Electric Potential

Electrical Potential Energy

A charge q moving in a constant electric field **E** experiences a force $F = qE$ from that field. Also, as we know from our study of work and energy, the work done on the charge by the field as it moves from point r_1 to r_2 is

$$W = \int_{r_1}^{r_2} F \cdot ds \qquad (1)$$

When F is the electrostatic force

$F = F_e = qE$ the work done then, becomes

$$W = \int_{r_1}^{r_2} qE \cdot ds = q \int_{r_1}^{r_2} E \cdot ds \qquad (2)$$

In Fig. 4.1, a charge is shown being moved from r_1 to r_2 along two different paths, with ds and E shown for a bit of each of the paths.

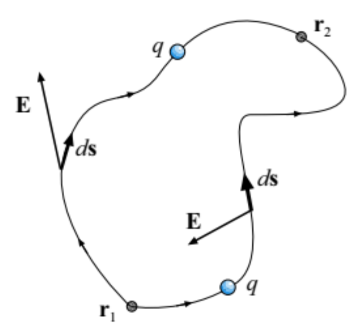

Figure 1: Charge is moved from r_1 to r_2 along two separate paths.

Now it turns out that from the mathematical form of the electrostatic force, the work done by the force does not depend on the path taken to get from r_1 to r_2.

As a result we say that the **electric force is conservative** and it allows us to calculate **an electric potential energy**, which as usual we will denote by U.

As before, only the changes in the potential ΔU have any real meaning, and the change in potential energy is the negative of the work done by the electric force:

$$\Delta U = -W = -q \int_{r_1}^{r_2} E \cdot ds \qquad (3)$$

Usually we will make the choice that the potential energy is zero when the charge is infinitely far away: $U_\infty = 0$.

Electric Potential

Recall how we developed the concept of the electric field E: The force on a charge q_0 is always proportional to F, so by dividing the charge by F we get something which we called electric field E can conveniently give the force on any charge.

$$E = \frac{F}{q}$$

Likewise, if we divide out the changes in the potential ΔU by charge q from Eq. 3 we get a function which we can use to get the change in potential energy for any charge (simply by multiplying by the charge). This new function is called the **electric potential**, V:

$$\Delta V = \frac{\Delta U}{q} \qquad (4)$$

where ΔU is the change in potential energy of a charge q. Then Eq.3 gives us the difference in **electrical potential** between points r_1 and r_2:

$$\Delta V = \frac{\Delta U}{q} = -\int_{r_1}^{r_2} E \cdot ds$$

$$\boxed{\Delta V = V_1 - V_2 = -\int_{r_1}^{r_2} E \cdot ds \qquad (5)}$$

The electric potential is a scalar. The unit of electric potential is a volt,

$$1\ volt = 1V = 1\ \frac{J}{C}$$

Of course, it is then true that a joule is equal to a coulomb. Volt

$$1\ J = 1C \times V$$

In general, multiplying a charge times a potential difference gives an energy. It often happens that we are multiplying an elementary charge (e) and a potential difference in volts. It is then convenient to use the unit of energy given by the product of e and a volt; this unit is called the **electron-volt**:

$$1\,\mathrm{eV} = (e)\cdot(1\,\mathrm{V}) = 1.60\times10^{-19}\,\mathrm{C}\cdot(1\,\mathrm{V}) = 1.60\times10^{-19}\,\mathrm{J}$$

To arrive at a function V (r) defined at all points we need to specify a point at which the potential V is zero. Often we will choose this point to be “infinity” (∞) that is, as we get very far away from the set of charges which give the electric field. So we can write:

$$V(r) = -\int_{\infty}^{r} E\cdot ds \qquad (6)$$

Finding electric field E from potential V

The definition of V an integral involving the E field implies that the electric field comes from V by taking derivatives:

$$\Delta V = -\int_{r_1}^{r_2} E\cdot ds$$

or

$$dV = -E\cdot ds \quad or \quad E = -\frac{dV}{ds}$$

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

These relations can be written as one equation using the notation for the gradient:

$$E = -\nabla V$$

Equipotential Surfaces

For a given configuration of charges, a set of points where the electric potential V(r) has a given value is called an **equipotential surface**. It takes no work to move a charged particle from one point on such a surface to another point on the surface, for then we have $\Delta V = 0$.

From the relations between E(r) and V (r) it follows that the field lines are perpendicular to the equipotential surfaces everywhere.

