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قسم الرياضيات/ المرحلة الاولى


مادة التفاضل والتكامل

# الفصل الثاني

# الدوال

# The Functions


اعضاء التدريس

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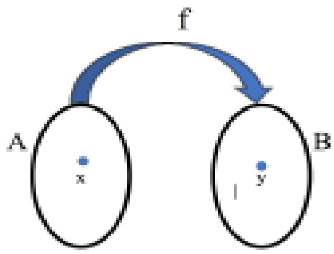
## CHAPTER TWO: The Functions

**Definition:** Let A and B be two non-empty sets, the relation that assigns to every element $x \in A$, with a unique value $y \in B$ is called a function i.e.

$$f: A \longrightarrow B; \forall x \in A \exists!\, y \in B \text{ such that } f(x) = y$$

Notes:

1 $A = \text{Domain} = D_f$ , $B = Co-$ domain $= Co - D_J$

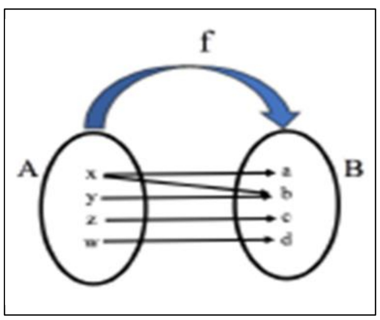



2 The set of all images $f(x) = y, \forall x \in D_f$ is called the Range of $f$.

i.e. $R_f = \{f(x) = y; x \in D_f\}$

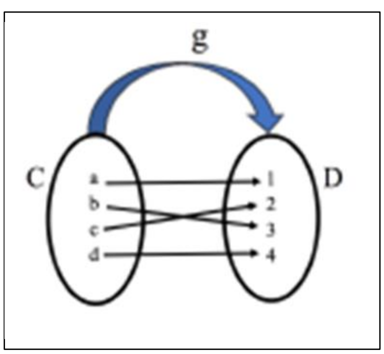
**Functions and Non-functions Examples:-**

**Example (1):**



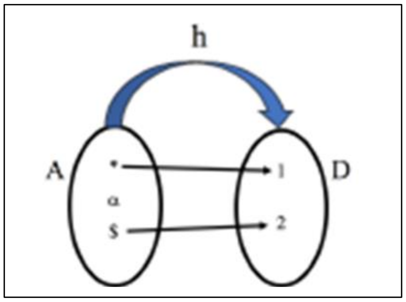

$f$ is not a function because $f(x) = a$ and $f(x) = b$ (i.e., $x$ has two images).

**Example (2):**



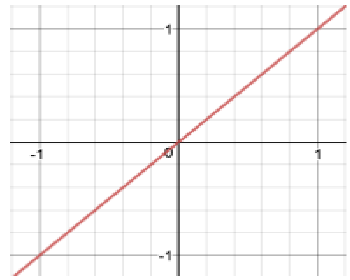

$g$ is a function and $R_g = \{1,2,3,4\}$.

**Example (3):**




$h$ is not a function because $\alpha \in A$ and $\alpha$ has not image.

**Example (4): Is $y = x$ a Linear function?**

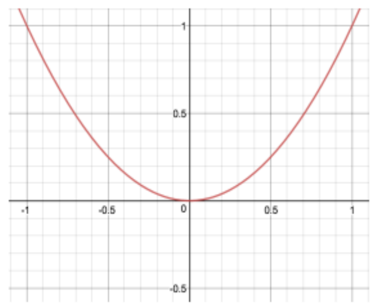


$y = f(x) = x$ , $f: \mathbb{R} \longrightarrow \mathbb{R}$

$D_f = \mathbb{R} = \{x \in \mathbb{R}: -\infty < x < \infty\}$

$R_f = \mathbb{R} = \{y \in \mathbb{R}: -\infty < y < \infty\}$

$y = x$ is a Linear function



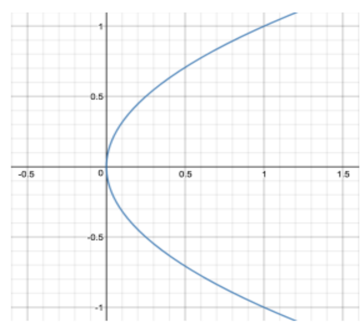
**Example (5): Is $y = x^2$ a Non- Linear function?**

$y = f(x) = x^2$ , $f: \mathbb{R} \longrightarrow [0, \infty)$

$D_f = \mathbb{R} = \{x \in \mathbb{R}: -\infty < x < \infty\}$

$R_f = \mathbb{R}^+ = \{y \in \mathbb{R}: y \geq 0\} = [0, \infty)$

$y = x^2$ is a Non- Linear function



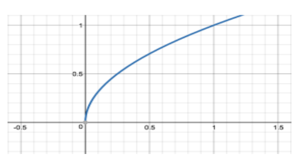
**Example (6): Is $y^2 = x$ a function?**

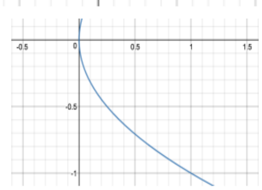
$y^2 = x \longrightarrow \sqrt{y^2} = \sqrt{x} \longrightarrow |y| = \mp\sqrt{x}$

$\forall x \in D_f, \exists \mp \sqrt{x}$ (i.e., there are two images for each x).

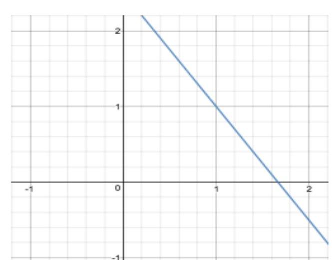
Hence, $y^2 = $ x is not a function.

However, $y_1 = +\sqrt{x}$ is a function.





Also, $y_2 = -\sqrt{x}$ is a function.

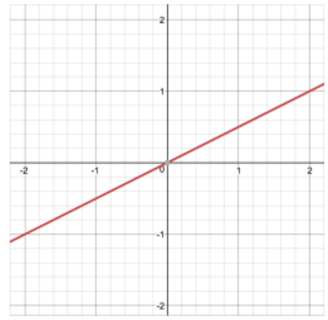


**Example (7): Is $2y + 3x = 5$ a function?**

$$2y + 3x = 5 \longrightarrow 2y = 5 - 3x \longrightarrow y = \frac{5 - 3x}{2}$$

Since for each value of $x$ there exists only one value of $y$, it is a function.

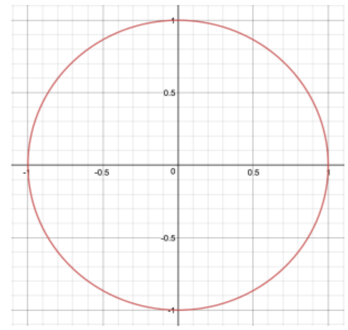
**Example (8): Is $\frac{x}{y} = 2$ a function?**



$$\frac{x}{y} = 2 \rightarrow y = \frac{1}{2}x$$

Since for each value of $x$ there exists only one value of $y$, it is a function.

**Example (9): Is $x^2 + y^2 = 1$ a function?**

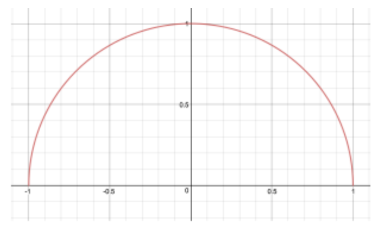


$$x^2 + y^2 = 1 \rightarrow y = \pm\sqrt{1 - x^2}$$

Since $\forall x \in D_f, \exists \mp \sqrt{1 - x^2}$ (i.e., there are two images for each value of $x$)

Hence, $x^2 + y^2 = 1$is not a function.

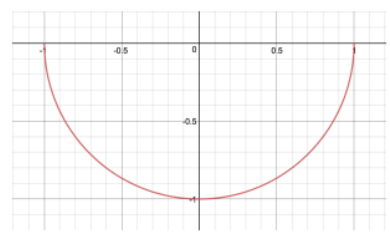
However, $y_1 = f_1(\mathrm{x}) = +\sqrt{1 - x^2}$ is a function.



$$1 - x^2 \geq 0 \rightarrow x^2 \leq 1 \rightarrow -1 < x < 1$$

$$D_{f_1} = [-1,1] \;\&\; R_{f_1} = [0,1]$$

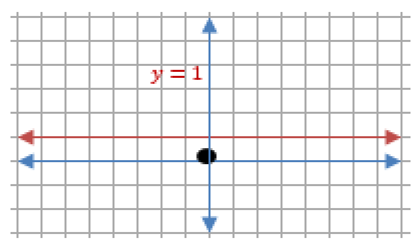
Also, $y_2 = f_2(\mathrm{x}) = -\sqrt{1 - x^2}$ is a function.



