



جامعة بغداد

كلية التربية للعلوم الصرفة ابن الهيثم

قسم الرياضيات / المرحلة الاولى

مادة التفاضل والتكامل

الفصل الخامس

الدوال المثلثية

Trigonometric Functions

اعضاء التدريس

- أ.م.د. ايمان محمد نعمه
- أ.م. اريج صلاح محمد
- م. حنان فاروق قاسم

CHAPTER FIVE: Trigonometric Functions

We will define six trigonometric functions in terms of the central angle θ drawn in the center circle (0,0) and radius r .

In the central angle θ with one of its sides is applied to the x -axis and the other side is drawn from the origin point and cut the circumference of the circle at the point $p(x, y)$, then:

- Sine: **$\sin \theta = \frac{y}{r}$**
- Cosine: **$\cos \theta = \frac{x}{r}$**
- Tangent: **$\tan \theta = \frac{y}{x}$**
- Cotangent: **$\cot \theta = \frac{x}{y}$**
- Secant: **$\sec \theta = \frac{r}{x}$**
- Cosecant: **$\csc \theta = \frac{r}{y}$**

From the previous definition definitions, a relation can be found between trigonometric functions as follows:

$$* \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$* \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} = \frac{\csc \theta}{\sec \theta}$$

$$* \sec \theta = \frac{1}{\cos \theta}$$

$$* \csc \theta = \frac{1}{\sin \theta}$$

And since the equation of the circle center (0,0) and radius r is:

$$x^2 + y^2 = r^2$$

$$\because x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

Trigonometric Identities

$$\Rightarrow r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2$$

$$\Rightarrow r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$$

$$\Rightarrow \cos^2 \theta + \sin^2 \theta = 1 \quad \dots (1)$$

Note: From the above equation, we can derive the following forms:

• If we divide eq.(1) by $\cos^2 \theta$:

$$\Rightarrow 1 + \tan^2 \theta = \sec^2 \theta$$

• If we divide eq.(1) by $\sin^2 \theta$:

$$\Rightarrow \cot^2 \theta + 1 = \csc^2 \theta$$

Laws of Sum and Subtract Two Angles:

Let A and B be any two angles, then:

$$* \cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$* \cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$* \sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$* \sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$* \tan(A \mp B) = \frac{\tan A \mp \tan B}{1 \pm \tan A \tan B}$$

Note: Now, we can use the laws of sum and subtract two angles to derive the following forms:

$$\bullet \sin(2\theta) = \sin(\theta + \theta) = \sin \theta \cos \theta + \sin \theta \cos \theta$$

$$\Rightarrow \sin(2\theta) = 2 \sin \theta \cos \theta \quad \dots (2)$$

$$\bullet \cos(2\theta) = \cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta$$

$$\Rightarrow \cos(2\theta) = \cos^2 \theta - \sin^2 \theta \quad \dots (3)$$

Note: From eq.(1) and eq (3), we can derive the following:

- eq.(1) + eq. (3) $\Rightarrow 2\cos^2 \theta = 1 + \cos 2\theta$
- eq.(1) - eq. (3) $\Rightarrow 2\sin^2 \theta = 1 - \cos 2\theta$

Remark: Trigonometric function are divided into two types (Odd Functions and Even Functions) as follows:

***Odd Functions:**

$$\sin(-\theta) = -\sin \theta$$

$$\tan(-\theta) = -\tan \theta$$

$$\cot(-\theta) = -\cot \theta$$

$$\csc(-\theta) = -\csc \theta$$

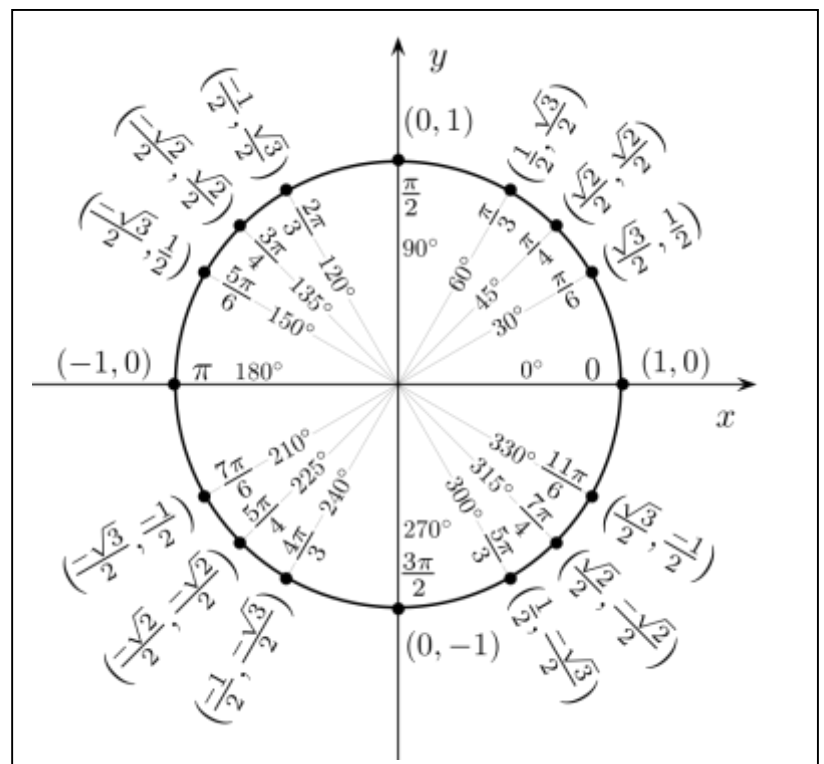
*** Even Functions:**

$$\cos(-\theta) = \cos \theta$$

$$\sec(-\theta) = \sec \theta$$

Rules: $\sin \theta$, $\cos \theta$ and $\tan \theta$ for some standard angles:

θ	$0 = 2\pi$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin \theta$	0	1	0	-1
$\cos \theta$	1	0	-1	0
$\tan \theta$	0	∞	0	$-\infty$



θ	$\frac{\pi}{6} = 30^\circ$	$\frac{\pi}{4} = 45^\circ$	$\frac{\pi}{3} = 60^\circ$
$\sin \theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos \theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan \theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Rules: The $\sin \theta$, $\cos \theta$, and $\tan \theta$ take positive and negative signs depends on the position in which quarter.

Problems (5.1): Dear student you can use the addition or subtraction laws for two angles for the functions *sin* and *cos* to prove that:

1. $\cos(\theta + 2\pi) = \cos(\theta)$
2. $\sin(\theta + 2\pi) = \sin(\theta)$
3. $\tan(\theta + 2\pi) = \tan(\theta)$
4. $\cot(\theta + 2\pi) = \cot(\theta)$
5. $\sec(\theta + 2\pi) = \sec(\theta)$
6. $\csc(\theta + 2\pi) = \csc(\theta)$
7. $\cos(\theta + \pi/2) = -\sin(\theta)$
8. $\cos(\theta - \pi/2) = \sin(\theta)$
9. $\sin(\theta + \pi/2) = \cos(\theta)$
10. $\sin(\theta - \pi/2) = -\cos(\theta)$

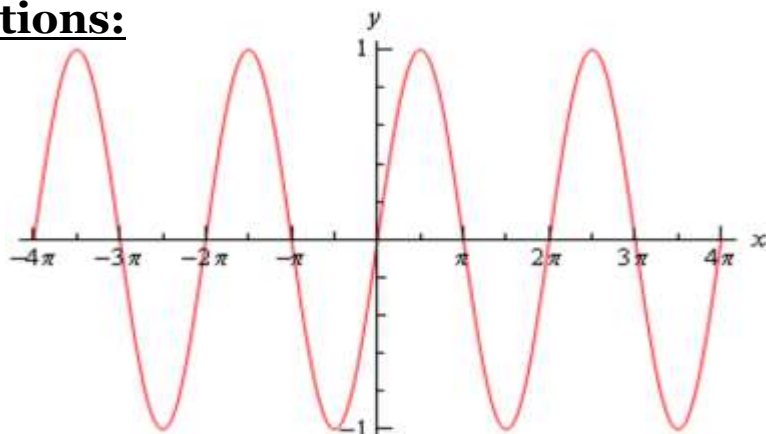
Graphs of Trigonometric Functions:

