Magnetic Scalar and Vector Potentials

We recall that some electrostatic field problems were simplified by relating the electric Potential V to the electric field intensity **E** ($\boldsymbol{E} = -\nabla V$). Similarly, we can define a potential associated with magnetostatic field B.

In fact, the magnetic potential could be scalar V_m and vector A.

To define V_m and A involves two important identities:

$$\nabla \times (\nabla V) = 0 \qquad (1)$$

$$\nabla \cdot (\nabla \times A) = 0 \qquad (2)$$

which must always hold for any scalar field V and vector field A.

Magnetic Scalar Potential

Just as $E = -\nabla V$, we define the *magnetic scalar potential* V_m (in amperes) as related to magnetic intensity H according to

$$H = -\nabla V_m \quad if \quad J = 0 \qquad (3)$$

The condition attached to this equation is important and will be explained.

Taking curl of both sides of eq. 3

$$\nabla \times H = -\nabla \times \nabla V_m \qquad (4)$$

Form of the **Ampere's law**

$$\nabla \times \vec{H} = \vec{J} \qquad (5)$$

Combining eq. (4) and eq. (5) gives

$$\vec{J} = \nabla \times \vec{H} = -\nabla \times \nabla V_m \qquad (6)$$

From identity 1, the curl of gradient potential must equal zero, then eq, 6 becomes

$$\vec{J} = \nabla \times \vec{H} = -\nabla \times \nabla V_m = 0 \qquad (7)$$

Thus the magnetic scalar potential V_m is only defined in a region where J = 0 as in eq. (3).

V_m satisfies Laplace's equation

We should also note that V_m satisfies Laplace's equation just as electric potential V does for electrostatic fields. To verify this:

$$B = \mu_0 H$$

Taking divergence of both side

$$\nabla \cdot B = \mu_0 \nabla \cdot H \qquad (8)$$

But $$\nabla \cdot B = 0 \qquad (9)$$

equating eq. 8 & 9

$$\mu_0 \nabla \cdot H = 0 \qquad (10)$$

from eq.3 $$H = -\nabla V_m$$

eq. 10 becomes

$$\nabla \cdot (-\nabla V_m) = 0$$

$$\boxed{\nabla^2 V_m = 0 \quad (J = 0) \qquad (11)}$$

That is V_m satisfy Laplace's equation.

Vector magnetic potential

Vector magnetic potential exists in regions where $J \neq 0$ is present. It is defined in such a way that its curl gives the magnetic flux density, that is,

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

where A = vector magnetic potential (wb/m)

Comparison between Electric Potential and Vector Magnetic Potential

As we saw in electrostatic, the electric potential V can be calculated for any charge distributions. Similarly, in magnetostatic the Vector Magnetic Potential A can be calculated for any current distributions.

Electric potential V		Vector magnetic potential A	
Unit	Volt V	Unit	$Tesla \cdot m,$ $or\ Wb/m$
Point charge	$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}$	Element current (Idl)	$A = \frac{\mu_0}{4\pi}\int \frac{Idl}{r}$
Line distribution of charge (ηdl)	$V_l = \frac{1}{4\pi\varepsilon_0}\int_l \frac{\eta dl}{r}$	Line carrying current l	$A = \frac{\mu_0}{4\pi}\int \frac{Idl}{r}$
Surface distribution σ	$V_s = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma\, dS}{r}$	Surface distribution K	$A = \frac{\mu_0}{4\pi}\int_S \frac{K\, dS}{r}$
Volume distribution ρ	$V_v = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho dV}{r}$	Volume distribution J	$A = \frac{\mu_0}{4\pi}\int_V \frac{JdV}{r}$

Where: η is the line charge density (C/m) , σ is the surface charge density $(\mathrm{C/m^2})$, ρ is the volume charge density $(\mathrm{C/m^3})$, K is the surface current density $(Amp./\mathrm{m^2})$, J is the volume current density $(Amp./\mathrm{m^3})$

Ex-1

The vector magnetic potential, A due to a direct current in a conductor in free space is given by $A = (x^2 + y^2)\hat{k}\ \mu wb/m^2$. Determine the magnetic field produced by the current element at (1, 2, 3).

Solution:

$A = (x^2 + y^2)\hat{k}\ \mu wb/m^2$
$\mathrm{B} = \nabla \times A$

$$\mathrm{B} = 10^{-6} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & (x^2 + y^2) \end{vmatrix}$$

$$= \left[\left(\frac{\partial}{\partial y}(x^2 + y^2) - 0\right)\hat{\imath} - \left(\frac{\partial}{\partial x}(x^2 + y^2) - 0\right)\hat{\jmath} + (0)\hat{k} \right] \times 10^{-6}$$

$$B = [\,2y\hat{\imath} - 2x\hat{\jmath}\,] \times 10^{-6}\ wb/m^2$$

$$B\ at\ (1,2,3) = [\,2(2)\hat{\imath} - 2(1)\hat{\jmath}\,] \times 10^{-6}\ wb/m^2$$

$$B = (4\hat{\imath} - 2\hat{\jmath}) \times 10^{-6}\ wb/m^2$$

Ex-2

A long, straight, current-carrying conductor ("wire"), calculate:

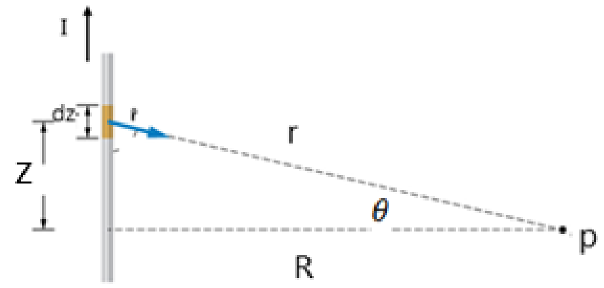
1- The magnetic vector potential $\vec{A}$ at point p at distance R away from it.