$1 - x^2 \geq 0 \rightarrow x^2 \leq 1 \rightarrow -1 \leq x \leq 1$ (same the above)

$$D_{f_2} = [-1,1] \;\&\; R_{f_2} = [-1,0]$$

**Example (10): Is $y = 1$ a function?**




$$y = f(x) = 1 \,,\, f: \mathbb{R} \longrightarrow \{1\}$$

$$D_f = \mathbb{R} = \{x \in \mathbb{R}: -\infty < x < \infty\}$$

$$R_f = \{y \in \mathbb{R}: y = 1\} = \{1\}$$

$y = 1$ is a constant function

## How to Find the Domain and the Rang of a Function?

**Remark (1):** The domain of all polynomials or odd roots is all real numbers.

**Example:** Find the domain and the rang of the following functions?

$1- \ f(x) = x^3 + 2x^2 + 3x - 5$ , $\qquad D_f = \mathbb{R} \,; R_f = \mathbb{R}$

$2 - g(x) = \sqrt[3]{x^7 - 1}$ , $\qquad D_g = \mathbb{R} \,; R_g = \mathbb{R}$

**Remark (2):** The domain of even root is all the real numbers such that the expression under the radical is greater than or equal to zero.

**Example (1):** Let $f(x) = \sqrt{x^2 - 4}$, find $D_f$ and $R_f$ ?
To find $D_f$ :

$x^2 - 4 \geq 0 \Longrightarrow (x-2)(x+2) \geq 0 \qquad$ +.+ $\quad$ -.-

Either: $x - 2 \geq 0 \wedge x + 2 \geq 0 \Longrightarrow x \geq 2 \wedge x \geq -2 \Longrightarrow [2, \infty)$

Or: $x - 2 \leq 0 \wedge x + 2 \leq 0 \Longrightarrow x \leq 2 \wedge x \leq -2 \Longrightarrow (-\infty, -2]$

Hence, $\qquad D_f = (-\infty, -2] \cup [2, \infty) = \mathbb{R} \setminus (-2,2)$
to find $R_f$:

$y = \sqrt{x^2 - 4} \rightarrow y^2 = x^2 - 4 \rightarrow x^2 = y^2 + 4 \rightarrow x = \sqrt{y^2 + 4}$
Since, y $\geq 0 \Longrightarrow y \in \mathbb{R}^+ \Longrightarrow R_f = \mathbb{R}^+$

**Example (2):** Let $g(x) = -\sqrt{2x - 1}$, find $D_g$ and $R_g$ ?
To find $D_g$:

$$2x - 1 \geq 0 \Longrightarrow 2x \geq 1 \Longrightarrow x \geq \frac{1}{2} \Longrightarrow D_g = \left[\frac{1}{2}, \infty\right)$$

To find $R_g$:

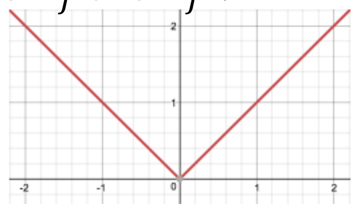
$$y = -\sqrt{2x - 1} \Longrightarrow y^2 = 2x - 1 \Longrightarrow 2x = y^2 + 1 \Longrightarrow x = \frac{y^2 + 1}{2}$$

$y \leq 0 \Longrightarrow y \in \mathbb{R}^- \Longrightarrow R_g = R^- = (-\infty, 0]$

**Definition:** The function that is defined by more than one formula (e.g., the function are written using the brace { } , signum function absolute value function) is called Piecewise function.

**Remark (3):** The domain of the Piecewise function are the restrictions of the functions.

**Example (1):** Let $f(x) = |x| = \sqrt{x^2} = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$ find $D_f$ and $R_f$ ?




$D_f = \mathbb{R}$ and $R_f = \mathbb{R}^+$

**Example (2):** Let $g(x) = \begin{cases} -1 & \text{if } x \leq 2 \\ 3 & \text{if } > 2 \end{cases}$ , find $D_g$ and $R_g$ ?

$D_g = \mathbb{R}$ and $R_g = \{-1,3\}$

**Example (3):** Let $h(x) = y = |x + 3|$, find $D_h$ and $R_h$ ?

since, $|x+3| = \begin{cases} x+3 & \text{if } x+3 > 0 \rightarrow x > -3 \\ 0 & \text{if } x+3 = 0 \rightarrow x = -3 \\ -(x+3) & \text{if } x+3 < 0 \rightarrow x < -3 \end{cases}$

$D_h = \mathbb{R}$ and $R_h = \mathbb{R}^+$

**Example (4):** Let $f(x) = \begin{cases} x & \text{if } x < -2 \\ x+1 & \text{if } -2 \leq x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$ , find $D_f$ and $R_f$ ?

$D_f = ?$

$x < -2 \;\vee\; -2 \leq x \leq 1 \;\vee\; x > 1$

$\Longrightarrow (-\infty,-2) \cup [-2,1] \cup (1,\infty) = \mathbb{R}$

$\Longrightarrow D_f = \mathbb{R}$

$R_f = ?$

$y < -2 \quad \vee \quad -1 \leq y \leq 2 \quad \vee \quad y > 1$

$\Longrightarrow (-\infty, -2) \cup [-1,2] \cup (1, \infty) = \mathbb{R} \setminus [-2, -1)$

$\Longrightarrow R_f = \mathbb{R} \setminus [-2, -1)$

**Example (5):** Let $w(t) = |t-2|$, find $D_w$ and $R_w$ ?

$$|t-2| = \begin{cases} t-2 & \text{if } t > 2 \\ 0 & \text{if } t = 2 \\ -(t-2) & \text{if } t < 2 \end{cases}$$

$D_w = \mathbb{R} \ and \ R_w = \mathbb{R}^+$

**Remark (4):** The domain of the Rational function is all the real number values except the value of $x$ which makes the denominator equal to zero.

**Example (1):** Let $(x) = \frac{x}{x^2-1}$ , find $D_f$ and $R_f$ ?

To find $D_f$

$x^2 - 1 \neq 0 \Longrightarrow x^2 \neq 1 \Longrightarrow \sqrt{x^2} \neq 1 \Longrightarrow |x| \neq 1 \Longrightarrow x \neq \mp 1$

$D_f = \mathbb{R} \setminus \{-1,1\}$

To find $R_f$

$$y = f(x) = \frac{x}{x^2-1} \Longrightarrow x = yx^2 - y \Longrightarrow yx^2 - x - y = 0$$

$$\Longrightarrow x = \frac{1 \mp \sqrt{1+4y^2}}{2y} \qquad \text{(Using } x = \frac{-b \mp \sqrt{b^2-4ac}}{2a} \text{ )}$$

Since $2y \neq 0 \Longrightarrow y \neq 0$,
and $1 + 4y^2 \geq 0 \Longrightarrow y^2 \geq \frac{-1}{4} \Longrightarrow y^2 \geq 0 \Longrightarrow y \in \mathbb{R}$
Hence, $R_f = \mathbb{R} \setminus \{0\}$

**Example (2):** Let $h(x) = \sqrt[3]{\frac{x+1}{x-2}}$ , find $D_h$ and $R_h$ ?

To find $D_h$

$\sqrt[3]{x-2} \neq 0 \Longrightarrow x-2 \neq 0 \Longrightarrow x \neq 2$

Hence, $D_h = \mathbb{R} \setminus \{2\}$

To find $R_h$

$$y^3 = \frac{x+1}{x-2} \Longrightarrow (x-2)y^3 = x+1$$

$\Longrightarrow xy^3 - 2y^3 - x - 1 = 0$

$\Rightarrow (y^3-1)x = 2y^3 + 1$

$$\Rightarrow x = \frac{2y^3+1}{y^3-1}$$

Let $y^3 - 1 \neq 0 \Longrightarrow y^3 \neq 1 \Longrightarrow y \neq 1$

Hence, $R_h = \mathbb{R} \setminus \{1\}$

**Problems (2.1)**: Find the domains and ranges for the following functions?

1 $f(x) = \sqrt{\frac{1}{x} - 2}$

2 $f(x) = \frac{1}{x^2+1} + 3$

3 $h(x) = \frac{\sqrt{x+1}}{x-1}$

4 $w(t) = \sqrt{x^2 + 25}$

5 $l(x) = \frac{x+1}{|x-5|}$

6 $g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$

7 $g(x) = \frac{2-x}{\sqrt{1-x}}$

**Algebraic of function:** Let $f$ and $g$ be two functions, then:

1. **Equality of functions**: $f$ and $g$ are equality $\Leftrightarrow D_f = D_g$ and $f(x) = g(x)$
2. **The Sum of functions**:

The sum of $f$ and $g$ is : $(f+g)(x) = f(x) + g(x)$

with the domain: $D_{f+g} = D_f \cap D_g$

3. **The Difference of functions**:

The difference between $f$ and $g$ is: $(f-g)(x) = f(x) - g(x)$

with the domain: $D_{f-g} = D_f \cap D_g$

4. **The Product of functions**:

The product of $f$ and $g$ is: $(f \cdot g)(x) = f(x) \cdot g(x)$

with the domain: $D_{f.\,g} = D_f \cap D_g$

5. **The Division of functions**:

The division of $f$ and $g$ is: $(f\,/\,g)(x) = \frac{f(x)}{g(x)}$

with the domain: $D_{f/g} = D_f \cap D_g \setminus \{x \in D_g : g(x) = 0\}$

**Similarly**, $(g\,/\,f)(x) = \frac{g(x)}{f(x)}$

$D_{g/f} = D_g \cap D_f \setminus \{x \in D_f : f(x) = 0\}$

**Example (1): Which of the following functions are equal to the function $\boldsymbol{f(x) = \frac{x-2x^2}{x}}$?**

1. $g(x) = 1 - 2x$

Solution: $D_g = \mathbb{R}$ ; $D_f = \mathbb{R} \setminus \{0\}$

Since, $D_f \neq D_g \Longrightarrow f \neq g$

2. $h(x) = \frac{x^2 - 2x^3}{x^2}$

Solution: $D_h = \mathbb{R} \setminus \{0\}$ ; $D_f = \mathbb{R} \setminus \{0\}$

$$h(x) = \frac{x^2 - 2x^2}{x^2} = \frac{x(x - 2x^2)}{xx} = \frac{x - 2x^2}{x} = f(x)$$

Since, $D_h = D_f$ and $h(x) = f(x) \Longrightarrow h = f$

3. $l(x) = \sqrt{1 - 4x + 4x^2}$

Solution: $\sqrt{1 - 4x + 4x^2} = \sqrt{(1 - 2x)(1 - 2x)} = \sqrt{(1 - 2x)^2} = |1 - 2x|$

$\Longrightarrow D_l = \mathbb{R}$

Since, $D_l \neq D_f \Longrightarrow l \neq f$

4. $w(x) = \frac{(x^3+x)(1-2x)}{x(1+x^2)}$

Solution: $x(1 + x^2) \neq 0 \Longrightarrow x \neq 0 \;\; \wedge \;\; 1 + x^2 \neq 0$ (i.g. $x^2 \neq 0$ which is always true)

$\Longrightarrow D_w = \mathbb{R} \setminus \{0\}$

$$w(x) = \frac{(x^3 + x)(1 - 2x)}{x(1 + x^2)} = \frac{x(x^2 + 1)(1 - 2x)}{x(1 + x^2)} = \frac{x - 2x^2}{x} = f(x)$$

Since, $D_w = D_f$ and $w(x) = f(x) \Longrightarrow w = f$

**Example (2): If $f(x) = \sqrt{x + 1}$ and $g(x) = \sqrt{4 - x^2}$, find $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$, $f(x)/g(x)$ and $g(x)/f(x)$ with domain for all.**

Solution:
Since, $f(x) = \sqrt{x + 1}$

$x + 1 \geq 0 \Longrightarrow x \geq -1 \Longrightarrow D_f = [-1, \infty)$

Since, $g(x) = \sqrt{4 - x^2}$

$4 - x^2 \geq 0 \Longrightarrow x^2 \leq 4 \Longrightarrow |x| \leq 2 \Longrightarrow -2 \leq x \leq 2 \Longrightarrow D_g = [-2,2]$

Now,

$(f + g)(x) = f(x) + g(x) = \sqrt{x + 1} + \sqrt{4 - x^2}$

$$(f-g)(x) = f(x) - g(x) = \sqrt{x+1} - \sqrt{4-x^2}$$

$$(f \cdot g)(x) = f(x) \cdot g(x) = \sqrt{x+1} \cdot \sqrt{4-x^2} = \sqrt{(x+1)(4-x^2)}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+1}}{\sqrt{4-x^2}} = \sqrt{\frac{x+1}{4-x^2}}$$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4-x^2}}{\sqrt{x+1}} = \sqrt{\frac{4-x^2}{x+1}}$$

Also,

$$D_{f+g} = D_{f-g} = D_{f \cdot g} = D_f \cap D_g = [-1, \infty) \cap [-2,2] = [-1,2]$$

$$D_{\frac{f}{g}} = D_f \cap D_g \setminus \{x \in [-2, 2]: g(x) = 0\}$$

$$= [-1,2] \setminus \left\{x \in D_g: \sqrt{4-x^2} = 0\right\}$$

$$= [-1,2] \setminus \{-2,2\}$$

$$= [-1,2)$$

$$D_{\frac{g}{f}} = D_f \cap D_g \setminus \{x \in D_f: f(x) = 0\}$$

$$= [-1,2] \setminus \{x \in [-1, \infty): \sqrt{x+1} = 0\}$$

$$= [-1,2] \setminus \{-1\}$$

$$= (-1,2]$$

**Problems (2.2):**

1)Check whether each of the following two functions equal or not?

(a) $f(x) = \frac{2x^2+4x}{6x^2}$, $g(x) = \frac{6x^3+12x^2}{6x^3}$ (b) $v(x) = \frac{\sqrt{x+1}}{x^3}$, $w(x) = \frac{\sqrt[3]{x^2-1}}{\sqrt{x^2}}$

(c) $h(x) = \frac{2x^2+3x^{-2}}{8x}$, $l(x) = \frac{2x^3+3x^{-1}}{8x^2}$

2) Find each of $f+g, f-g, f\cdot g, f/g, g/f$, then find the domain of each of them?

a) $f(x)=\frac{x}{x+1}$, $\quad g(x)=\frac{x-1}{\sqrt{x}}$ $\qquad$ c) $f(x)=x^3+x$, $\quad g(x)=\frac{1}{\sqrt{x+1}}$

b) $f(x)=x^2$, $\ g(x)=x+1$

**Composition Function:**

Let $f(x)$ and $g(x)$ be two functions such that $R_g \subseteq D_f$, then there exist a function $(f\circ g)(x)$ define in the following formula:

$$(f\circ g)(x)=f(g(x))$$

$D_{f\circ g}=\{x: x\in D_g \wedge g(x)\in D_f\}$

Similarly, if $R_f \subseteq D_g$, then there exist a function $(g\circ f)(x)$ define in the following formula:

$$(g\circ f)(x)=g(f(x))$$

$D_{g\circ f}=\{x: x\in D_f \wedge f(x)\in D_g\}$

**Note:** $(f\circ g)(x)\neq(g\circ f)(x)$

**Example (1):**

Let $f(x)=\sqrt{x}$ and $g(x)=x^2+1$, find $f\circ g$ and $g\circ f$ ?

**Solution:**

First, we are going to find the domain and range for $f(x)$ and $g(x)$,

To find $D_f$: $f(x)=\sqrt{x}\rightarrow D_f=\mathbb{R}^+=[0,\infty)$

To find $R_f$: $y=\sqrt{x}\rightarrow y^2=x\rightarrow R_f=\mathbb{R}^+=[0,\infty)$

Also,

To find $D_g$: $g(x)=x^2+1\rightarrow D_g=\mathbb{R}=(-\infty,\infty)$

To find $R_g$:

$y = x^2 + 1 \rightarrow x^2 = y - 1 \rightarrow x = \mp\sqrt{y-1} \rightarrow y - 1 \geq 0 \rightarrow y \geq 1 \rightarrow R_g = [1, \infty)$

To find $f \circ \mathrm{g}$ : Since $R_g = [1, \infty) \sqsubseteq [0, \infty) = D_f$, so $f \circ \mathrm{g}$ exist.

$(f \circ \mathrm{g})(\mathrm{x}) = f\big(\mathrm{g}(\mathrm{x})\big) = \sqrt{g(x)} = \sqrt{x^2 + 1}$

$D_{f\circ \mathrm{g}} = \big\{x : x \in D_g \wedge g(x) \in D_f\big\} = \{x : x \in \mathbb{R} \wedge x^2 + 1 \in \mathbb{R}^+\}$

$\quad = \{x : x \in \mathbb{R} \wedge x \in \mathbb{R}\} = \mathbb{R}$

Since $x^2 + 1 \geq 0 \rightarrow x^2 \geq -1$ which is always true and hence $x \in \mathbb{R}$

To find $g \circ f$ : Since $R_f = [0, \infty) \subseteq [0, \infty) = D_g$, so $g \circ f$ exist.

$(g \circ f\ )(\mathrm{x}) = \mathrm{g}\big(f(\mathrm{x})\big) = \left(\sqrt{x}\right)^2 + 1 = |x| + 1$

$D_{g\circ f} = \big\{x : x \in D_f \wedge f(x) \in D_g\big\} = \big\{x : x \in \mathbb{R}^+ \wedge \sqrt{x} \in \mathbb{R}\big\}$

$\quad = \{x : x \in \mathbb{R}^+ \wedge x \in \mathbb{R}^+\} = \mathbb{R}^+$

Since $x \geq 0 \rightarrow x \in \mathbb{R}^+$

**Example (2): Let $f(x) = \sqrt{x-4}$ and $g(x) = \frac{x+1}{3-x}$, find $f \circ \mathrm{g}$ and $g \circ f$ ?**

Solution: First, we are going to find the domain and range for $f(x)$ and $g(x)$.