1. $y = \sin \theta$

Domain: $= \mathbb{R}$

Range: $= [-1, 1]$

Period: $= 2\pi$

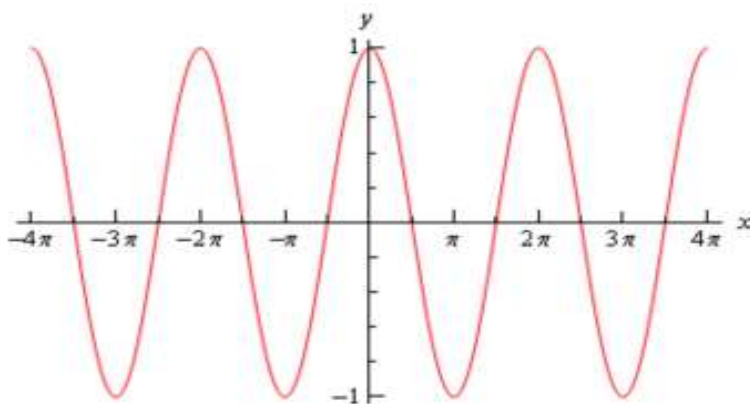


2. $y = \cos \theta$

Domain: $= \mathbb{R}$

Range: $= [-1, 1]$

Period: $= 2\pi$

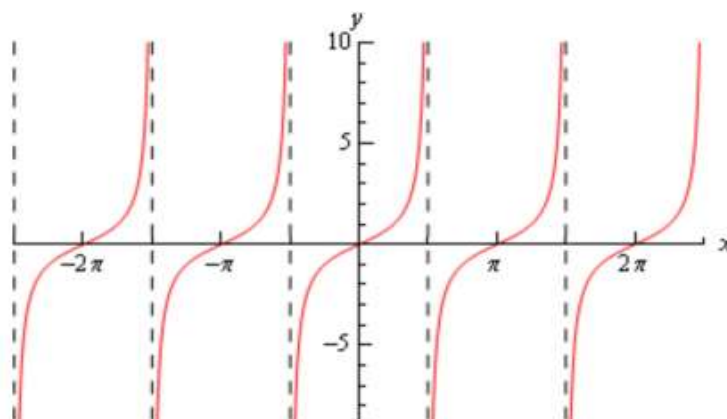


3. $y = \tan \theta$

Domain: $= \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi : n = 0, \pm 1, \pm 2, \dots \right\}$

Range: $= \mathbb{R}$

Period: $= \pi$

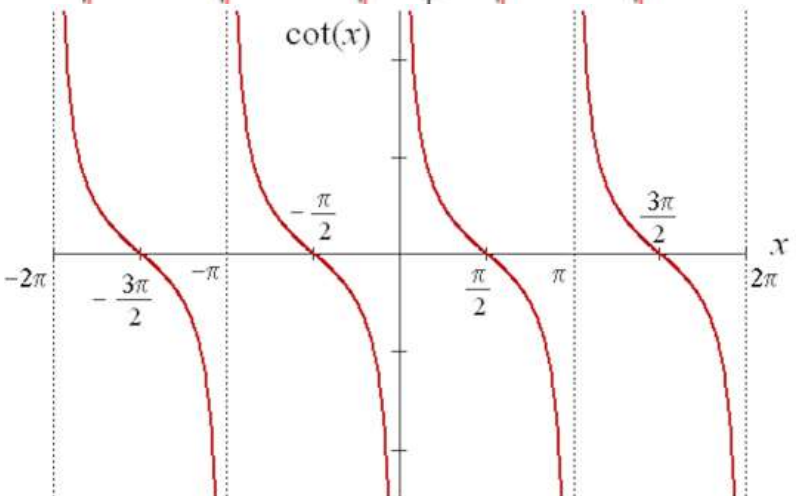


4. $y = \cot \theta$

Domain: $= \mathbb{R} \setminus \{n\pi : n = 0, \pm 1, \pm 2, \dots\}$

Range: $= \mathbb{R}$

Period: $= \pi$

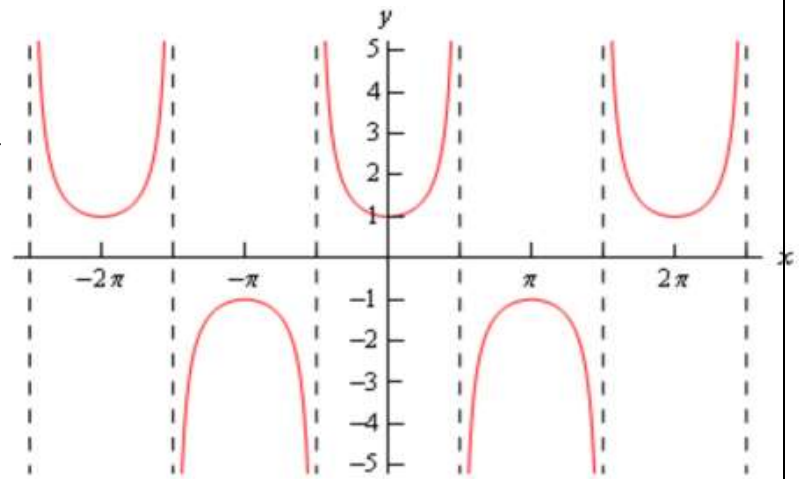


5. $y = \sec \theta$

Domain: $= \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi : n = 0, \pm 1, \pm 2, \dots \right\}$

