Magnetostatics

Up until now, we have been discussing electrostatics, which deals with physics of the electric field created by **static charges**. We will now look into a different phenomenon, that of production and properties of magnetic field, whose source is **steady current**, i.e., of charges in motion. An essential difference between the electrostatics and magnetostatics is that electric charges can be isolated, i.e., there exist positive and negative charges which can exist by themselves. Unlike this situation, magnetic charges (which are known as magnetic monopoles) cannot exist in isolation, every north magnetic pole is always associated with a south pole, so that the net magnetic charge is always zero.

Lorentz force law

Sources of magnetic field are steady currents. In such a field a **moving charge experiences a sidewise force.** Recall that an electric field exerts a force on a charge, irrespective of whether the charge is moving or static. Magnetic, field, on the other hand, exerts a force only on charges that are moving.

$$F_{mag} = Q(v \times B)$$

Under the combined action of electric and magnetic fields, a charge experiences, what is known as **Lorentz force**,

$$\boxed{\vec{F} = Q\left[\vec{E} + (v \times \vec{B})\right] \qquad (1)}$$

where the field $\vec{B}$ is known by various names, such as, "**magnetic field of induction", "magnetic flux density**", or simply, as we will be referring to it in this course as the "**magnetic field**" .

Notes:

• Particles moving parallel to the magnetic field do not experience a magnetic force.

• The magnetic force cannot do work, since it is always perpendicular to the motion of the particle. It cannot change the speed of a particle, only its direction.

Biot Savart Law

The field due to the infinitesimal charge element is given by Coulomb's law, which is an inverse square law. We take a similar approach to calculate the magnetic field due to a charge distribution.

Using *small* straight wires containing currents and compass magnets, Oersted, Biot, and Savart experimentally found the following properties:
- The magnetic field is directly proportional to the length *dl* of the small wire.

$$d\boldsymbol{B} \propto dl$$

- The magnetic field is directly proportional to the electrical current *I* in the wire.
- The magnetic field is inversely proportional to the square of the distance *r* from the wire.
- The magnetic field points in the direction normal to the plane in which the wire and observation point lie.
- Expressing each of these experimental observations one by one

$$dB \propto dl$$

$$dB \propto I\,dl$$

$$dB \propto \frac{I\,dl}{r^2}$$

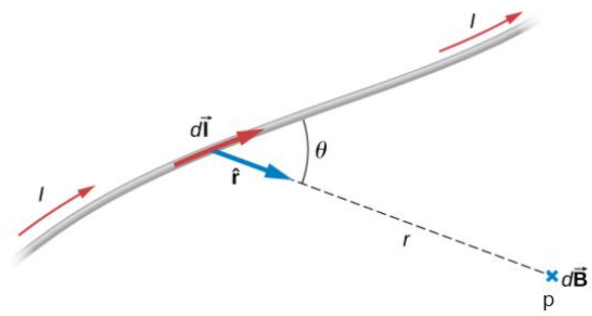
$$d\,\mathbf{B} \propto \frac{I\,dl}{r^2}(\hat{\mathbf{l}} \times \hat{\mathbf{r}})$$

$$d\,\mathbf{B} = k\frac{I\,dl}{r^2}(\hat{\mathbf{l}} \times \hat{\mathbf{r}})$$

The **Biot-Savart law** states that at any point *P*, the magnetic field $\vec{B}$ due to an element of a current-carrying wire *I* is given by

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l} \times \hat{r}}{r^2} \qquad (2)}$$

where the constant μ_0 is known as the **permeability of free space** it's value in SI unit is exactly $\mu_0 = 4\pi \times 10^{-7}\ T \cdot m/A$



The magnitude of the magnetic field $\vec{B}$ is given by

$$dB = \frac{\mu_0}{4\pi}\frac{Idlsin\theta}{r^2} \qquad (3)$$

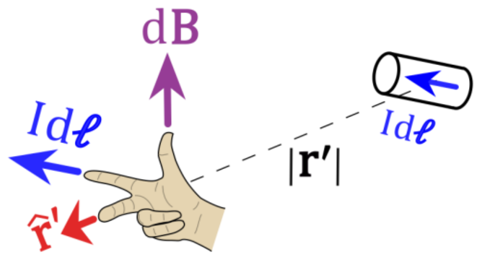
where θ is the angle between $d\vec{l}$ and $\hat{r}$. Notice that if $\theta = 0$, then $\vec{B} = 0$.