To find $D_f$: $x - 4 \geq 0 \rightarrow x \geq 4 \rightarrow D_f = [4, \infty)$

To find $R_f$: $y = \sqrt{x-4} \rightarrow y^2 = x - 4 \rightarrow x = y^2 + 4 \rightarrow R_f = \mathbb{R}^+$

Also,

To find $D_g$: $3 - x \neq 0 \rightarrow x \neq 3 \rightarrow D_g = \mathbb{R}\backslash\{3\}$

To find $R_g$:

$$y = \frac{x+1}{3-x} \rightarrow x + 1 = 3y - xy \rightarrow x + xy = 3y - 1 \rightarrow x = \frac{3y-1}{1+y}$$

So, $y + 1 \neq 0 \rightarrow y \neq -1 \rightarrow R_g = \mathbb{R}\backslash\{-1\}$

To find $f \circ g$ : Since $R_g = \mathbb{R} \backslash \{-1\} \nsubseteq [4, \infty) = D_f$, so $f \circ g$ does not exist.

To find $g \circ f$ : Since $R_f = \mathbb{R}^+ \nsubseteq \mathbb{R} \backslash \{3\} = D_g$, so $g \circ f$ does not exist.

**Problems (2.3)**: find $f \circ g$ and $g \circ f$ for the following functions**:-**

1. $f(x) = |x|, \quad g(x) = -x$
2. $f(x) = \frac{x}{x+2}, \quad g(x) = \frac{x-1}{x}$
3. $f(x) = \sqrt{x-1}, \quad g(x) = \sqrt{1-x}$
4. $f(x) = x+1, \quad g(x) = 2x$
5. $f(x) = -\sqrt{x}, \quad g(x) = x^2+1$
6. $f(x) = 2x+4, \quad g(x) = \frac{1}{2}x - 2$
7. $f(x) = x^2, \quad g(x) = 2x+3$
8. $f(x) = x^3, \quad g(x) = \sqrt{1-x}$

## The Greatest Integer Function:

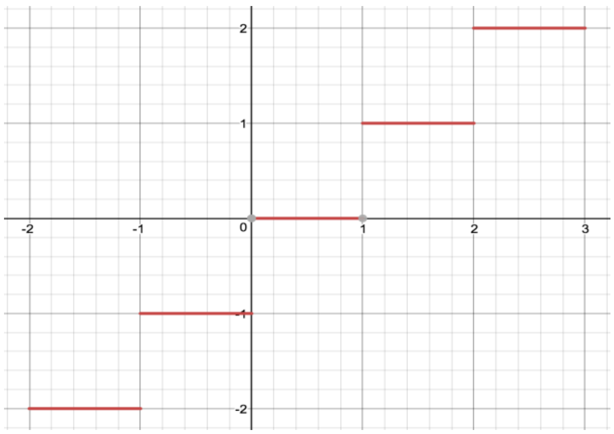
The function whose value at any number $x$ is the greatest integer less than or equal to $x$ is called the greatest integer function. It is denoted by “[ ]“ such that $[x] \leq x$.

Examples: $[2] = 2$ , $[1.5] = 1$ , $[-1.5] = -2$ , $[3.4] = 3$

**Example (1)**: Sketch a graph for the following function:

$$f(x) = [x], \forall x \in [-2,3)$$

| $x$ | $y = [x]$ | closed point | open point |
|---|---|---|---|
| $-2 \leq x < -1$ | -2 | $(-2,-2)$ | $(-1,-2)$ |
| $-1 \leq x < 0$ | -1 | $(-1,-1)$ | $(0,-1)$ |
| $0 \leq x < 1$ | 0 | $(0,0)$ | $(1,0)$ |
| $1 \leq x < 2$ | 1 | $(1,1)$ | $(2,1)$ |
| $2 \leq x < 3$ | 2 | $(2,2)$ | $(3,2)$ |



From the above table, we can see that:

$$D_f = [-2,3) \text{ and } R_f = \{-2,-1,0,1,2\}$$

**Note:** In general, $f(x) = [x] = n, \forall n \in \mathbb{I}, \forall x \in [n, n+1)$ is called "Step Function".

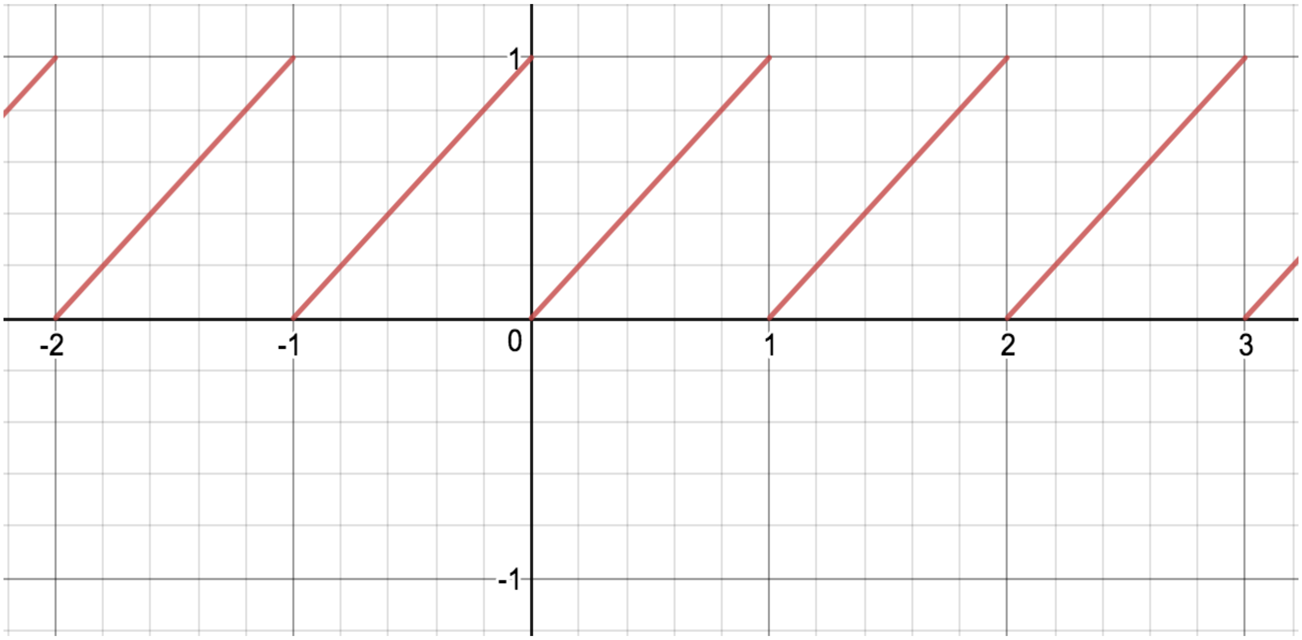
**Example (2):** Sketch a graph for the following function:

$$f(x) = x - [x], \forall x \in [-3,3].$$

| $x$ | $[x]$ | $y = x - [x]$ | closed point | open point |
|---|---|---|---|---|
| $-3 \leq x < -2$ | -3 | $x + 3$ | $(-3,0)$ | $(-2,1)$ |
| $-2 \leq x < -1$ | -2 | $x + 2$ | $(-2,0)$ | $(-1,1)$ |
| $-1 \leq x < 0$ | -1 | $x + 1$ | $(-1,0)$ | $(0,1)$ |
| $0 \leq x < 1$ | 0 | $x$ | $(0,0)$ | $(1,1)$ |
| $1 \leq x < 2$ | 1 | $x - 1$ | $(1,0)$ | $(2,1)$ |
| $2 \leq x < 3$ | 2 | $x - 2$ | $(2,0)$ | $(3,1)$ |
| $x = 3$ | 3 | $x - 3$ | $(3,0)$ | |

From the above table, we can see that:

$$D_f = [-3,3] \text{ and } R_f = [0,1)$$



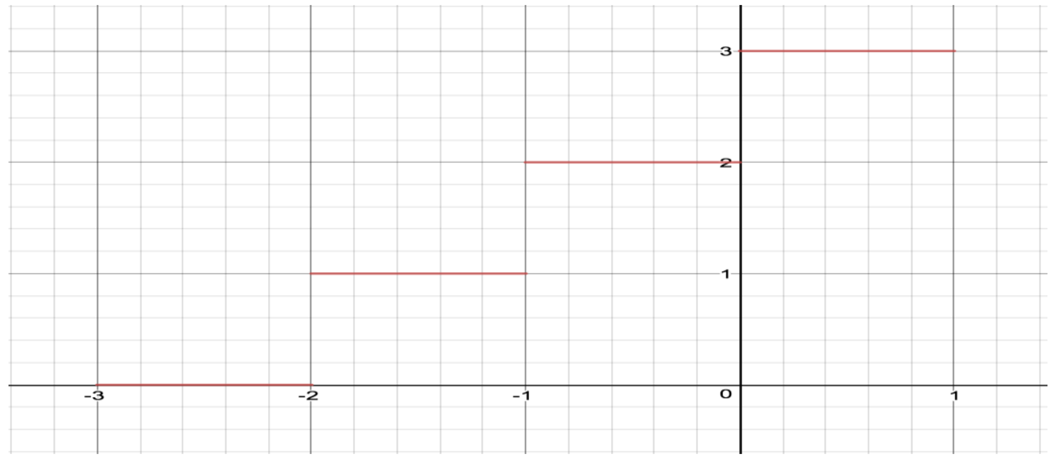

**Example (3):** Sketch a graph for the following function:

$$f(x) = [3 + x], \forall x \in [-3,1).$$

| $x$ | $3 + x$ | $y = [3 + x]$ | closed point | open point |
|---|---|---|---|---|
| $-3 \leq x < -2$ | $0 \leq 3 + x < 1$ | $0$ | $(-3,0)$ | $(-2,0)$ |
| $-2 \leq x < -1$ | $1 \leq 3 + x < 2$ | $1$ | $(-2,1)$ | $(-1,1)$ |
| $-1 \leq x < 0$ | $2 \leq 3 + x < 3$ | $2$ | $(-1,2)$ | $(0,2)$ |
| $0 \leq x < 1$ | $3 \leq 3 + x < 4$ | $3$ | $(0,3)$ | $(1,3)$ |