Range: $= \mathbb{R} \setminus (-1, 1)$

Period: $= 2\pi$

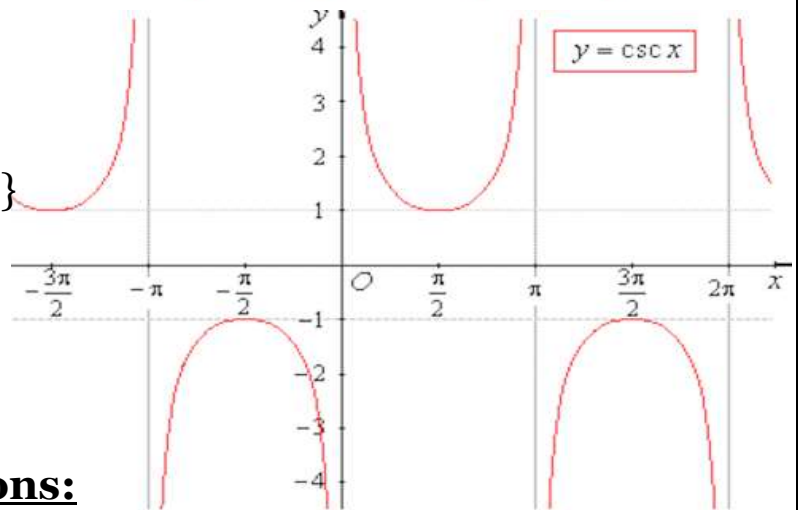


6. $y = \csc \theta$

Domain: $= \mathbb{R} \setminus \{n\pi : n = 0, \pm 1, \pm 2, \dots\}$

Range: $= \mathbb{R} \setminus (-1, 1)$

Period: $= 2\pi$



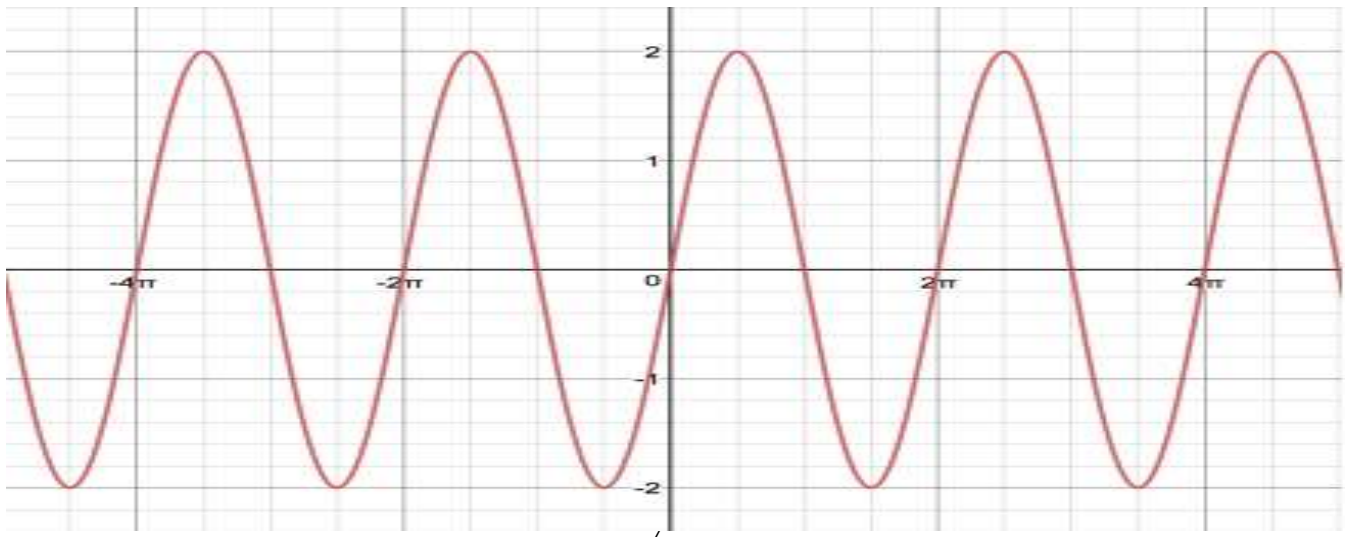
Shifting Trigonometric Functions:

Examples: Plot the following functions:

(1) $y = 2\sin \theta$

$$\because D_{\sin \theta} = \mathbb{R} \Rightarrow D_{2\sin \theta} = \mathbb{R}$$

$$\because R_{\sin \theta} = [-1, 1] \Rightarrow R_{2\sin \theta} = [-2, 2]$$

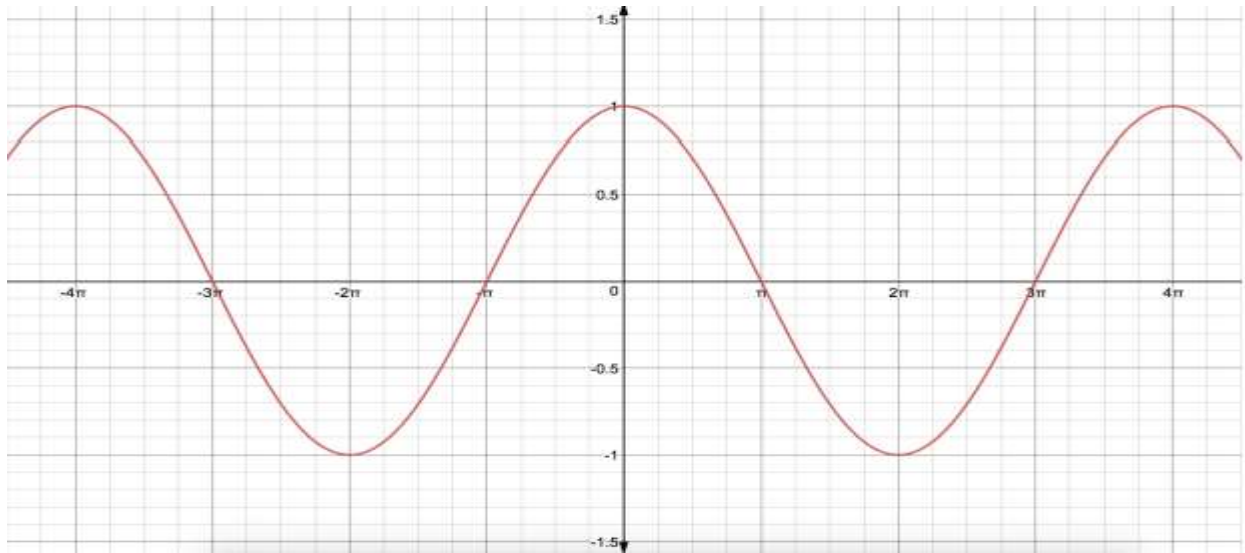


$$(2) \ y = \cos \frac{\theta}{2}$$

$$\because D_{\cos \theta} := \mathbb{R} \Rightarrow D_{\cos \frac{\theta}{2}} := \mathbb{R}$$

$$\text{but, } \because -2\pi \leq \frac{\theta}{2} \leq 2\pi \Rightarrow -4\pi \leq \theta \leq 4\pi$$

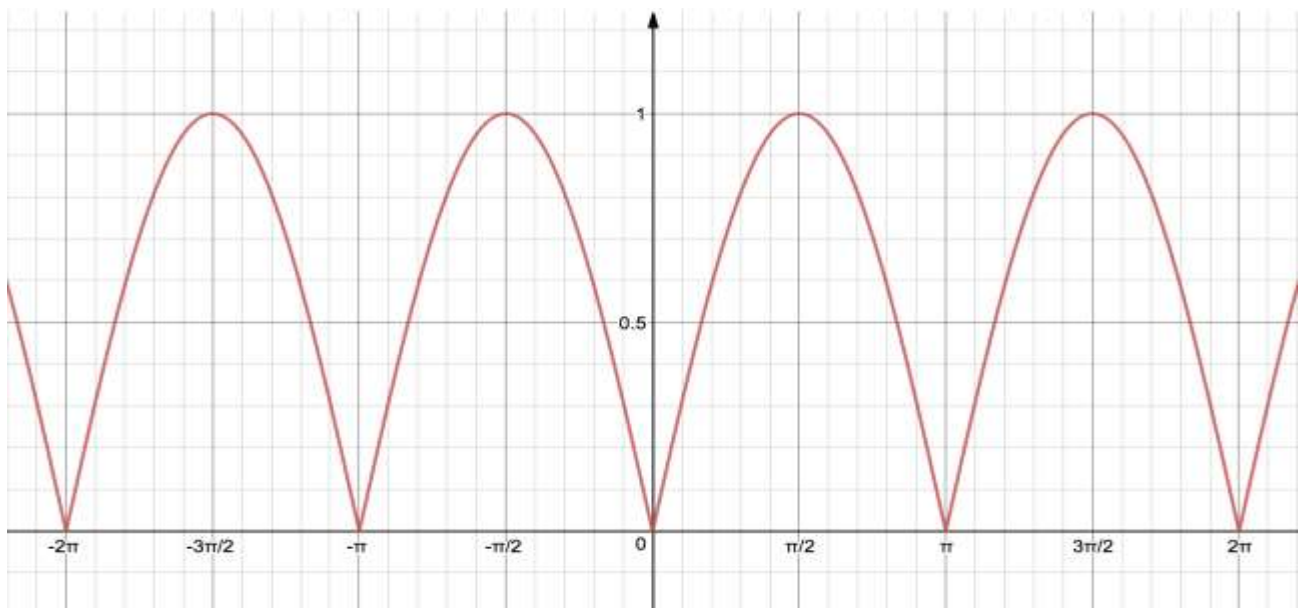
$$\because R_{\cos \theta} := [-1, 1] \Rightarrow R_{\cos \frac{\theta}{2}} := [-1, 1]$$



$$(3) \ y = |\sin \theta|$$

$$\because D_{\sin \theta} := \mathbb{R} \Rightarrow D_{|\sin \theta|} := \mathbb{R}$$