2- The magnetic flux density $\vec{B}$ at point p.

Solution

1- The magnetic vector potential $\vec{A}$

Consider an element $\hat{z}dz$ on the wire at a height z above the *xy*-plane. (The length of this element is *dz*; the unit vector $\hat{z}$ just indicates its direction). Consider also a point P in the *xy*-plane at a distance *r* from the wire.



The contribution to the magnetic vector potential is therefore

$$\vec{A}_z = \hat{k}\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\frac{Idz}{r} = 2\ \times\hat{k}\frac{\mu_0}{4\pi}\int_{0}^{\infty}\frac{Idz}{r} = \ \hat{k}\frac{\mu_0}{2\pi}\int_{0}^{\infty}\frac{Idz}{r} \qquad (1)$$

because of the wire is infinite length, then the limit of integration from $(-\infty\ to + \ \infty\ )$.

The distance of P from the element *dz* is

$$r = \sqrt{z^2 + R^2} \qquad (2)$$

Sub. eq. 2 in 1

$$\vec{A}_z = \hat{k}\frac{\mu_0 I}{2\pi}\int_{0}^{\infty}\frac{dz}{\sqrt{z^2 + R^2}} \qquad (3)$$

From the fig.

$$tan\theta = \frac{z}{R} \rightarrow z = R\ tan\theta$$

$$dz = R\ sec^2\theta\ d\theta$$

eq. 3 becomes

$$\vec{A}_z = \hat{k}\frac{\mu_0 I}{2\pi}\int_{0}^{\infty}\frac{R\ sec^2\theta\ d\theta}{\sqrt{R^2 tan^2\theta + R^2}} = \hat{k}\frac{\mu_0 I}{2\pi}\int_{0}^{\infty}\frac{R\ sec^2\theta\ d\theta}{R\sqrt{tan^2\theta + 1}}$$

$$= \hat{k}\frac{\mu_0 I}{2\pi}\int_0^{\infty}\frac{R\, sec^2\theta\, d\theta}{R \sec\theta}$$

$$\vec{A}_z = \hat{k}\frac{\mu_0 I}{2\pi}\int_0^{\infty} sec\,\theta\, d\theta \qquad (4)$$

the limit of integration should be from 0 to θ

$$\vec{A}_z = \hat{k}\frac{\mu_0 I}{2\pi}\int_0^{\theta} sec\,\theta\, d\theta = \hat{k}\frac{\mu_0 I}{2\pi}\; ln(sec\,\theta + tan\,\theta)|_0^{\theta}$$

$$\vec{A}_z = \hat{k}\frac{\mu_0 I}{2\pi}\; ln(sec\,\theta + tan\,\theta) - ln(1)$$

$$\vec{A}_z = \hat{k}\frac{\mu_0 I}{2\pi}\; ln(sec\,\theta + tan\,\theta) \qquad (5)$$

$$sec\,\theta = \frac{1}{cos\,\theta} = \frac{1}{\frac{R}{r}} = \frac{r}{R} = \frac{\sqrt{l^2 + R^2}}{R}$$

$$tan\,\theta = \frac{l}{R}$$

$$\vec{A}_z = \hat{k}\frac{\mu_0 I}{2\pi}\; ln\left(\frac{\sqrt{l^2 + R^2}}{R} + \frac{l}{R}\right)$$

$$\vec{A}_z = \hat{k}\frac{\mu_0 I}{2\pi}\; ln\left(\frac{\sqrt{l^2 + R^2} + l}{R}\right) \qquad (5)$$

for $\gg R$, then eq. 5 becomes

$$\vec{A}_z = \hat{k}\frac{\mu_0 I}{2\pi}\; ln\left(\frac{2l}{R}\right) = \hat{k}\;\frac{\mu_0 I}{2\pi}\left(\, ln(2l) - \; ln(R)\right)$$

2- The magnetic flux $\vec{B}$

Because of

$$\mathrm{B} = \nabla \times \mathrm{A}$$

We'll work in cylindrical coordinates, and the symbols (r, φ, z) will denote the unit orthogonal vectors.

$$\nabla \times \mathrm{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\varphi} + \left(\frac{1}{r}\frac{\partial (rA_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \varphi}\right)\hat{z}$$

The magnetic vector potential whose only z-component $\vec{A}_z$

$$\nabla \times \mathrm{A} = \frac{1}{r}\frac{\partial A_z}{\partial \varphi}\hat{r} - \frac{\partial A_z}{\partial r}\hat{\varphi}$$

Becouse of $\vec{A}_z$ depends on radial r only, then $\frac{\partial A_z}{\partial \varphi} = 0$, hence

$$\nabla \times \mathrm{A} = -\frac{\partial A_z}{\partial r}\hat{\varphi}$$

$$\vec{A}_z = \hat{k}\,\frac{\mu_0 I}{2\pi}\left(\,ln(2l) - \,ln(R)\right)$$

$$B = \nabla \times A = -\frac{\partial \vec{A}}{\partial r}\hat{\varphi} \;= -\hat{\varphi}\,\frac{\mu_0 I}{2\pi}\frac{\partial}{\partial r}\left(\,ln(2l) - \,ln(R)\right)$$

$$B = \nabla - \hat{\varphi}\,\frac{\mu_0 I}{2\pi}\frac{\partial}{\partial r}\left(0 - \frac{1}{R}\right)$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi R}\hat{\varphi}}$$

Which is the same result that we get from Biot-Savart’ and Amper’ laws.