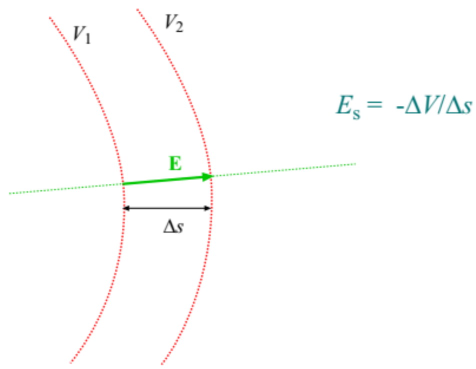

Figure 2: Equipotential surfaces

Principal of Superposition

The law of superposition applies to electrical potential as well as to the electrical field:

$$V = V_1 + V_2 + V_3 + \cdots$$

where each V_i is the potential due to a separate charge distribution. It follows that

$$\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3 + \cdots$$

This means that we can use the expression for the electric potential of a point charge to build up the potential of arbitrary charge distributions just as we can use Coulomb's law to find field expressions for arbitrary charge distributions.

$$V_a = \sum_{i=1}^{n} V_i = k \sum_{i=1}^{n} \frac{q_i}{r_{ia}}$$

We can also produce an integral from this:

$$V_a = \int dV_a = k \int \frac{dq}{r}$$

$$V = k \int \frac{dq}{r} \qquad (12)$$

There are 2 differences between this expression for V and that we use for *E*. First, the potential drops off as $\frac{1}{r}$ rather than $\frac{1}{r^2}$. This is significant, but doesn't make a huge difference in calculation. The fact that *V* is a scalar and E is a vector, however, can make these integrals much easier to evaluate than their field counterparts.

Examples

Point charge

Find the electrical potential of a point charge.

Solution:

From Eq. 5

$$\Delta V = V_a - V_b = -\int_a^b E \cdot ds$$

The electric field for a point charge is

$$\boldsymbol{E} = k \frac{Q}{r^2} \hat{r}$$

then

$$V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{\ell} = -kQ\int_a^b \frac{dr}{r^2} = -kQ\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

$$V_{ba} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

when the point b at infinity $r_2 \rightarrow \infty$, then $V_b \rightarrow 0$

$$V = k\frac{Q}{r}$$

This is the electric potential of point charge.

Ex-2. An infinite nonconducting sheet has a surface charge density $\sigma = 0.10\mu C/m^2$ on one side. How far apart are equipotential surfaces whose potentials differ by 50 V?

Solution:

The electric field due a nonconducting sheet of charge is given by:

$$E = \frac{\sigma}{2\epsilon_0}$$

Suppose the sheet lies in the xy plane, and the E field is uniform then $E_x = E_y = 0$, the only E_z component for electric field which is perpendicular to the surface of sheet

$$\begin{aligned} E_z &= \frac{\sigma}{2\epsilon_0} \\ &= \frac{(0.10\times 10^{-6}\,\frac{\mathrm{C}}{\mathrm{m}^2}}{2(8.85\times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m}^2})} = 5.64\times 10^3\,\frac{\mathrm{N}}{\mathrm{C}} \end{aligned}$$

Now, we have

$$E_z = \frac{\partial V}{\partial z} = -5.64 \times 10^{12} \frac{N}{C}$$

and when the rate of change of some quantity is constant we can write the relation in terms of finite changes, that is, with "Δ"s:

$$\frac{\Delta V}{\Delta z} = -5.64 \times 10^{12} \frac{N}{C}$$

and from this result we can find the change in z corresponding to any change in V . If we are interested in ΔV = 50 V, then

$$\Delta z = -\frac{\Delta V}{E_z} = -\frac{(50\,\mathrm{V})}{(5.64 \times 10^{3}\,\frac{\mathrm{N}}{\mathrm{C}})} = -8.8 \times 10^{-3}\,\mathrm{m} = -8.8\,\mathrm{mm}$$

Ex. 3:

What are (a) the charge and (b) the charge density on the surface of a conducting sphere of radius 0.15 m whose potential is 200 V (with V = 0 at infinity)?

Solution:

Suppose the radius of the sphere is *R*

Outside sphere $r \geq R$

In this case, the Gaussian surface is a sphere of radius $r \geq R$. Since the radius of the "Gaussian sphere" is greater than the radius of the spherical shell, all the charge is enclosed:

$$q_{enc} = Q$$

by applying Gauss's law, we obtain

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

$$E \oint d\vec{A} = E(A) = E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} = k\frac{Q}{r^2} \quad , \quad r > R$$

This equation holds for r ≥ R (outside sphere).