$$\because R_{\sin \theta} := [-1, 1] \Rightarrow R_{|\sin \theta|} := [0, 1]$$

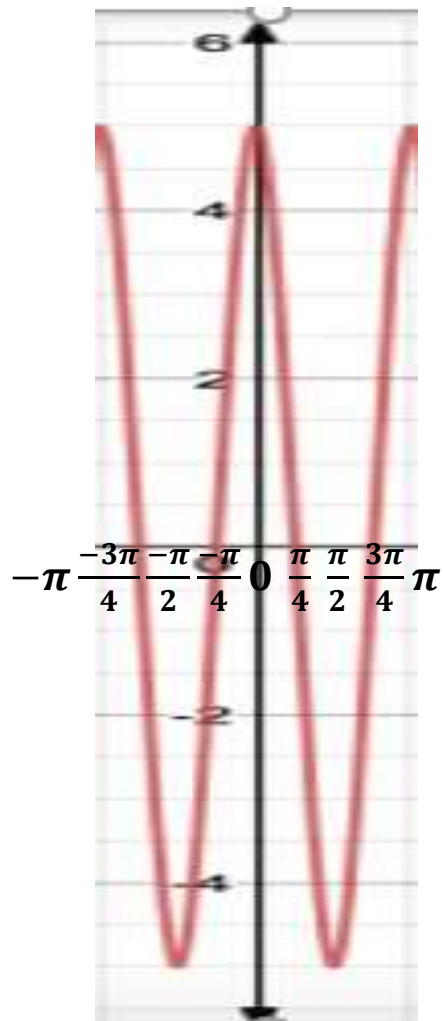


(4) $y = 5\cos(2\theta)$

$\because D_{\cos \theta} := \mathbb{R} \Rightarrow D_{5\cos(2\theta)} := \mathbb{R}$

but, $\because -2\pi \leq 2\theta \leq 2\pi \Rightarrow -\pi \leq \theta \leq \pi$

$\because R_{\cos} := [-1,1] \Rightarrow R_{5\cos(2\theta)} := [-5,5]$



Problems (5.2):

Q1/ Sketch the graph the following functions:

1) $y = \sin\left(\frac{\theta}{2}\right)$

7) $y = 1 + \cos(-x)$

2) $y = \cos(3\theta)$

8) $y = \cos\left(x - \frac{\pi}{2}\right) - 1$

3) $y = 1 + \sin(\theta)$

9) $y = \cos\left(\frac{x}{2}\right) + 2$

4) $y = \frac{1+\cos(2\theta)}{2}$

10) $y = |\cos x| - 1$

5) $y = |\sin(4\theta)|$

11) $y = -\sin(-x)$

6) $y = 2\sin(\theta + \pi)$

Q2/ Prove that:

a) $\sin^2 \theta + \cos^2 \theta + \tan^2 \theta = \sec^2 \theta$ b) $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \cdot \sin^2 \theta$

c) $\sqrt{\frac{1+\tan^2 \theta}{1+\cot^2 \theta}} = \tan \theta$

d) $\sqrt{\frac{1+\cos \theta}{1-\cos \theta}} = \csc \theta + \cot \theta$

e) $\frac{1-\tan^2 \theta}{1+\tan^2 \theta} = \cos 2\theta$

Limits of Trigonometric Functions:

Theorems:

$$1. \quad \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$2. \quad \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$$

Result: $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1$

proof:

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\cos \theta}}{\theta} \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta} \right) \\ &= \left(\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \right) \cdot \left(\lim_{\theta \rightarrow 0} \frac{1}{\cos \theta} \right) \\ &= (1) \cdot \left(\frac{1}{\cos(0)} \right) = 1 \end{aligned}$$

Examples: Find the following limits?

$$(1) \quad \lim_{t \rightarrow 0} \frac{\sin 3t}{t}$$

$$\lim_{t \rightarrow 0} \frac{\sin 3t}{t} = \lim_{t \rightarrow 0} \frac{3 \cdot \sin 3t}{3 \cdot t} = 3 \cdot \lim_{t \rightarrow 0} \frac{\sin 3t}{3t} = 3 \cdot 1 = 3$$

$$(2) \quad \lim_{x \rightarrow \infty} x \cdot \sin \frac{1}{x}$$

$$\text{Let } y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$$

$$\therefore x \rightarrow \infty \Rightarrow y \rightarrow 0$$

Hence, $\lim_{x \rightarrow \infty} x \cdot \sin \frac{1}{x} = \lim_{y \rightarrow 0} \frac{1}{y} \cdot \sin y = \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$

(3) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{2x}$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{2x} = \lim_{x \rightarrow 0} \frac{-1}{2} \cdot \frac{\cos x - 1}{x} = \frac{-1}{2} \cdot \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = \frac{-1}{2} \cdot 0 = 0$$

(4) $\lim_{h \rightarrow 2} \frac{\cos\left(\frac{\pi}{h}\right)}{h - 2}$

$$\lim_{h \rightarrow 2} \frac{\cos\left(\frac{\pi}{h}\right)}{h - 2} = \frac{0}{0}$$

$$\begin{aligned} \therefore \lim_{h \rightarrow 2} \frac{\cos\left(\frac{\pi}{h}\right)}{h - 2} &\stackrel{L'R}{=} \lim_{h \rightarrow 2} \frac{-\sin\left(\frac{\pi}{h}\right) \cdot \frac{h \cdot 0 - \pi \cdot 1}{h^2}}{1 - 0} = -\lim_{h \rightarrow 2} \left(-\sin\left(\frac{\pi}{h}\right) \cdot \frac{\pi}{h^2} \right) \\ &= -\sin\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{4} = -1 \cdot \frac{-\pi}{4} = \frac{\pi}{4} \end{aligned}$$

(5) $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{1 - \sin \theta}{1 + \cos(2\theta)}$

$$\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{1 - \sin \theta}{1 + \cos(2\theta)} = \frac{0}{0}$$

$$\begin{aligned} \therefore \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{1 - \sin \theta}{1 + \cos(2\theta)} &\stackrel{L'R}{=} \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{-\cos \theta}{-2\sin(2\theta)} \\ &= \frac{1}{2} \cdot \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\cos \theta}{\sin(2\theta)} \\ &\stackrel{L'R}{=} \frac{-1}{2} \cdot \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{-\sin \theta}{2\cos(2\theta)} \\ &= \frac{-1}{4} \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin \theta}{\cos(2\theta)} = \frac{-1}{4} \cdot \frac{\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)} = \frac{-1}{4} \cdot \frac{1}{-1} = \frac{1}{4} \end{aligned}$$

Problems (5.3): Evaluate the following limits, if it exist?

1. $\lim_{w \rightarrow 0} \frac{3\sin(5w)}{7w}$

2. $\lim_{x \rightarrow 0} \frac{\sin(3x)}{x\cos(5x)}$

3. $\lim_{t \rightarrow 0} \frac{5t}{\tan(6t)}$

4. $\lim_{\theta \rightarrow 0} \frac{\sec \theta - \cos \theta}{\theta^2}$

5. $\lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{2x^2}$

6. $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta \cdot \sin \theta}$

7. $\lim_{z \rightarrow 0} (\tan(2z) \cdot \csc(4z))$

8. $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2}$

9. $\lim_{t \rightarrow 0} \frac{-2\tan t}{5t}$

10. $\lim_{y \rightarrow 0} \frac{1 - \cos y}{\sin y}$

Q2/Proof the following:

(a) $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin \theta - 1}{\cos \theta} = 0$

(b) $\lim_{w \rightarrow -4} \frac{\sin(\pi w)}{w^2 - 16} = -\frac{\pi}{8}$

Differentiation of Trigonometric Functions:

Let u be a function of x , then:

1. $\frac{d}{dx}(\sin(u)) = \cos(u) \cdot \frac{du}{dx}$

2. $\frac{d}{dx}(\cos(u)) = -\sin(u) \cdot \frac{du}{dx}$

3. $\frac{d}{dx}(\tan(u)) = \sec^2(u) \cdot \frac{du}{dx}$

4. $\frac{d}{dx}(\cot(u)) = -\csc^2(u) \cdot \frac{du}{dx}$

5. $\frac{d}{dx}(\sec(u)) = \sec(u) \cdot \tan(u) \cdot \frac{du}{dx}$

$$6. \quad \frac{d}{dx}(\csc(u)) = -\csc(u) \cdot \cot(u) \cdot \frac{du}{dx}$$

Examples: Find the derivatives of the following functions?