The magnetic field due to a finite length of current-carrying wire is found by integrating e**quation 2** along the wire, giving us the usual form of the Biot-Savart law.

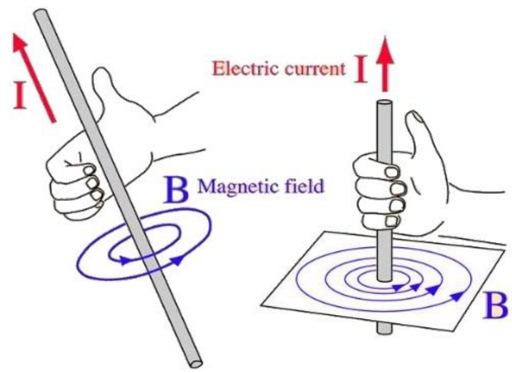
$$\int dB = \frac{\mu_0 I}{4\pi}\int \frac{dl \times \hat{r}}{r^2}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l} \times \hat{r}}{r^2} \qquad (4)$$

The direction of the magnetic field $\vec{B}$ is determined by applying the right-hand rule to the vector product $d\vec{l} \times \hat{r}$.



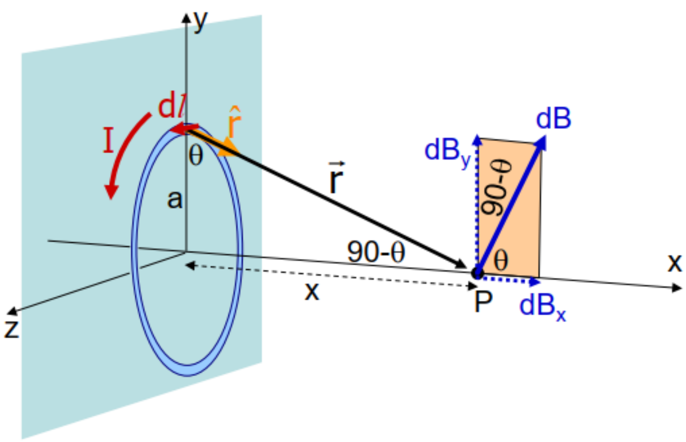
Also, the direction of the magnetic field due to a long straight wire can be determined by the right-hand rule. If you direct your right thumb along the direction of the current in the wire, then the fingers of your right hand curl in the direction of the magnetic field.



Ex. 1:

A circular loop wire with current *I*, radius *a* in yz plane. Calculate the B-field at following cases:

1- At a distance *x* along its axis from the center.
2- At the center of the loop.
3- For coil consist of N loop.



Solution:

From The Biot-Savart law

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\, d\vec{l} \times \hat{r}}{r^2} \qquad (1)$$

Because of $d\vec{l}$ and $\hat{r}$ are perpendicular to each other, then $dl \times \hat{r} = dl$.

Therefore

$$dB = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}$$

From the figure we find

$$r^2 = x^2 + a^2 \qquad (2)$$

Sub. Eq. 2 in Eq.1

$$dB = \frac{\mu_0}{4\pi}\frac{Idl}{(x^2 + a^2)} \qquad (3)$$

From the symmetric of the problem, the algebraic summation of all y-components of the magnetic field are equal to zero

$$\sum dB_y = 0$$

while the x-components of the magnetic field are equal to:

$$dB_x = \frac{\mu_0}{4\pi} \frac{Idl}{(x^2 + a^2)} \cos\theta$$

From the fig. $\cos\theta = \frac{a}{(x^2 + a^2)^{1/2}}$

Then, $dB_x = \frac{\mu_0}{4\pi} \frac{Idl}{(x^2 + a^2)} \cos\theta = \frac{\mu_0}{4\pi} \frac{Idl}{(x^2 + a^2)} \frac{a}{(x^2 + a^2)^{1/2}}$