The electric potential

$$\Delta V = V_a - V_b = -\int_a^b E \cdot ds$$

In this case $ds = dr$, then

$$\Delta V = V_\infty - V_R = -kQ \int_\infty^r \frac{dr}{r^2}$$

with the condition V = 0 at infinity, we would get the same result for V as we would for a point charge Q, namely:

$$V = k\frac{Q}{\mathrm{r}} \quad , \quad r > R \, , \qquad (outside\ sphere)$$

Then at the sphere's surface (r = R) we have:

$$V = k\frac{Q}{\mathrm{R}}$$

Solve for Q and plug in the numbers:

$$Q = \frac{VR}{k}$$

$$Q = \frac{200 \times 0.15}{9 \times 10^9} = 3.3 \times 10^{-9}\ C$$

The charge on the sphere is 3.3×10^{-9} C

(b) The charge density

To get the charge density, divide the charge by the surface are of the sphere:

$$\sigma = \frac{Q}{4\pi R^2} = \frac{(3.3 \times 10^{-9}\,\mathrm{C})}{4\pi(0.15\,\mathrm{m})2} = 1.2 \times 10^{-8}\,\frac{\mathrm{C}}{\mathrm{m}^2}$$

The charge density on the sphere's surface is 1.2×10^{-8} m/C^2.

Ex. 3:

The electric potential at points in an *xy* plane is given by

$$V = \left(2.0\,\frac{V}{m^2}\right)x^2 - \left(3.0\,\frac{V}{m^2}\right)y^2$$

What are the magnitude and direction of the electric field at the point (3.0, 2.0)m?

Solution:

The electric potential

$$V = \left(2.0\,\frac{V}{m^2}\right)x^2 - \left(3.0\,\frac{V}{m^2}\right)y^2$$

The electric field is given by

$$E_x = -\frac{\partial V}{\partial x} = -4\,x\,V/m$$

and

$$E_y = -\frac{\partial V}{\partial y} = 6\,y\,V/m$$

Plugging in the given values of x = 3.0 m and y = 2.0 m we get:

$$E_x = 4\,(3) = -12\,V/m^2$$

$$E_y = 6\,(2) = 12\,V/m^2$$

So the magnitude of the E field at the given is

$$E = \sqrt{(12.0)^2 + (12.0)^2}\,\frac{\mathrm{V}}{\mathrm{m}} = 17\,\frac{\mathrm{V}}{\mathrm{m}}$$

and its direction

$$\theta = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}(1.0) = 135°$$

where for θ we have made the proper choice so that it lies in the second quadrant.

Ex. 4:

In the rectangle of Fig. 3, the sides have lengths 5.0 cm and 15 cm, $q_1 = -5\ \mu C$ and $q_2 = +2\ \mu C$. What are the electric potentials (a) at corner A and (b) corner B? (c) How much work is required to move a third charge $q_3 = +3\ \mu C$ from B to A along a diagonal of the rectangle?

Solution

(a) The electric potential at corner A.

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2}\right] \\ &= (8.99 \times 10^9\ \tfrac{\text{N·m}^2}{\text{C}^2})\left[\frac{(-5.0 \times 10^{-6}\ \text{C})}{(15 \times 10^{-2}\ \text{m})} + \frac{(+2.0 \times 10^{-6}\ \text{C})}{(5.0 \times 10^{-2}\ \text{m})}\right] = 6.0 \times 10^4\ \text{V} \end{aligned}$$

(b) The electric potential at corner B.

$$V = (8.99 \times 10^9\ \tfrac{\text{N·m}^2}{\text{C}^2})\left[\frac{(-5.0 \times 10^{-6}\ \text{C})}{(5.0 \times 10^{-2}\ \text{m})} + \frac{(+2.0 \times 10^{-6}\ \text{C})}{(15 \times 10^{-2}\ \text{m})}\right] = -7.8 \times 10^5\ \text{V}$$

(c) The work is required to move a charge $q_3 = +3\ \mu C$ from B to A.

The potential difference ΔV as we move from point B to point A:

$$\Delta V = V_A - V_B = 6.0 \times 10^4\ \text{V} - (-7.8 \times 10^5\ \text{V}) = 8.4 \times 10^5\ \text{V}$$

The change in potential energy for a +3.0 μC charge to move from B to A is

$$\Delta U = q\Delta V = (3.0 \times 10^{-6}\ \text{C})(8.4 \times 10^{5}\ \text{V}) = 2.5\ \text{J}$$

the work done is equal to the negative change in potential energy

$$W = -\Delta U = -2.5\ J$$

the work done is equal to $W = -2.5\ J$