$$(1) \quad y = \frac{\sin x}{x}$$

$$\frac{dy}{dx} = \frac{x \cdot \cos x - \sin x - 1}{x^2} = \frac{x \cos x - \sin x}{x^2}$$

$$(2) \quad y = \frac{2}{\cos(3t)}$$

$$\frac{dy}{dt} = \frac{\cos(3t) \cdot 0 - 2 \cdot (-\sin(3t) \cdot 3)}{\cos^2(3t)} = \frac{6\sin(3t)}{\cos^2(3t)}$$

$$(3) \quad y = \cot(z^2)$$

$$\frac{dy}{dz} = -\csc^2(z^2) \cdot 2z$$

$$(4) \quad y = \sec^2(5x)$$

$$\frac{dy}{dx} = 2 \cdot \sec(5x) \cdot \sec(5x) \cdot \tan(5x) \cdot 5 = 10\sec^2(5x) \cdot \tan(5x)$$

$$(5) \quad y = \sin(\cos w)$$

$$\frac{dy}{dw} = \cos(\cos w) \cdot (-\sin w) \cdot 1 = \cos(\cos w) \cdot -\sin w$$

Problems (5.4): Find the derivative of the following functions?

$$1. \quad y = \left(\frac{\sin \sqrt{x}}{\sqrt{x}} \right)^3$$

$$2. \quad y = 4\cos^2(-3w)$$

$$3. \quad y = \sin^2\left(\frac{3}{z}\right) + \cos^2(z^2)$$

$$4. \quad y = \sqrt[3]{9x + \cos(2x)}$$

$$5. \quad y = \frac{\sqrt{2t}}{\cos(3t)}$$

$$6. \quad y = x^3 \cdot \sin(2x^2 + 3)$$

$$7. \quad y = (\cos^2(1+t) + \sqrt{t+5})^5$$

$$8. \quad y = \frac{7\sqrt{\sec(3\theta)}}{\theta^2}$$

$$9. \quad y = 2 \sin\left(\frac{z}{2}\right) - \left(z \cos\left(\frac{2}{z}\right)\right)^3$$

$$10. \quad y = \sin(3t) \cdot \cos(5t^2)$$

Implicit Differentiation of Trigonometric Functions:

Examples: Find y' of the following functions?

$$(1) \quad x \sin(2y) = y \cdot \cos(2x)$$

$$\Rightarrow x \cdot \cos(2y) \cdot 2y' + \sin(2y) \cdot 1 = y \cdot (-\sin(2x)) \cdot 2 + \cos(2x) \cdot y'$$

$$\Rightarrow x \cdot \cos(2y) \cdot 2y' - \cos(2x) \cdot y' = y \cdot (-\sin(2x)) \cdot 2 - \sin(2y)$$

$$\Rightarrow (2x \cdot \cos(2y) - \cos(2x)) \cdot y' = -2y \cdot \sin(2x) - \sin(2y)$$

$$\Rightarrow y' = \frac{-2y \cdot \sin(2x) - \sin(2y)}{(2x \cdot \cos(2y) - \cos(2x))}$$

$$(2) \quad \cot(xy) + xy = 0$$

$$\Rightarrow -\csc^2(xy)(xy' + y \cdot 1) + xy' + y = 0$$

$$\Rightarrow -x \csc^2(xy)y' - y \csc^2(xy) + xy' + y = 0$$

$$\Rightarrow (-x \csc^2(xy) + x)y' = y \csc^2(xy) - y$$

$$\Rightarrow y' = \frac{y \csc^2(xy) - y}{-x \csc^2(xy) + x}$$

$$\Rightarrow y' = \frac{y(\cos^2(xy) - 1)}{-x(\csc^2(xy) - 1)}$$

$$\Rightarrow y' = -\frac{y}{x}$$

Problems (5.5): Find y' of the following functions?

$$1. \quad y \sin x + x \sin y = y^2$$

$$2. \quad \sec^2 y + \csc^2 y = 4x$$

$$3. \quad y = \tan(x + y)$$

$$4. \quad y^2 = \sin^4(2x) + \cos^4(2x)$$

$$5. \cos(x^2 y^2) = x$$

$$6. x^2 y = \frac{\cot y}{1 + \cos y}$$

$$7. \sqrt{xy} + \csc(-xy) = y$$

$$8. y(3 + \tan y)^{\frac{1}{3}} = x + 5$$

$$9. y = \tan y + \sec^2(xy) + \cot(x^2 + y^2)$$

$$10. x^2 = \sin y + \sin(2y)$$

The Inverse Trigonometric Functions:

Suppose f be a one-to-one (i.e. 1-1) and onto function.

$$f: X \rightarrow Y$$

- f is 1 - 1 $\Leftrightarrow x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$$\stackrel{OR}{\Leftrightarrow} f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

- f is onto $\Leftrightarrow \forall y \in Y \exists x \in X \ni y = f(x)$

- $f: X \rightarrow Y \ni y = f(x) \Leftrightarrow f^{-1}: Y \rightarrow X \ni x = f^{-1}(y)$

(1) The Inverse of Sine Function:

Let $y = \sin(x)$, $\sin(x): \mathbb{R} \rightarrow [-1, 1]$

We are going to define a new function which is inverse sine, and we denote it by $\sin^{-1}$ or arcsin.

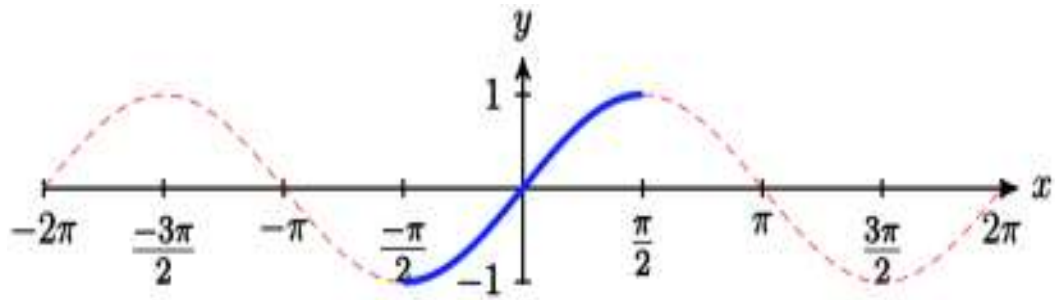
$$\therefore \sin^{-1} y = \sin^{-1}(\sin x) \Rightarrow \sin^{-1} y = x$$

$$\therefore y = \sin x \Leftrightarrow x = \sin^{-1} y$$

$$\sin(x): \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow [-1, 1]$$

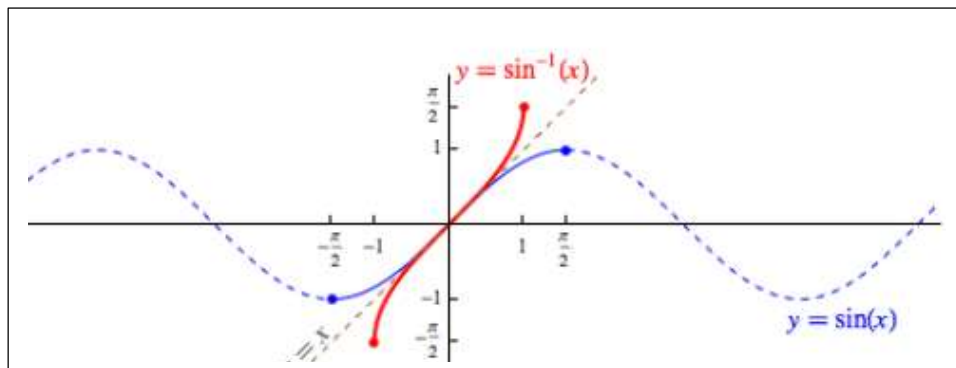
$$\therefore \sin \text{ is } 1 - 1 \text{ and onto} \Rightarrow \exists \sin^{-1} \ni \sin^{-1}: [-1, 1] \Rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$D_{\sin^{-1}} = [-1, 1] = R_{\sin x}$$



$$R_{\sin^{-1}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = D_{\sin x}$$