From the above table, we can see that:

$$D_f = [-3,1] \text{ and } R_f = \{0,1,2,3\}$$



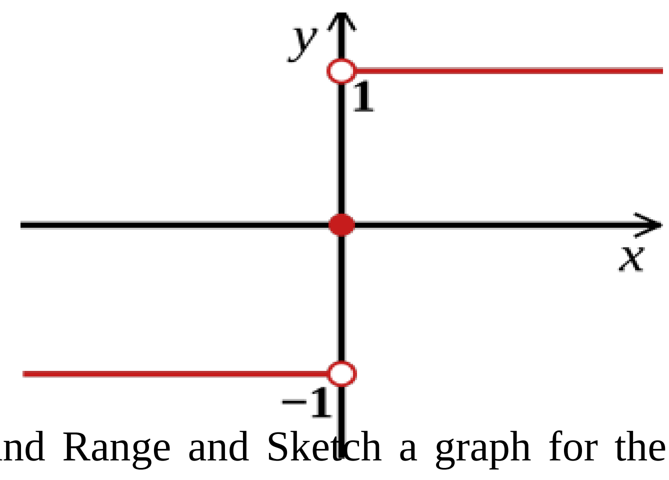

## Signum Function:

We denoted to the signum function by " $\text{Sgn}(x)$ ", and it is defined as follows:

$$\text{Sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

$D_f = \mathbb{R}$

$R_f = \{-1,0,1\}$




**Example (1)**: Find the Domain and Range and Sketch a graph for the following function:

$$f(t) = \text{Sgn}(t^2 + 1)$$

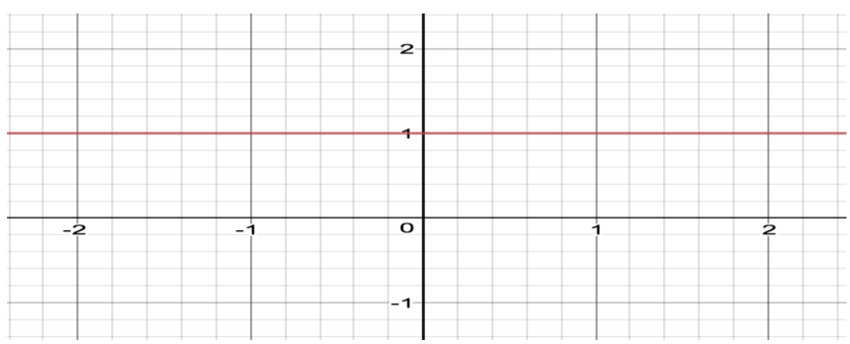
Solution:

$$f(t) = \text{Sgn}(t^2 + 1) = \begin{cases} 1 & \text{if } t^2 + 1 > 0 \Rightarrow t^2 \geq -1 \Rightarrow t^2 \geq 0 \Rightarrow t \in \mathbb{R} \\ 0 & \text{if } t^2 + 1 = 0 \Rightarrow t^2 = -1, \text{Contradiction} \\ -1 & \text{if } t^2 + 1 < 0 \Rightarrow t^2 < -1, \text{Contradiction} \end{cases}$$

Hence, $\text{Sgn}(t^2 + 1) = 1, \forall t \in \mathbb{R}$

$D_f = \mathbb{R}$

$R_f = 1$



**Example (2):** Find the Domain and Range and Sketch a graph for the following function:

$$g(t) = t\text{Sgn}(t)$$

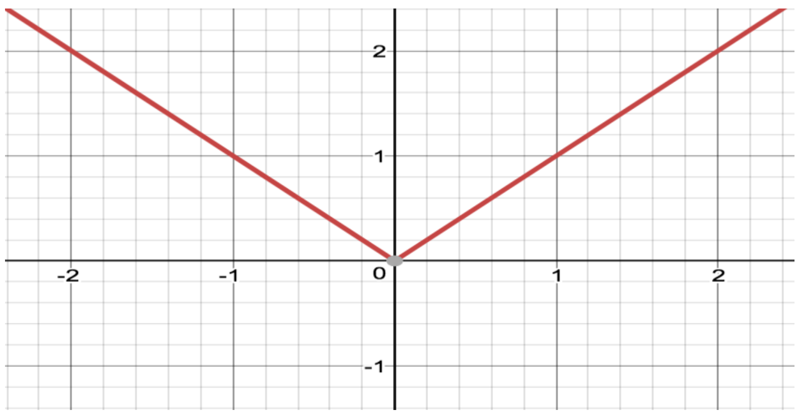
Solution:

$$g(x) = \text{t Sgn}(t) = t * \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases} = \begin{cases} t & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -t & \text{if } t < 0 \end{cases} = |t|$$

Hence, $g(t) = t\ \text{Sgn}(t) = |t|, \forall t \in \mathbb{R}$

$D_f = \mathbb{R}$

$R_f = \mathbb{R}^+$




## Odd Function:

A function $f(x)$ is called an odd function if $f(-x) = -f(x)$
Examples:

- $f(x) = x^3$
  $\because f(-x) = (-x)^3 = -x^3 = -f(x) \Longrightarrow f(x)$ is an odd function.
- $g(x) = x^4$
  $\because g(-x) = (-x)^4 = x^4 \neq -g(x) \Longrightarrow g(x)$ is not an odd function.

Note: For odd function, $D_f = R_f = \mathbb{R}$.

## Even Function:

A function $f(x)$ is called an Even function if $f(-x) = f(x)$
Examples:

- $\mathrm{h(x)} = x^2$
  $\because h(-x) = (-x)^2 = x^2 = \mathrm{h}(x) \Longrightarrow f(x)$ is an even function.
- $t(x) = x^5$
  $\because t(-x) = (-x)^5 = -x^5 \neq \mathrm{t}(x) \Longrightarrow t(x)$ is not an even function.

Note: For even function, $D_f = \mathbb{R}$, but $R_f = \mathbb{R}^+$

## Shifting Function:

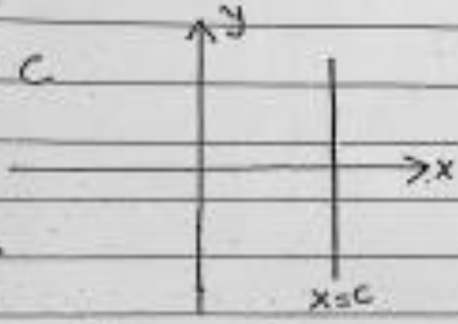
Let $y = f(x)$ s.t. $x \in \mathbb{R}$, and let $c \in \mathbb{R}^+$, then:

1. $g(x) = f(x) + c$ [Shifting to the **top** $c$ unit]
2. $g(x) = f(x) - c$ [Shifting to the **bottom** $c$ unit]
3. $g(x) = f(x + c)$ [ Shifting to the **left** $c$ unit]
4. $g(x) = f(x - c)$ [Shifting to the **right** $c$ unit]
5. $g(x) = -f(x)$ [ reflect around $x$-axes]
6. $g(x) = f(-x)$ [reflect around $y$-axes]

اشكال دوال بصورة عامة :

{1} $x = c$

خط شاقولي




{2} $y = c$

تسمى دالة ثابتة

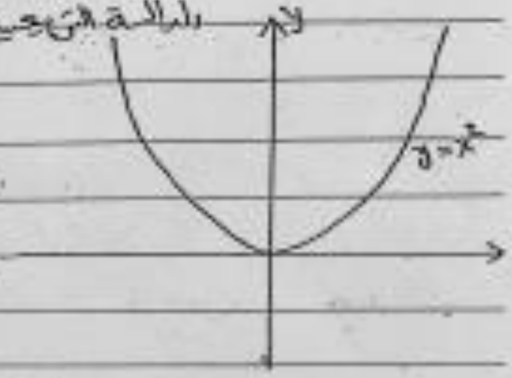
خط افقي


{3} $f(x)=y=x$

الدالة خطية .