$$dB_x = \frac{\mu_0}{4\pi} \frac{Iadl}{(x^2 + a^2)^{3/2}}$$

Integrate both side

$$B_x = \int_{ring} dB_x$$

$$B_x = \frac{\mu_0}{4\pi} \frac{Iadl}{(x^2 + a^2)^{\frac{3}{2}}} \int_{ring} dl$$

$$B_x = \frac{\mu_0}{4\pi} \frac{Ia}{(x^2 + a^2)^{3/2}} (l)$$

where $l = 2\pi a$ is the circumference of the ring

$$B_x = \frac{\mu_0}{4\pi} \frac{Ia}{(x^2 + a^2)^{3/2}} 2\pi a$$

$$B_x = \frac{\mu_0}{2} \frac{I\,a^2}{(x^2 + a^2)^{3/2}}$$

2- at the center of the loop.

$$x = 0$$

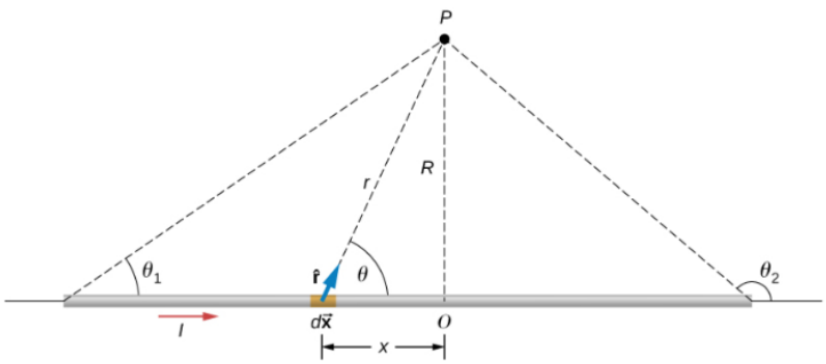
$$B_{x,center} = \frac{\mu_0}{2} \frac{I\ a^2}{(a^2\ )^{3/2}} = \frac{\mu_0}{2} \frac{I\ a^2}{a^3} = \frac{\mu_0 I}{2a}$$

4- For coil consist of N loop.

$$B_x = \frac{\mu_0 NI}{2a}$$

Ex. 2

A thin, straight wire carrying a current *I* is placed along the x-axis, as shown in Figure Evaluate the magnetic field at p.



Solution:

From The Biot-Savart law eq. 3 we have,

$$B = \frac{\mu_0 I}{4\pi} \int \frac{Isin\theta dx}{r^2}$$

$$B = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} \frac{I sin\theta dx}{r^2}$$

From symmetry about O point

$$B = 2 \times \frac{\mu_0 I}{4\pi} \int_{0}^{\infty} \frac{I sin\theta dx}{r^2}$$

$$B = \frac{\mu_0 I}{2\pi} \int_{0}^{\infty} \frac{I sin\theta dx}{r^2} \qquad (1)$$

From the fig.

$$r = \sqrt{x^2 + R^2}$$

$$\sin\theta = \frac{R}{\sqrt{x^2 + R^2}}.$$

Substitute in eq. 1

$$B = \frac{\mu_0 I}{2\pi} \int_{0}^{\infty} \frac{R dx}{(x^2 + R^2)^{3/2}} \qquad (2)$$

From the fig. $tan\theta = \frac{R}{x} \rightarrow x = \frac{R}{tan\theta} = R\, cot\theta$

$$dx = -R\, csc^2\theta\, d\theta$$

The limit of the integration, $x = 0 \rightarrow \theta = \frac{\pi}{2}$

$$x = \infty \rightarrow \theta = 0$$

Substitute in eq. 2

$$B = \frac{\mu_0 I}{2\pi} \int_{\frac{\pi}{2}}^{0} \frac{-R \cdot R\, csc^2\theta\, d\theta}{(R^2 cot^2\theta + R^2)^{3/2}}$$

$$B = \frac{-\mu_0 I}{2\pi} \int_{\frac{\pi}{2}}^{0} \frac{R^2\, csc^2\theta\, d\theta}{R^3(cot^2\theta + 1)^{3/2}} = \frac{-\mu_0 I}{2\pi R} \int_{\frac{\pi}{2}}^{0} \frac{csc^2\theta\, d\theta}{(csc^2\theta)^{3/2}}$$