Note: $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$



Remark: $\sin^{-1}$ is an odd function. (i.e., $\sin^{-1}(-x) = -\sin^{-1}(x)$)

(2) The Inverse of Cosine Function:

Let $y = \cos(x)$, $\cos(x): \mathbb{R} \rightarrow [-1, 1]$

We are going to define a new function which is inverse cosine, and we denote it by $\cos^{-1}$ or arccos.

$$\therefore \cos^{-1} y = \cos^{-1}(\cos x) \Rightarrow \cos^{-1} y = x$$

$$\therefore y = \cos x \Leftrightarrow x = \cos^{-1} y$$

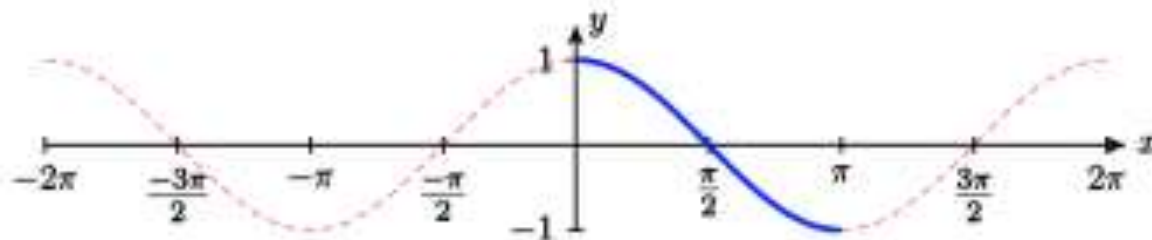
$$\cos: [0, \pi] \Rightarrow [-1, 1]$$

$\because \cos$ is $1-1$ and onto $\Rightarrow \exists \cos^{-1} \ni$

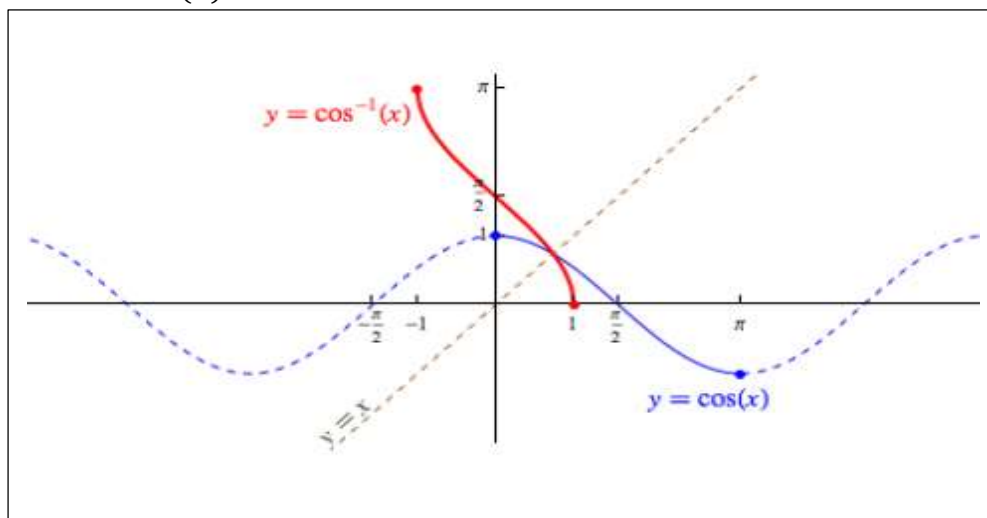
$$\cos^{-1}: [-1, 1] \Rightarrow [0, \pi]$$

$$D_{\cos^{-1}} = [-1, 1] = R_{\cos x}$$

$$R_{\cos^{-1}} = [0, \pi] = D_{\cos x}$$



Note: $\cos^{-1}(x) \neq \frac{1}{\cos(x)}$



Remark: $\cos^{-1}$ is neither even nor odd function.

Note: $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

(3) The Inverse of Tangent Function:

Let $y = \tan(x)$, $\tan(x): \mathbb{R} \setminus \left\{x: x = \frac{\pi}{2} + n\pi; n \in \mathbb{I}\right\} \rightarrow \mathbb{R}$

We are going to define a new function which is inverse tangent, and we denote it by $\tan^{-1}$ or arctan.

$$\therefore \tan^{-1} y = \tan^{-1}(\tan x) \Rightarrow \tan^{-1} y = x$$

$$\therefore y = \tan x \Leftrightarrow x = \tan^{-1} y$$

$$\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$$

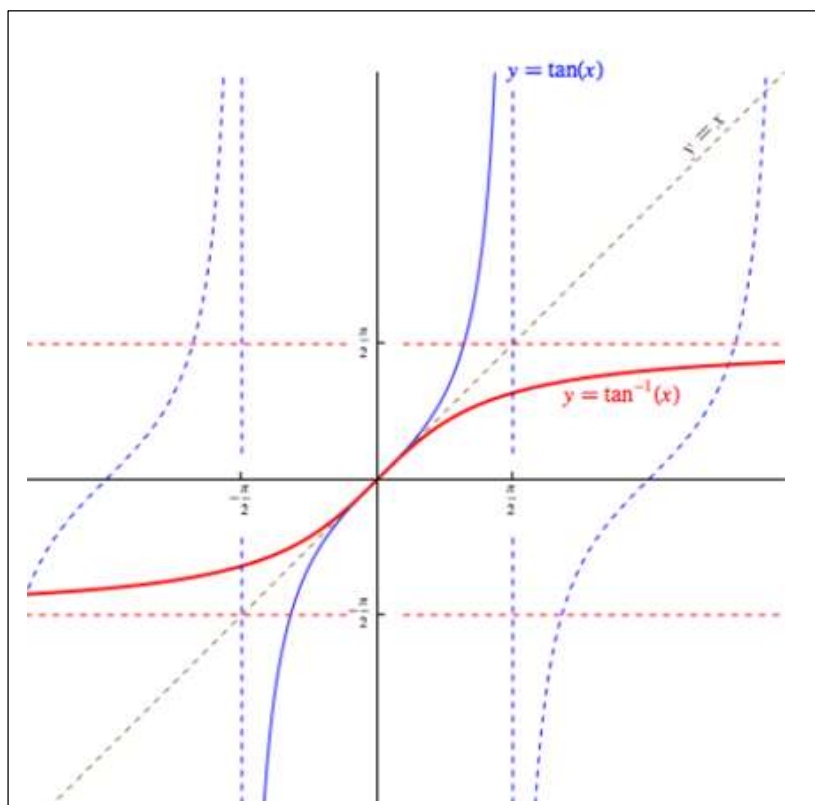
$$\because \tan \text{ is } 1-1 \text{ and onto, } \Rightarrow \exists \tan^{-1} \ni$$

$$\tan^{-1}: \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$D_{\tan^{-1}} = \mathbb{R} = R_{\tan x}$$

$$R_{\tan^{-1}} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = D_{\tan x}$$

Note: $\tan^{-1}(x) \neq \frac{1}{\tan(x)}$



Remark: $\tan^{-1}$ is an odd function. *i.e.*, $(\tan^{-1}(-x) = -\tan^{-1}(x))$

The Derivative of Inverse Trigonometric Functions:

Let u be a function of x , then:

$$1. \quad \frac{d}{dx} \left(\sin^{-1}(u) \right) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$2. \quad \frac{d}{dx} \left(\cos^{-1}(u) \right) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$3. \quad \frac{d}{dx} \left(\tan^{-1}(u) \right) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$4. \quad \frac{d}{dx} \left(\cot^{-1}(u) \right) = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$$

$$5. \quad \frac{d}{dx} \left(\sec^{-1}(u) \right) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$6. \quad \frac{d}{dx} \left(\csc^{-1}(u) \right) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Examples: Find the derivatives of the following functions:

$$(1) \quad f(x) = \sin^{-1}(x^2)$$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

$$(2) \quad g(t) = \cos^{-1}(\sqrt{t})$$

$$\Rightarrow g'(t) = \frac{-1}{\sqrt{1-(\sqrt{t})^2}} \cdot \frac{1}{2} t^{-\frac{1}{2}}$$

$$(3) \quad y = \sin^{-1} \sqrt{1-\sqrt{\theta}}$$

$$\Rightarrow y' = \frac{1}{\sqrt{1-(\sqrt{1-\sqrt{\theta}})^2}} \cdot \frac{1}{2} (1-\sqrt{\theta})^{-\frac{1}{2}} \cdot \frac{1}{2} \theta^{-\frac{1}{2}}$$

$$(4) f(x) = \cot^{-1} \left(\frac{1-x}{1+x} \right)$$

$$\Rightarrow f'(x) = \frac{1}{1 + \left(\frac{1-x}{1+x} \right)^2} \cdot \frac{(1+x) \cdot (-1) - (1-x) \cdot 1}{(1+x)^2}$$

$$(5) g(x) = \sec^{-1} \left(\frac{\sqrt{1+x^2}}{x} \right)$$

$$\Rightarrow g'(x) = \frac{1}{\left| \frac{\sqrt{1+x^2}}{x} \right| \sqrt{\left(\frac{1+x^2}{x} \right)^2 - 1}} \cdot \frac{x \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x - \sqrt{1+x^2} \cdot 1}{x^2}$$

$$(6) y = x \cdot \csc^{-1} \left(\frac{1}{x} \right) + \sqrt{1-x^2}$$

$$\Rightarrow y' = x \cdot \frac{-1}{\left| \frac{1}{x} \right| \sqrt{\frac{1}{x^2} - 1}} \cdot \frac{x \cdot 0 - 1 \cdot 1}{x^2} + \csc^{-1} \left(\frac{1}{x} \right) \cdot 1 + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x$$

Problems (5.6):

1 Find y' of the following functions?

$$(a) y = \sin^{-1} \left(\frac{x-1}{x+1} \right)$$

$$(b) y = \theta \cdot (\sin^{-1}(\theta))^2 - 2\theta + 2\sqrt{1-\theta} \cdot \sin^{-1}(\theta)$$

$$(c) y = t \cdot \cos^{-1}(2t) - \frac{1}{2}\sqrt{1-4t^2}$$

$$(d) y = \frac{\cos^{-1}(2x)}{\sqrt{1+4x^2}}$$

$$(e) y = \cos^{-1} \left(\frac{3}{t} \right) + \frac{t}{1-t^2}$$

$$(f) y = \sec^{-1}(\sqrt{w^2 + 4})$$

$$(g) y = \sin(\tan^{-1} x)$$

$$(h) y = \tan^{-1}(3\tan 2z)$$

$$(i) y = \sec^{-1}(5x^2)$$

$$(j) y = \frac{\cot^{-1}(3\theta)}{1 + \theta^2}$$

2 Find y' of the following functions?

$$(a) x \sin y + x^3 = \tan^{-1} y$$

$$(b) \sin^{-1}(xy) = \cos^{-1}(x + y)$$

$$(c) \cos 3y - \cos^{-1}(y) - \frac{\sin x}{2x} = 0$$

Algebra of Inverse Trigonometric Functions:

Some properties for the inverse trigonometric functions:

$$* \cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)$$

$$* \sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$$

$$* \csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$$

Examples: Evaluate the following:

1 $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = ?$

$$\therefore \cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \cos \cos^{-1}\left(\frac{1}{2}\right) = I\left(\frac{1}{2}\right) = \frac{1}{2}$$

2 $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = ?$

$$\text{Let } \alpha = \cos^{-1}\frac{\sqrt{2}}{2}$$

$$\Rightarrow \alpha = \cos^{-1}\frac{1}{\sqrt{2}} \Rightarrow \cos \alpha = \frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{\pi}{4} = 45^\circ$$

$$\therefore \sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

3 $\csc(\sec^{-1}(2)) = ?$

$$\text{Let } \alpha = \sec^{-1}(2)$$

$$\Rightarrow \alpha = \sec^{-1}(2) = \cos^{-1}\left(\frac{1}{2}\right) \Rightarrow \cos \alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3} = 60^\circ$$

$$\therefore \csc(\sec^{-1}(2)) = \csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

4 $\cot\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = ?$

$$\text{Let } \alpha = \sin^{-1}\frac{1}{2} \Rightarrow \sin \alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{6}$$

$$\therefore \cot\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \cot\left(\frac{\pi}{6}\right) = \cot\frac{\pi}{6}$$

$$= \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$5 \cos \left(\sin^{-1} \left(\frac{8}{10} \right) \right) = ?$$

$$\text{Let } \alpha = \sin^{-1} \left(\frac{8}{10} \right) \Rightarrow \sin \alpha = \frac{8}{10} \Rightarrow \sin^2 \alpha = \frac{64}{100}$$

$$\therefore \cos^2 \alpha = 1 - \sin^2 \alpha \Rightarrow \cos^2 \alpha = 1 - \frac{64}{100} = \frac{36}{100}$$

$$\Rightarrow \cos(\alpha) = \frac{6}{10}$$

$$6 \cos \left(\sin^{-1} \left(\frac{1}{3} \right) - \tan^{-1} \left(\frac{1}{2} \right) \right) = ?$$

$$\text{Let } \alpha = \sin^{-1} \left(\frac{1}{3} \right) \Rightarrow \sin \alpha = \frac{1}{3}$$

$$\text{Let } \beta = \tan^{-1} \left(\frac{1}{2} \right) \Rightarrow \tan \beta = \frac{1}{2}$$

$$\therefore \cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\therefore \cos \left(\sin^{-1} \left(\frac{1}{3} \right) - \tan^{-1} \left(\frac{1}{2} \right) \right) = \cos(\alpha - \beta)$$

$$= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$= \frac{2\sqrt{2}}{3} \cdot \frac{2}{\sqrt{5}} + \frac{1}{3} \cdot \frac{1}{\sqrt{5}} = \frac{4\sqrt{2} + 1}{3\sqrt{5}}$$

Problems (5.7): Evaluate the following?

$$1. \sec \left(\cos^{-1} \left(\frac{1}{2} \right) \right)$$

$$4. \csc \left(\sin^{-1} \left(\frac{1}{\sqrt{2}} \right) \right)$$

$$2. \cos(\cot^{-1}(1))$$

$$5. \cos^{-1} \left(-\sin \left(\frac{\pi}{6} \right) \right)$$

$$3. \tan \left(\sin^{-1} \left(-\frac{1}{2} \right) \right)$$

$$6. \cos \left(\cos^{-1} \left(\frac{3}{4} \right) - \cot^{-1} \left(\frac{1}{4} \right) \right)$$