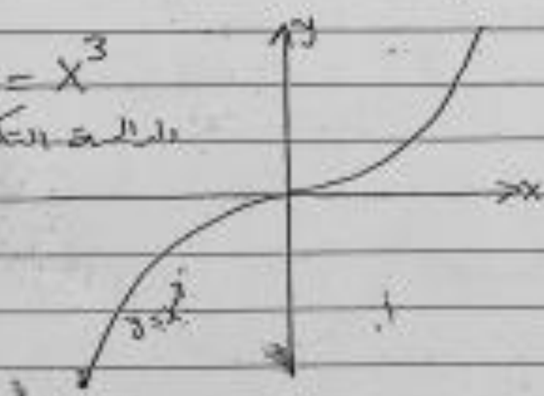

{4} $y = x^2$

الدالة التربيعية



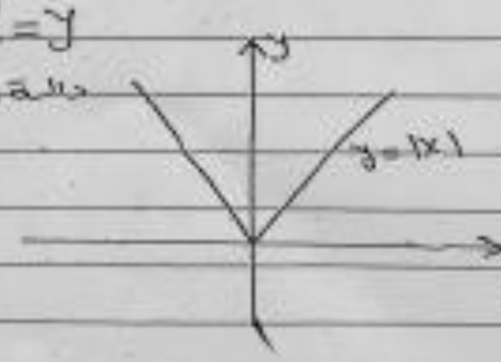

{5} $y = x^3$

الدالة التكعيبية



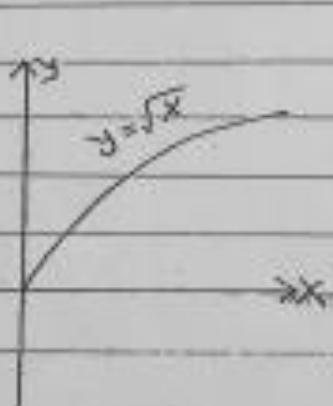

{6} $|x| = y$

دالة المطلق



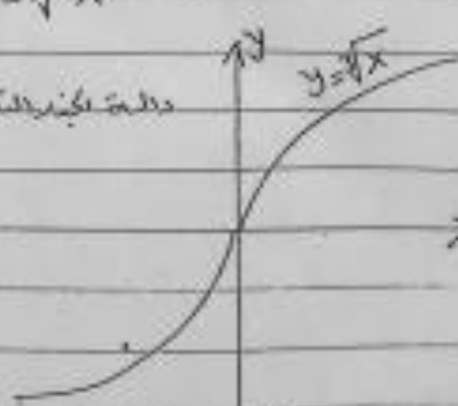

{7} $y = \sqrt{x}$

دالة الجذر التربيعي



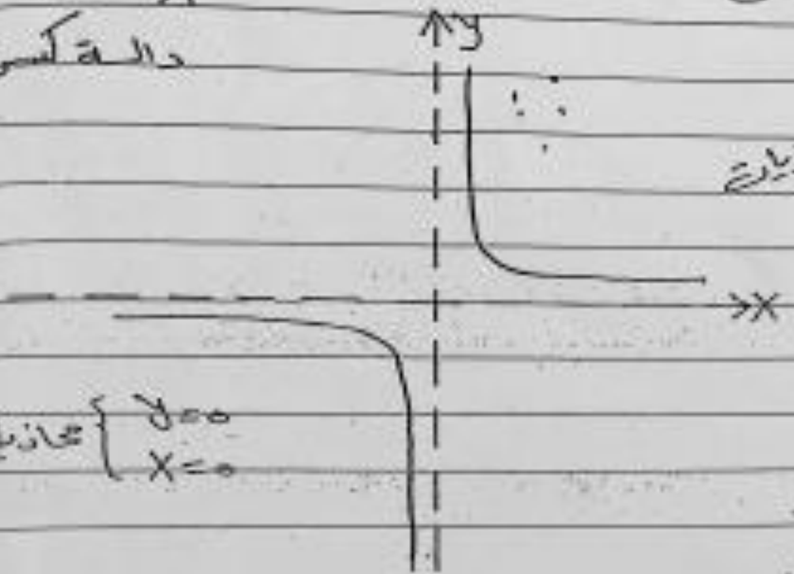

{8} $y = \sqrt[3]{x}$

دالة الجذر التكعيبي



(9) $y = \frac{1}{x}$

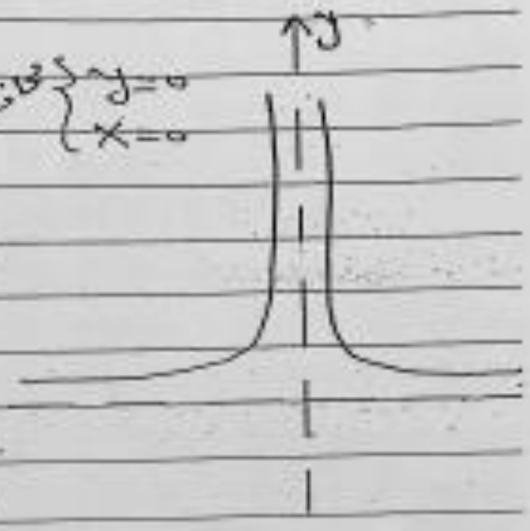
دالة كسرية




محاذيات $\begin{cases} y = 0 \\ x = 0 \end{cases}$

(10) $y = \frac{1}{x^2}$

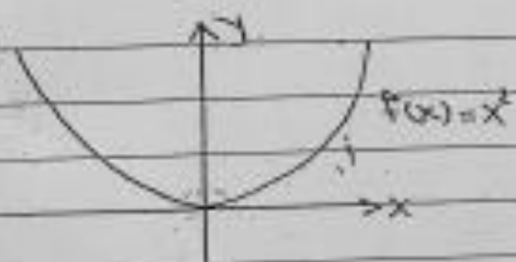
محاذيات $\begin{cases} y = 0 \\ x = 0 \end{cases}$




مثال (1): ارسم مخطط الدالة $f(x) = y = x^2$ ثم جد:

$g(x) = f(x)+1$ , $k(x) = f(x)-1$ , $h(x) = f(x+1)$ , $t(x) = f(x-1)$ , $L(x) = -f(x)$ , $m(x) = f(-x)$

