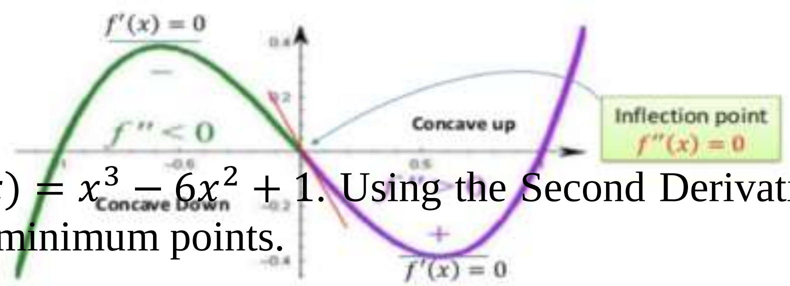
<u>Definition:</u> The graph of a differentiable function $y = f(x)$ is concave up on an interval where y' is increasing and is concave down on an interval where y' is decreasing .

<u>Definition:</u> An inflection point is a point on a curve where the curve change from being concave down (going up, then down) to concave up (going down, then up), or the other way around.

<u>Second Derivative Test for Local Maxima and Minima</u>

If $f'(x) = 0$ and $f''(x) > 0$ then f has a local maximum at x=c. (i.e. f is concave up)

If $f'(x) = 0$ and $f''(x) < 0$ then f has a local minimum at x=c. (i.e. f is concave down)



<u>Example 1:</u> Let $f(x) = x^3 - 6x^2 + 1$. Using the Second Derivative Test, find the local maximum and minimum points.

Solution:
First, we need to find the critical points ($f'(x) = 0$):

$$\because f(x) = x^3 - 6x^2 + 1 \Rightarrow f'(x) = 3x^2 - 12x$$
$$f'(x) = 0 \Rightarrow 3x^2 - 12x = 0 \Rightarrow 3x(x - 4) = 0$$

Hence, $f(x)$ has critical points at $x = 0,4$.

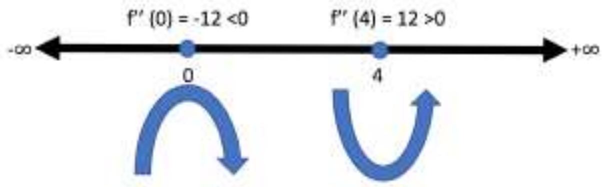
$$\because f'(x) = 3x^2 - 12x \Rightarrow f''(x) = 6x - 12$$
$$f''(x) = 0 \Rightarrow 6x - 12 = 0 \Rightarrow x = 2$$

$$\because f''(0) = -12 \Rightarrow f(x) \text{ "Concave Down" on } (-\infty, 2),$$

and has local Maximum at " $x = 0$ ".

$$\because f''(4) = 12 \Longrightarrow f(x) \text{ "Concave Up" on } (2, \infty),$$

and has local Minimum at " $x = 4$ ". $f(x)$ has an inflection point at $x = 2$ because the function concave down then concave up.



Example 2: Let $f(x) = x^3 - 6x^2 + 9x - 8$ on $(0,5)$. Using the Second Derivative Test, find the local maximum and minimum points.

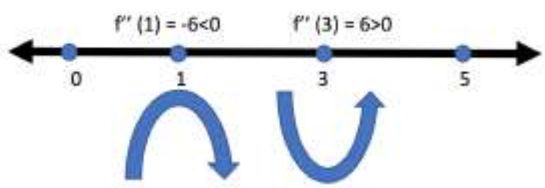
Solution:
First, we need to find the critical prints ($f'(x) = 0$):

$$\because f(x) = x^3 - 6x^2 + 9x - 8 \Longrightarrow f'(x) = 3x^2 - 12x + 9$$

$$f'(x) = 0 \Longrightarrow 3x^2 - 12x + 9 = 0$$
$$\Rightarrow 3(x^2 - 4x + 3) = 0 \Longrightarrow (x-1)(x-3) = 0$$

Hence, $f(x)$ has critical points at $x = 1{,}3$.

$$\because f'(x) = 3x^2 - 12x + 9 \Longrightarrow f''(x) = 6x - 12$$
$$f''(x) = 0 \Longrightarrow 6x - 12 = 0 \Longrightarrow x = 2$$



$\because f''(1) = -6 \Longrightarrow f(x)$ "Concave Down" on (0,2), and has local Maximum at " $x = 1$ ".
$\because f''(3) = 6 \Longrightarrow f(x)$ "Concave Up" on $(2,5)$,and has local Minimum at " $x = 3$ ".

f(x) has inflection point at x=2 because the function concave down then concave up.

Problems (4.8):

1. By using the First Derivative Test, check whether the critical points are local maximum or minimum points, and specify the increasing and decreasing intervals.
   (a) $f(x) = \frac{1}{2}x^2 - x, x \in [0,2]$
   (b) $f(x) = \frac{x^2}{3} + \frac{5}{2}x^2 + 6x, x \in \mathbb{R}$
   (c) $f(x) = x^3 - x, x \in [-2,2]$
2. By using the Second Derivative Test, check whether the critical points are local maximum or minimum points, and specify the concave up and concave down intervals.
   (a) $f(x) = \frac{x^3}{6} - \frac{x^2}{2} - 4, x \in [-2,5]$
   (b) $f(x) = \frac{x^3}{3} + \frac{5}{2}x^2 + 6x, x \in \mathbb{R}$
   (c) $f(x) = \frac{x^4}{12} - \frac{x^2}{6} - x^2, x \in [-3,3]$