Hyperbolic Functions:

$$1. \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$2. \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$3. \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$4. \coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$5. \operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}$$

$$6. \operatorname{csch}(x) = \frac{2}{e^x - e^{-x}}$$

Remarks:

$$* \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$* \coth(x) = \frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)}$$

$$* \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

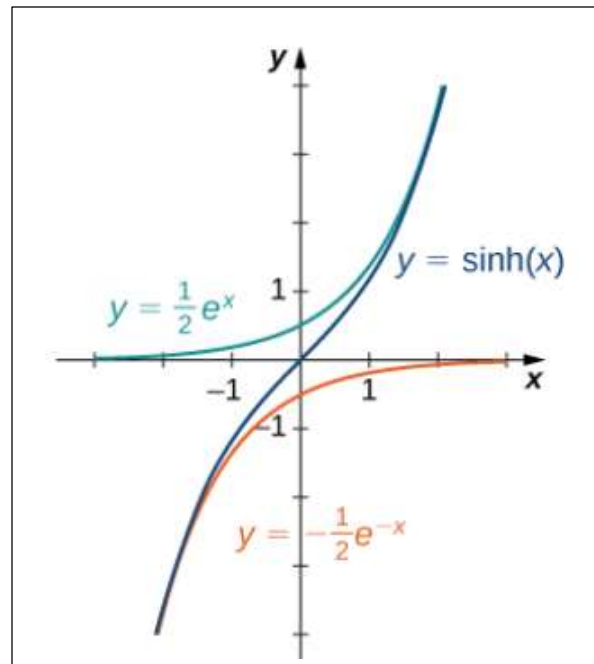
$$* \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

The Graph of Hyperbolic Functions:

$$1. y = \sinh(x)$$

Domain := $\mathbb{R}$

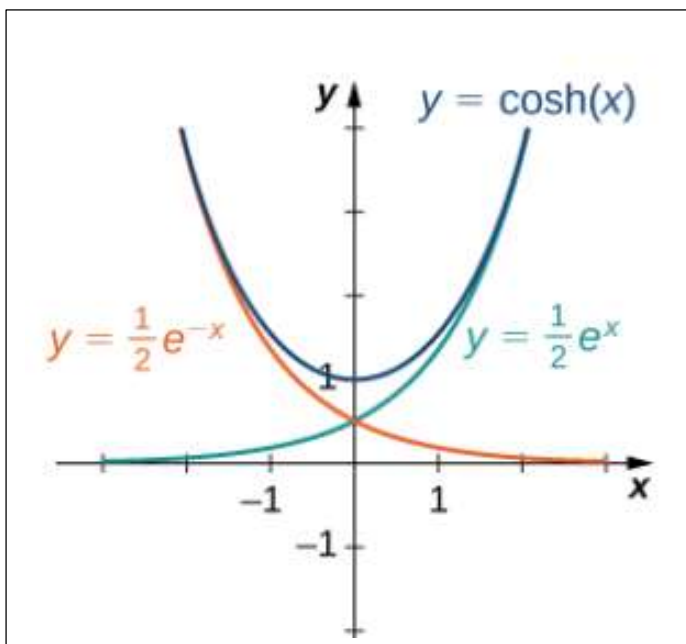
Range := $\mathbb{R}$



2. $y = \cosh(x)$

Domain: $=\mathbb{R}$

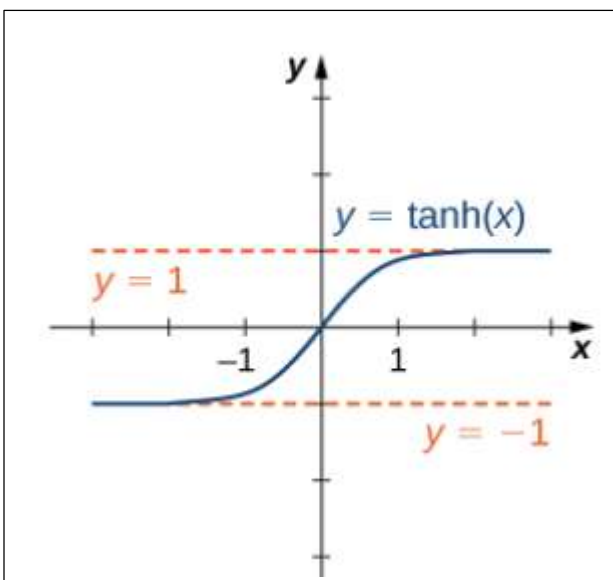
Range $:= [1, \infty)$



3. $y = \tanh(x)$

Domain: $=\mathbb{R}$

Range $:= (-1, 1)$



Some Facts about Hyperbolic Functions:

1. $\cosh^2(x) - \sinh^2(x) = 1$

2. $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

3. $\coth^2(x) - 1 = \operatorname{csch}^2(x)$

$$\begin{aligned} 4. \cosh(-x) &= \cosh(x) \text{ "Even Function"}, \\ \sinh(-x) &= -\sinh(x) \text{ "Odd Function"}, \\ \tanh(-x) &= -\tanh(x) \text{ "Odd Function"} \end{aligned}$$

$$\begin{aligned} 5. \cosh(x) + \sinh(x) &= e^x, \\ \cosh(x) - \sinh(x) &= e^{-x} \end{aligned}$$

$$\begin{aligned} 6. \cosh(x + y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y), \\ \sinh(x + y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y), \\ \tanh(x + y) &= \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)} \end{aligned}$$

$$\begin{aligned} 7. \cosh^2(x) &= \frac{1}{2}(\cosh(2x) + 1), \\ \sinh^2(x) &= \frac{1}{2}(\cosh(2x) - 1) \end{aligned}$$

$$\begin{aligned} 8. \cosh(2x) &= \cosh^2(x) + \sinh^2(x), \\ \sinh(2x) &= 2\sinh(x)\cosh(x) \end{aligned}$$

The Derivative of Hyperbolic Functions:

Let u be a function of x , then:

$$1 \quad \frac{d}{dx}(\sinh(u)) = \cosh(u) \cdot \frac{du}{dx}$$

$$2 \quad \frac{d}{dx}(\cosh(u)) = \sinh(u) \cdot \frac{du}{dx}$$

$$3 \quad \frac{d}{dx}(\tanh(u)) = \operatorname{sech}^2(u) \cdot \frac{du}{dx}$$

$$4 \quad \frac{d}{dx}(\coth(u)) = -\operatorname{csch}^2(u) \cdot \frac{du}{dx}$$

$$5 \quad \frac{d}{dx}(\operatorname{sech}(u)) = -\operatorname{sech}(u) \cdot \tanh(u) \cdot \frac{du}{dx}$$

$$6 \quad \frac{d}{dx}(\operatorname{csch}(u)) = -\operatorname{csch}(u) \cdot \coth(u) \cdot \frac{du}{dx}$$

Examples: Find the derivatives of the following functions:

- $\sinh(3x)$

$$\Rightarrow y' = 3\cosh(3x)$$

- $y = \cosh^2(5x)$

$$\Rightarrow y' = 2\cosh(5x) \cdot \sinh(5x) \cdot 5$$

- $\tanh(2x)$

$$\Rightarrow y' = \operatorname{sech}^2(2x) \cdot 2$$

- $y = \coth(\tan(x))$

$$\Rightarrow y' = -\operatorname{csch}^2(\tan x) \cdot \sec^2 x$$

- $y = \operatorname{sech}^3 x$

$$\Rightarrow y' = 3\operatorname{sech}^2(x) \cdot (-\operatorname{sech}(x)\tanh(x) \cdot 1)$$

- $y = 4\operatorname{csch}\left(\frac{x}{4}\right)$

$$\Rightarrow y' = 4 \cdot \left(-\operatorname{csch}\left(\frac{x}{4}\right)\right) \cdot \coth\left(\frac{x}{4}\right) \cdot \frac{1}{4}$$

Problems (5.8): Find y' of the following:

1 $y = \frac{\cosh(x)}{x}$

8. $\sinh(y) = \sec(x)$

2 $y = \sinh^2(3w)$

8. $y = \coth\left(\frac{1}{\theta}\right)$

3 $y = e^w \cdot \cosh(w)$

9. $y^2 + x \cosh y + \sinh^2 x = 50$

4 $\sin^{-1}(x) = \operatorname{sech}(y)$

10. $y = \cosh^2(5x) - \sinh^2(5x)$

5 $y = \tanh\left(\frac{4t+1}{5}\right)$

11. $y = \operatorname{csch}^3(\sqrt{2x})$

6 $\tan(x) = \tanh^2(y)$

12. $\sinh(y) = \tanh(x)$

7 $x = \cosh(\cos(y))$