sol

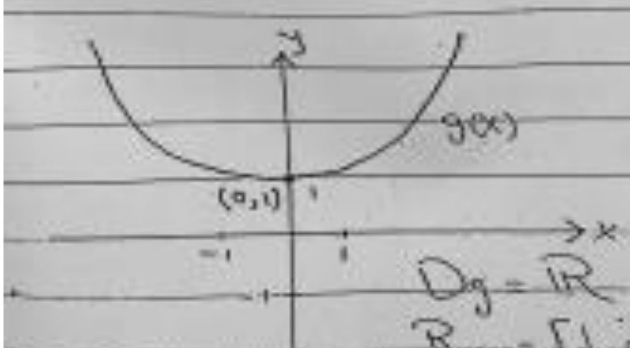



$D_f = \mathbb{R}$

$R_f = \mathbb{R}^+$

$= [0, \infty)$

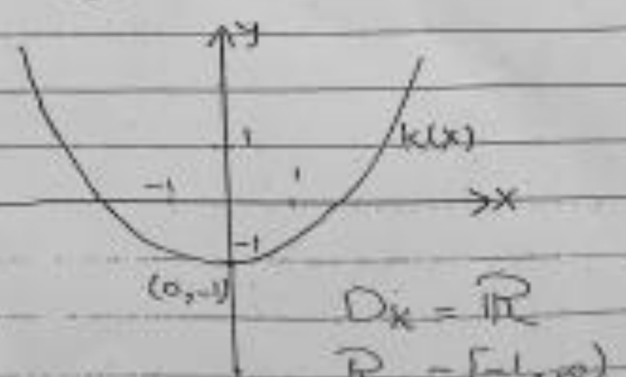
(1) $g(x) = f(x)+1 = x^2+1$




$D_g = \mathbb{R}$

$R_g = [1, \infty)$

(2) $k(x) = f(x)-1 = x^2-1$

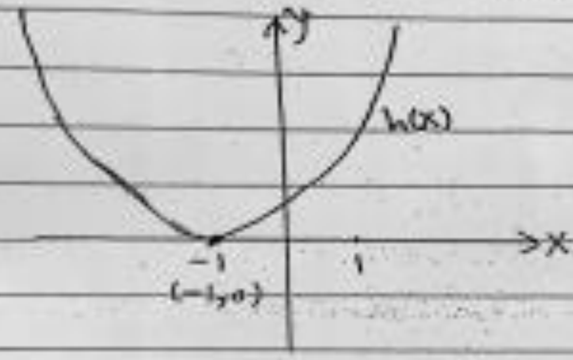



$D_k = \mathbb{R}$

$R_k = [-1, \infty)$

$x+1=0 \Rightarrow x=-1$

{3} $h(x) = f(x+1) = (x+1)^2$

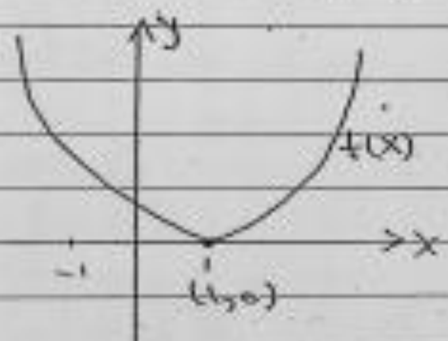



$D_h = \mathbb{R}$

$R_h = \mathbb{R}^+ = [0, \infty)$

$x-1=0 \Rightarrow x=1$

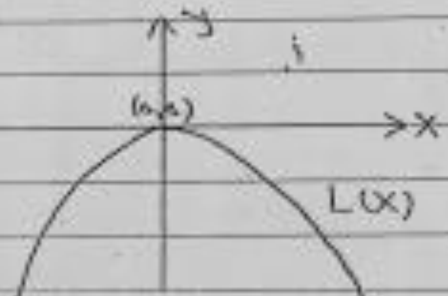
{4} $t(x) = f(x-1) = (x-1)^2$




$D_t = \mathbb{R}$

$R_t = \mathbb{R}^+ = [0, \infty)$

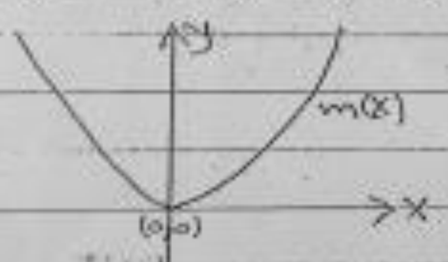
{5} $L(x) = -f(x) = -x^2$




$D_L = \mathbb{R}$

$R_L = \mathbb{R}^- = (-\infty, 0]$

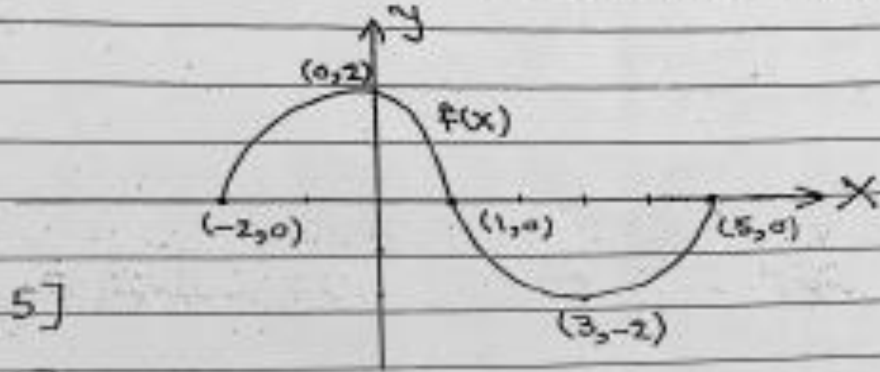
{6} $m(x) = f(-x) = (-x)^2$




$D_m = \mathbb{R}$

$R_m = \mathbb{R}^+ = [0, \infty)$

مثال (٤): ارسم مخطط الدالة $f(3-x)$ اذا كانت $f(x)$ معطاه في الشكل المجاور :

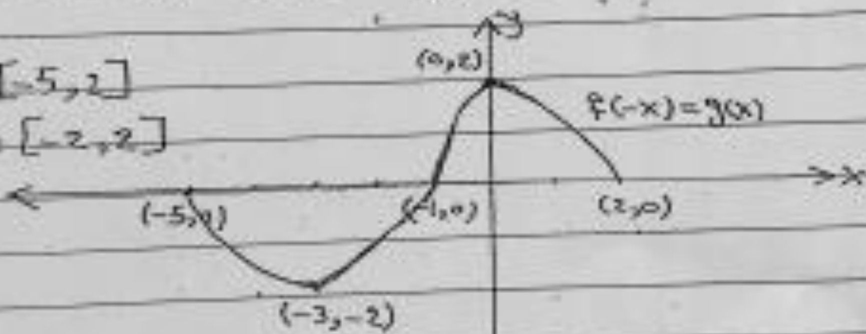



$D_f = [-2,5]$

$R_f = [-2,2]$

sol

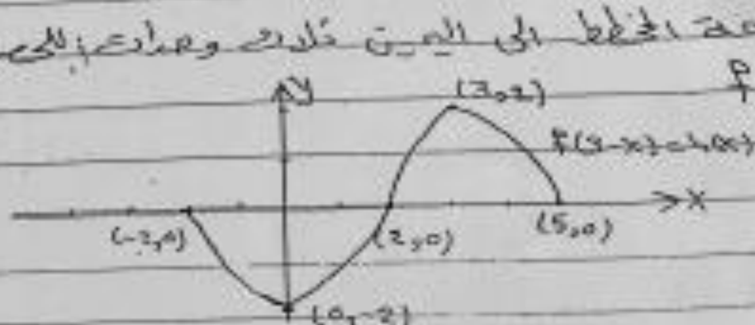
① نعكس المخطط حول محور y للحصول على مخطط $f(-x)$




$D_g = [-5,2]$

$R_g = [-2,2]$

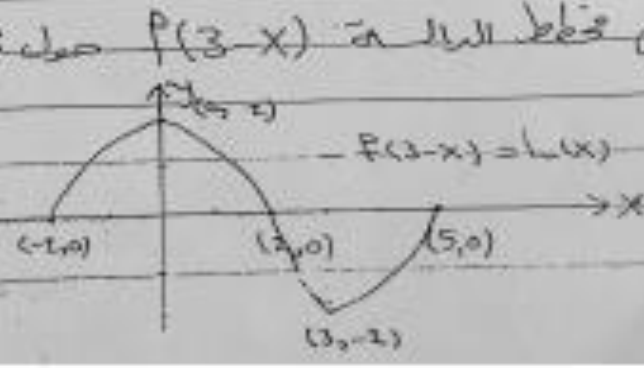
② ازاحة المخطط الى اليمين ثلاث وحدات للحصول على مخطط الدالة $f(3-x)$