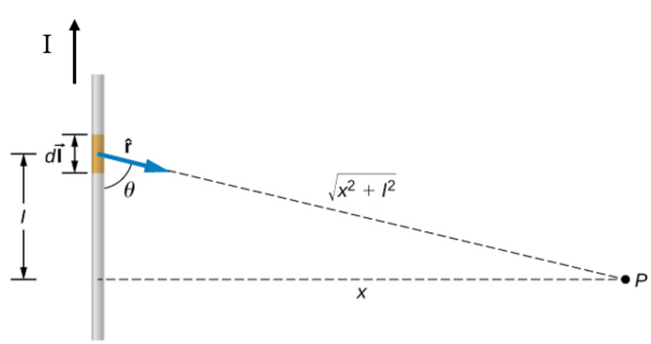
$$B = \frac{-\mu_0 I}{2\pi R} \int_{\frac{\pi}{2}}^{0} \frac{csc^2\theta\, d\theta}{csc^3\theta} = \frac{-\mu_0 I}{2\pi R} \int_{\frac{\pi}{2}}^{0} \frac{d\theta}{csc\theta}$$

$$B = \frac{-\mu_0 I}{2\pi R} \int_{\frac{\pi}{2}}^{0} \sin\theta\, d\theta = \frac{-\mu_0 I}{2\pi R} (-cos\theta\,]_{\frac{\pi}{2}}^{0}$$

$$B = \frac{\mu_0 I}{2\pi R}$$

Ex. 3

A short wire of length 1.0 cm carries a current of 2.0 A in the vertical direction (**Figure**). The rest of the wire is shielded so it does not add to the magnetic field produced by the wire. Calculate the magnetic field at point *P*, which is 1 meter from the wire in the *x*-direction.



Solution:

From The Biot-Savart law

$$dB = \frac{\mu_0}{4\pi}\frac{I \cdot dl sin\theta}{r^2}$$

$$B = \frac{\mu_0}{4\pi}\frac{I \cdot \Delta l sin\theta}{r^2}$$

From the fig.

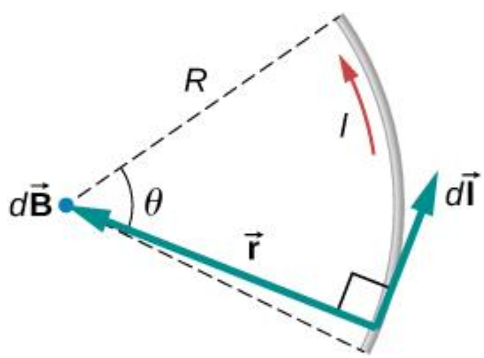
$$\theta = tan^{-1}\left(\frac{1}{0.01}\right) = 89.4^0$$

$$B = \frac{\mu_0}{4\pi}\frac{I \cdot \Delta l sin\theta}{r^2} = \frac{4\pi \times 10^{-7}}{4\pi}\left(\frac{2 \times 0.01 \times \sin(89.4^0)}{1^2}\right)$$

$$B = 2 \times 10^{-9}\, T$$

Ex. 4

A wire carries a current *I* in a circular arc with radius *R* swept through an arbitrary angle θ (**Figure**). Calculate the magnetic field at the center of this arc at point *P*.



Solution:

From The Biot-Savart law

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \hat{r}}{r^2} \qquad (1)$$

Because of $d\vec{l}$ and $\hat{r}$ are perpendicular to each other, then $dl \times \hat{r} = dl$.

Therefore

$$dB = \frac{\mu_0}{4\pi} \frac{Idl}{r^2}$$

Integrate both side

$$B = \frac{\mu_0 I}{4\pi} \int \frac{dl}{r^2}$$

From the arc length $dl = rd\theta$, then

$$B = \frac{\mu_0 I}{4\pi} \int \frac{rd\theta}{r^2} = \frac{\mu_0 I}{4\pi r} \int d\theta = \frac{\mu_0 I}{4\pi r} \int_0^{\theta} d\theta$$

$$B = \frac{\mu_0 I\theta}{4\pi r}$$