$D_h = [-2,5]$

$R_h = [-2,2]$

③ نعكس مخطط الدالة $f(3-x)$ حول محور السينات




$D_L = [-2,5]$

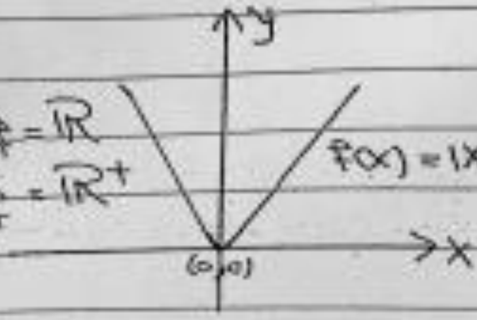
$R_L = [-2,2]$

مثال (٣): أرسم مخطط الدالة $y = -|2-x| + 4$

{1} $f(x) = |x|$

$D_f = \mathbb{R}$

$R_f = \mathbb{R}^+$

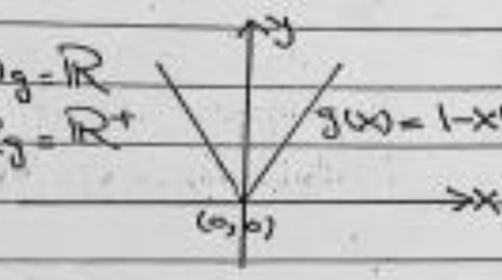



{2} $g(x) = |-x|$

نعكس مخطط الدالة حول محور y

$D_g = \mathbb{R}$

$R_g = \mathbb{R}^+$



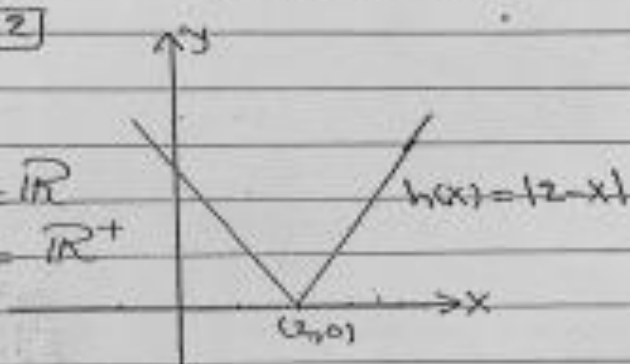

{3} $h(x) = |2-x|$

$2-x=0$     إزاحة نحو اليمين

$\boxed{x=2}$

$D_h = \mathbb{R}$

$R_h = \mathbb{R}^+$

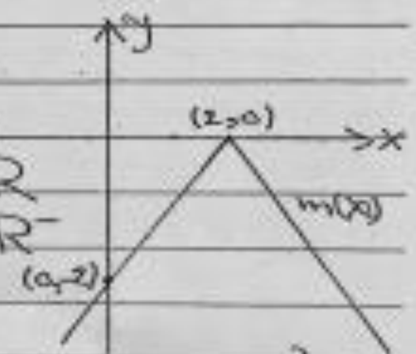



{4} $m(x) = -|2-x|$

نعكس مخطط الدالة حول محور x

$D_m = \mathbb{R}$

$R_m = \mathbb{R}^-$



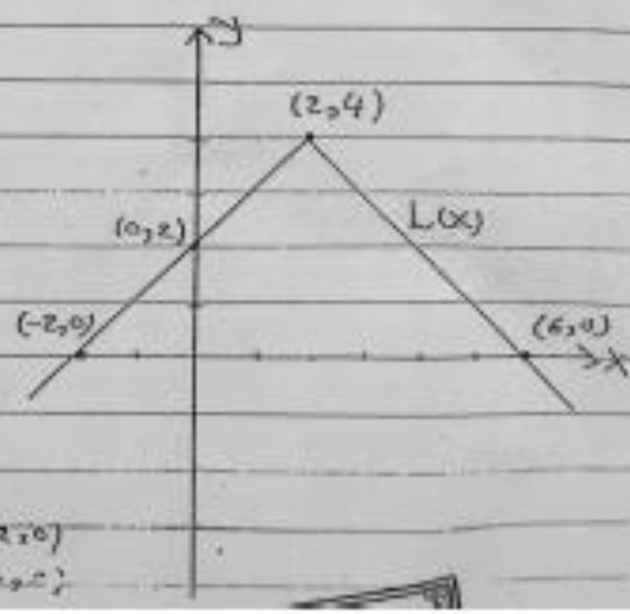

$x=0$

$\Rightarrow y = -|2-0| = -|2| = -2 \Rightarrow (0,-2)$

{5} $L(x) = -|2-x| + 4$

$D_L = \mathbb{R}$

$R_L = (-\infty, 4]$

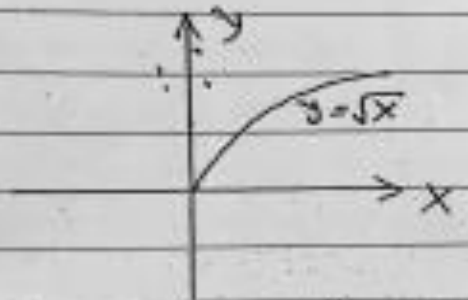



$$\left.\begin{array}{l} x=0 \\ y=-|2-0|+4=2 \end{array}\right\} \Rightarrow (0,2)$$

$y=0$

$$|2-x|=4 \quad \left.\begin{array}{l} 2-x=4 \Rightarrow x=-2 \\ 2-x=-4 \Rightarrow x=6 \end{array}\right\} \Rightarrow \begin{array}{l}(-2,0) \\ (6,0)\end{array}$$

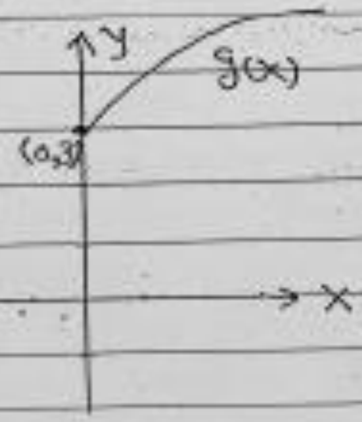
مثال (٤): الشكل الآتي يمثل مخططاً للدالة $y=\sqrt{x}$ ، أرسم مخططاً لكل من الدوال الآتية:




$D_f = \mathbb{R}^+$

$R_f = \mathbb{R}^+$

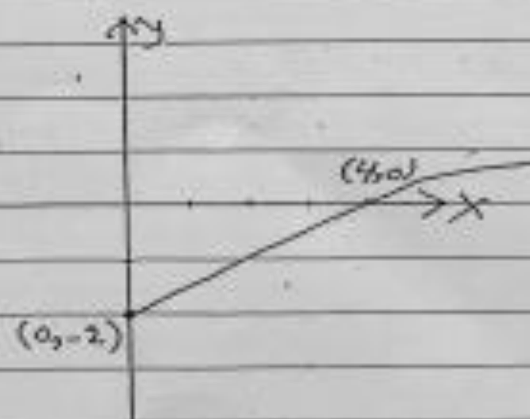
{1} $g(x) = \sqrt{x} + 3$




$D_g = \mathbb{R}^+$

$R_g = [3, \infty)$

{2} $h(x) = \sqrt{x} - 2$




$y = 0 \Rightarrow 0 = \sqrt{x} - 2$

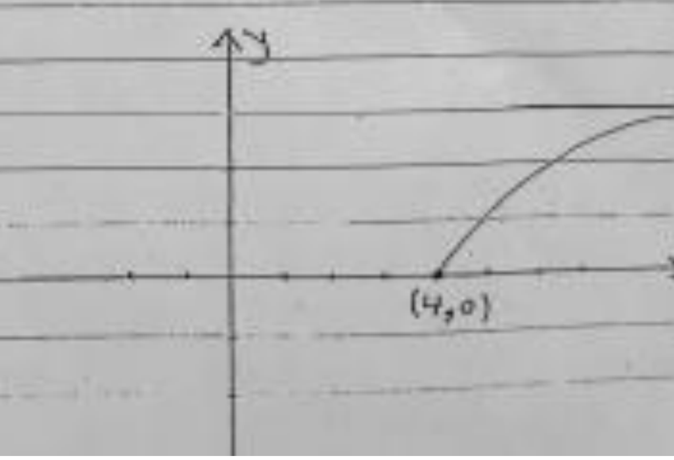
$\Rightarrow \sqrt{x} = 2$

$\Rightarrow x = 4 \Rightarrow (4,0)$

$D_h = \mathbb{R}^+$

$R_h = [-2, \infty)$

{3} $m(x) = \sqrt{x-4}$




$x - 4 = 0 \Rightarrow x = 4$

$D_m = [4, \infty)$

$R_m = \mathbb{R}^+$

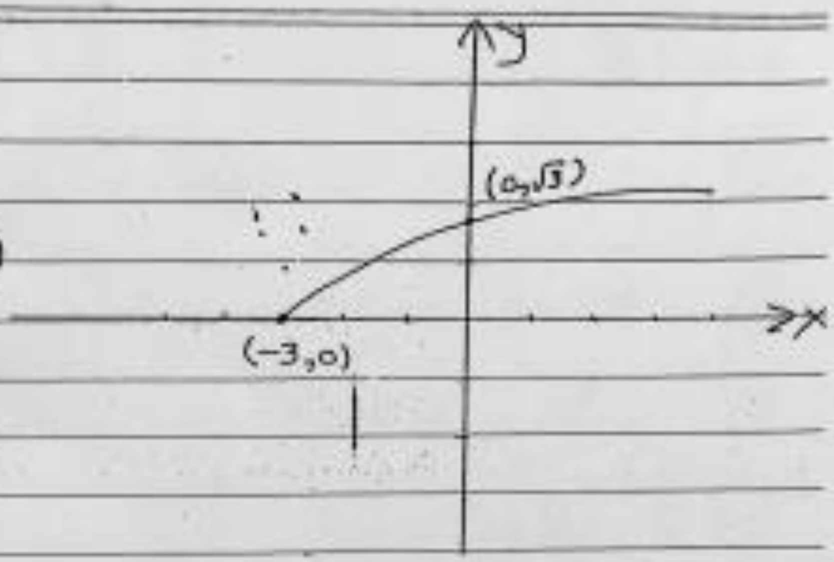
4} $\sqrt{x+3} = t(x)$

$x+3=0 \Rightarrow x=-3$

$x=0 \Rightarrow y=\sqrt{3} \Rightarrow (0,\sqrt{3})$

$D_t = [-3,\infty)$

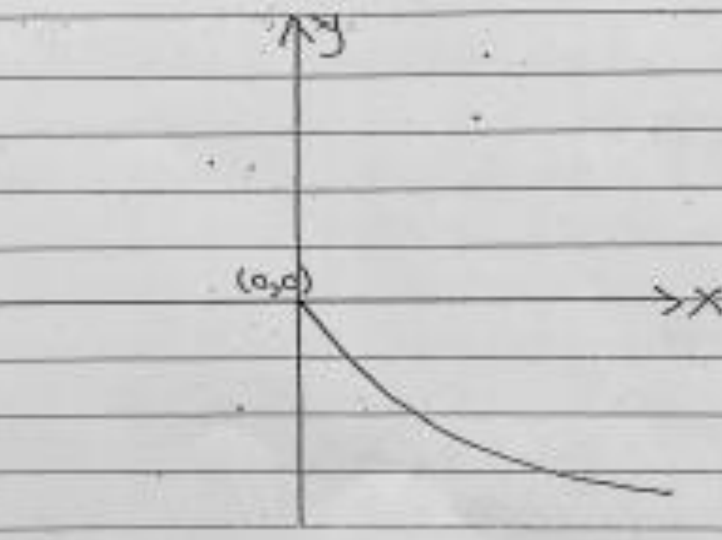
$R_t = \mathbb{R}^+$




5} $h(x) = -\sqrt{x}$

$D_h = \mathbb{R}^+$

$R_h = \mathbb{R}^-$
$= (-\infty, 0]$




6} $k(x) = \sqrt{-x}$

$D_k = \mathbb{R}^-$
$= (-\infty, 0]$

$R_k = \mathbb{R}^+$